A TREATISE

ON

ELECTRICITY AND MAGNETISM

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AND

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BY

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ELECTRICITY AND MAGNETISM

PART I.

METHODS OF MEASUREMENT.

CHAPTER I.

MEASUREMENT OF ANGLES.

656. Principal Kinds of Measurements. — The various magnitudes which are met with in physical measurements may all be referred to the three fundamental mechanical units—the units of *length*, of *mass* or *weight*, and of *time*.

We need not dwell here on the measurement of rectilinear lengths, nor on that of weight; these are the most frequent physical operations, and they present no particular difficulty when we do not attempt to reach the ultimate limits of accuracy. We shall restrict ourselves to mentioning some of the corrections which they necessitate.

The scale used for measuring lengths is ordinarily divided in millimetres. Whatever may be the value of the divisions, fractions of divisions are estimated either by a vernier or by a micrometric telescope. It is clear that the accuracy of any particular mode of subdivision cannot exceed the accuracy with which the scale has been graduated.

If we denote by ϵ the value of a division of the scale at the temperature 0°, then if λ is the coefficient of linear expansion of the scale, this length at the temperature t will be equal to $\epsilon(1 + \lambda t)$.

If a length λ measured on the scale at the temperature t, is represented by n divisions, its true length is

$$l = n\epsilon(1 + \lambda t).$$

If we assume that the value of the divisions of the scale at a temperature t_0, but little different from ordinary temperatures, is ϵ_0, we shall have

$$\epsilon(1+\lambda t_0)=\epsilon_0,$$

and therefore

$$l=n\epsilon_0\frac{1+\lambda t}{1+\lambda t_0}=n\epsilon_0\left[1+\lambda(t-t_0)\right].$$

The values of the coefficient λ vary from 8.10^{-6} to 20.10^{-6} for ordinary metals; a change of temperature of 10°, would at most necessitate a correction of 2 in ten thousand, which in most cases could be neglected.

If the object to be measured is itself at the temperature t^0, and we desire to know its length l_0 at zero, then if λ is the coefficient of expansion,

$$l_0=\frac{l}{1+\lambda' t'}=n\epsilon_0\left[1+\lambda(t-t_0)\right]\left[1-\lambda' t'\right]=n\epsilon\left[1+\lambda(t-t_0)-\lambda' t'\right].$$

This second correction is of the same order of importance as the first.

657. The *balance* enables us to ascertain the equality of two apparent weights. If P is the true weight of the body weighed, π the actual weights which balance it, D, Δ, and δ the specific gravities of the body, of the weights, and of air, we have

$$P\left(1-\frac{\delta}{D}\right)=\pi\left(1-\frac{\delta}{\Delta}\right);$$

from which, with sufficient approximation,

$$P=\pi\left[1-\delta\left(\frac{1}{\Delta}-\frac{1}{D}\right)\right].$$

In order to give an idea of the importance of this correction, the values of D, of $\frac{\delta}{D}$ and of the expression $\frac{\delta}{\Delta}-\frac{\delta}{D}$ for a few bodies,

have been collated in the following table, air being supposed at 0°, and under the pressure of 760 mm.

			$\frac{\delta}{\Delta} - \frac{\delta}{D}$	
	D	$\frac{\delta}{D}$	Weight in Brass.	Weight in Platinum.
Platinum........	21·30	0060.10^{-6}	94.10^{-6}	0
Mercury.........	13·59	0095.10^{-6}	59.10^{-6}	-15.10^{-6}
Copper...........	8·85	0146.10^{-6}	8.10^{-6}	-86.10^{-6}
Brass............	8·40	0154.10^{-6}	0.10^{-6}	-94.10^{-6}
Aluminum......	2·73	0473.10^{-6}	-319.10^{-6}	-413.10^{-6}
Water............	1·00	1293.10^{-6}	-1139.10^{-6}	-1233.10^{-6}

With brass weights, the term of correction in weighings with the metals is far below the ten-thousandth, except for platinum on the one hand and aluminum on the other. There is no occasion to take into account changes arising from alterations in temperature, for they do not affect the corrections themselves to the extent of thousandths.

The corrections in the weighings are relatively much more important when the body we have in view forms part of a much heavier system, and the weight is to be obtained from them by difference—as in chemical analyses, for instance.

658. MEASUREMENT OF ANGLES.—The evaluation of an angle, which is in reality an abstract number, amounts to the measurement of a length, or more exactly to the comparison of two lengths. As this problem is often met with, and in the most varied conditions, it may be useful to examine its general features.

In most cases the angle is measured by the displacement of telescope movable over a divided circle. The telescope has: 1st, a *line of sight*, defined by a limb or by a cross wire telescope, which is alternately brought in the direction of the two sides of the angle; 2nd, an *index*, or a *vernier*, the displacement of which is observed in reference to the division of the circle.

The accuracy of the measurements depends on the precision with which the line of sight is divided, and, on the other hand, on the graduation of the circle. In a well constructed instrument, the approximation of the sighting should be at least equal to that of the graduation. If, for instance, the circle with the vernier measures 10 seconds, the telescope which serves for the sighting should not have an error greater than 10 seconds.

659. When we look through a telescope at a luminous point without apparent magnitude standing out against a dark ground—

a star in the sky, for instance—the image produced at the focus of the telescope is not a point. Owing to diffraction, this image consists of central circular spot surrounded by a dark circle, and then by a series of concentric circles alternately light and dark.

In order to distinguish two stars which are very close, the central images which correspond to each of them must be separated, or at any rate must not trench too much on each other. According to the laws of diffraction, the apparent angle of the central spot seen from the optical centre of the object-glass (or from the mirror in the case of the reflecting telescope) is inversely as the diameter of the object-glass. We cannot *à priori* define the extent to which the central spots overlap—that is to say, the smallest angle of two stars, beyond which the eye could not affirm the existence as distinct objects of two adjacent stars; but it is certain that this limiting angle is inversely as the diameter of the object-glass; it is *the angle of penetration;* the inverse of the angle of penetration is called the *optical power.* The optical power is accordingly proportional to the diameter of the object-glass.

To determine the optical power by experiment, Foucault* recommends the use of a grating, consisting of equidistant parallel white lines, separated by dark lines of the same thickness. The greatest distance is measured at which the grating must be placed so that the lines are still distinctly seen by means of the object-glass, provided with a suitable eye-piece; the apparent angle of two consecutive lines measures the penetration of the instrument. Foucault found that a telescope of 14 centimetres aperture will show lines whose distance does not exceed 1″, whatever be its focal distance. It follows from this, that the constant ratio of the optical power of an object-glass to the diameter of this object-glass, expressed in centimetres, is equal to the inverse of the product of 14 by the arc of a second—that is, about 15,000. The result would not be quite the same if we estimated the optical power of a telescope by the separation of two adjacent stars of the same magnitude. We can understand that it is quite possible, in measuring angles, to direct the line of the cross wire on the image of an object with a less error than the angle of penetration of the telescope, especially when we can multiply the sightings. This is particularly the case in transit observations and in geodetic triangulations; the error of the sighting is then not more than one-tenth of the angle of penetration of the telescope.

* *Annales de l'Observatoire de Paris*, Vol. v. 1858. *Œuvres*, p. 259.

660. In most cases, however, the approximation indicated by the optical power is not exceeded, or even attained, especially in the case of moving images, or of observations which cannot be repeated at will; any practical miscalculation will be avoided if we assume that an object-glass of 16 centimetres diameter can measure at least 1″.

On the other hand, a divided circle 80 cm. in diameter, of good construction, gives 2″ directly, when read by a vernier, and the complementary part with a less error than 1″; a circle of this diameter is therefore comparable with a telescope of 16 cm. aperture.

In these conditions the ratio of the diameter of the circle to that of the object-glass is equal to 5. This ratio ought not to change if we desire to maintain the same concordance between the two organs for any given limit of accuracy.

This rule being admitted, it is interesting to examine what is the absolute degree of approximation obtainable in reading verniers. The value of an angle of 1″ is 0·0000048; on a circumference of 80 cm. in diameter it corresponds to a length of 0·002 mm. This is the limit which the errors of the graduation must not exceed, unless we make a special examination of the divisions of the circle.

661. It frequently happens that we have to determine the rotation which a movable system spontaneously undergoes about its axis in the course of an experiment.

Such, for instance, is the case of a magnetised needle suspended in a cup about a vertical pivot, or of a dip needle movable about a cylindrical axis which turns in a plane, or of the beam of a balance oscillating about a knife-edge, or of any apparatus suspended by one or more wires. It is not possible to provide these objects with verniers, which move along a circular division, for we ought to avoid all friction, and verniers are only really useful when they can be placed in contact with the scales. On the other hand, a telescope mounted on the movable arrangement would needlessly increase the weight.

The simplest plan is to provide the apparatus with a very movable index in the shape of a long pointed needle, which moves over a divided scale. The needle should then be sighted in a plane perpendicular to the circle; in any other direction it would be projected on a division different to the first, and what is called the *parallax* error would be produced.

This latter error is sometimes avoided by dividing the circle on a plate of silvered glass; the movable index being placed above, it is viewed so that its image is covered, and the corresponding division is read off.

This mode of sighting is far from giving the same degree of accuracy as the use of verniers. With a circle 16 centimetres in diameter it is difficult to sight to less than 1′; this is about the case with Gambey's dip circle. In the same conditions a vernier would give 5″.

A very ingenious arrangement has been used by MM. Brunner for dip circles. The vertical divided circle is movable about an axis which coincides with that of the needle, and carries a small concave mirror, the centre of curvature of which describes the same circumference as the end of the needle. When the point of the needle is near the centre of curvature of the mirror, a reversed image is produced in the same plane as the needle itself, and the adjustment consists in bringing these two images in the prolongation of each other. It is easy to estimate the degree of accuracy of which this method is capable. If the diameter of the mirror is 1·6 cm., its angle of penetration is 10″, which, for a curvature of 5 centimetres,

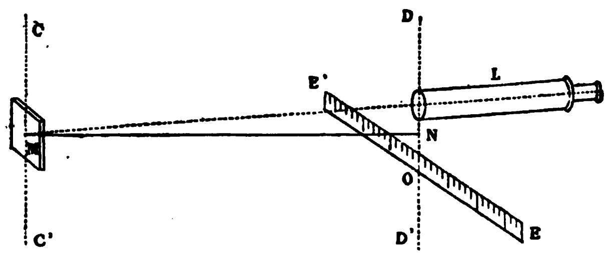


Fig. 128.

corresponds to an absolute length of 0·0025 mm.; the approximation is virtually the same as with the verniers.

662. Mirror Method.—In order to improve the mode of measuring rotations, Poggendorff* conceived the ingenious idea of attaching to the movable body itself a mirror by which the displacement of the image of an external body is observed. This method came rapidly into general use in consequence of the beautiful researches of Gauss and Weber.

Let us suppose that the movable part turns about a vertical axis, and carries a plane mirror passing through the axis.

Let M be the mirror (Fig. 128), MN the direction of the perpendicular at its centre, when the movable system is in equilibrium, or in that position which is taken as the initial one, and CC′DD′ the vertical plane through this normal and the axis of rotation. A

* Poggendorff. *Pogg. Ann.*, Vol. VII., p. 121. 1826.

horizontal divided scale, EE′, is placed at a certain distance from the mirror, below the horizontal MN, and perpendicular to its direction. A little above the scale is a telescope, L, movable about a horizontal axis, so that its optical axis describes the plane CC′DD′. The telescope is directed on the mirror, and it is adjusted so as to give a sharp image of the scale as seen by reflection. The conditions of adjustment will be satisfied for the telescope, if, by a simple turn of the eye-piece and a rotation about the horizontal axis, the wire by which the mirror is suspended, and the image of a plumb-line, DD′, stretched in front of the middle of the object-glass formed in this mirror, are made to coincide successively with the cross wires; the scale is in adjustment if its two ends, E and E′ are at the same distance from the mirror, and if during the oscillations the image of these two ends is at the same height in the plane of the network.

The position of the mirror is defined by the division of the scale which makes its image on the cross wires of the telescope. If the

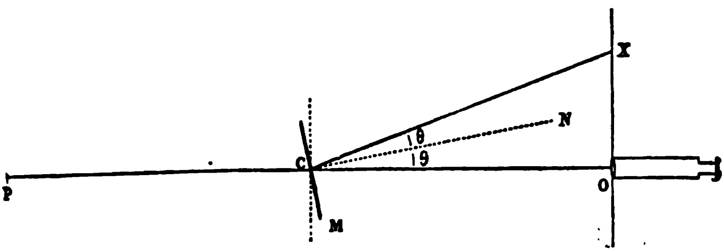


Fig. 129.

adjustment is perfect, the zero of the division corresponds to that division of the scale which in the telescope is concealed by the plumb-line DD′.

The divisions of the scale may be numbered on each side, counting from the centre; but in order to avoid the readings of contrary signs and of change of directions, it is better to place the zero of graduation at one end of the scale.

663. Calculation of the Deflection.—Let us take as plane of figure the horizontal plane which passes through the centre of the mirror (Fig. 129). Let C be the axis of rotation; let the lines CO, CN, and CX, representing the vertical planes which pass respectively through the optical axis of the telescope, through the actual position of the perpendicular to the mirror, and through that division of the scale the image of which coincides with the cross wires. The two

angles OCN and NCX are equal from the laws of reflection; OCN is moreover the angle through which the mirror has been turned. If this be called θ, and if x and p are the distances from zero of the divisions in X and O, and if d is the distance CO, we have

$$(1) \qquad \tan 2\theta = \frac{OX}{OC} = \frac{x-p}{d}.$$

If the surface of the mirror M, instead of being on the axis of rotation itself, is at a distance $\rho = CM$ (Fig. 130), it is easy to

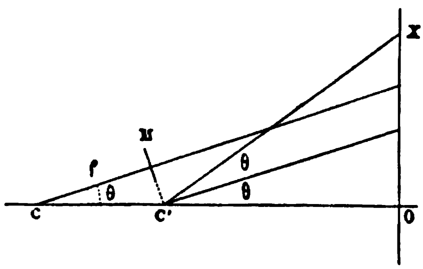


Fig. 130.

see that the division X seen through the telescope for a deflection θ, is the same as if the mirror had been turned through an angle θ, about the point C′, and that we have

$$(2) \qquad \tan 2\theta = \frac{x-p}{d - \dfrac{\rho}{\cos\theta}}.$$

As the deflections are always small, if the distance ρ of the mirror from the axis is small in comparison with the distance of the scale, we may put unity for $\cos\theta$ in this expression.

For very small deflections we have approximately

$$(3) \qquad \tan\theta = \theta = \frac{1}{2}\,\frac{x-p}{d-\rho}.$$

When the approximation given by this simple formula is insufficient, and we wish to avoid the calculation of trigonometrical lines,

we may expand the value of tan θ in a series. If m stands for the expression $\frac{1}{2}\frac{x-p}{d-\rho}$, we get

$$\tan\theta = m\left[1 - m^2 + 2m^4\right] = m - m^3 + 2m^5.$$

For a deflection of 15° the error made by neglecting the further terms of the series is less than $\frac{1}{10000}$.

664. If the mirror is a glass one, silvered at the back, or, more generally, if a glass plate is interposed between the telescope and the reflecting surface, so as to be traversed by the rays in a direction near the perpendicular, a correction is necessary for the value of the distance $d-\rho$. If e is the thickness, and n the refractive index of the plate, the rays which traverse it behave as if they had passed through a layer of air whose thickness is $\frac{e}{n}$; the distance of the scale from the reflecting surface should then be diminished by $e-\frac{e}{n}$ or $e\,\frac{n-1}{n}$.

665. GRADUATION OF THE SCALE.—The most direct method of obtaining the angular graduation of the scale would be to observe the deflection of the image which corresponds to a rotation of the mirror, measured on a divided circle. If the movable system is not submitted to any other action tending to give it a determinate direction, and if the wire is attached at the top to an arrangement movable over a circle divided horizontally, which has the same axis as the wire, it is sufficient to turn the whole system through a known angle, and to observe at the same time the displacement of the image in the telescope. In the case in which the system to which the mirror is fixed is acted on by an extraneous force, like a magnet in a magnetic field, it will be necessary to fix temporarily the mirror in respect of the rest of the apparatus, or to provide this with a fixed mirror situate at the same distance from the axis as the movable mirror.

As the tangent of the deflection is given by the ratio of the two lengths, it is sufficient to measure directly in the same unit the scale and its distance.

The operation, which consists in measuring the distance from a scale to a movable mirror, presents some difficulties when the error has to be less than a ten-thousandth, which corresponds to an approximation of 0·1 mm. for a distance of one metre. For this

reason we are often led to increase the distance from the mirror to the scale to an inconvenient extent.

666. These difficulties may be lessened, and at the same time the errors, due to small displacements of the axis during the experiments, may be eliminated by using* two mirrors, directly opposite, and two parallel scales. If ρ and d, ρ' and d', are the distances from the axis of each of the mirrors and of the corresponding scale, and if two observers note the deflections x and x' produced by the same rotation, we shall have

$$\tan 2\theta = \frac{x-p}{d-\rho} = \frac{x'-p'}{d'-\rho'} = \frac{(x+x')-(p+p')}{(d+d')-(\rho+\rho')}.$$

The sum $d+d'$ is the distance of the two scales which are fixed; $\rho+\rho'$ is the distance of the two mirrors which are connected with each other. This latter distance may be reduced to the thickness of a glass plate silvered on both sides.

667. Limit of Accuracy in the Mirror Method.—The accuracy obtainable in the method of reflection depends simply on the telescope and on the dimensions of the mirror, and it is often wrongly thought that the accuracy may be improved by unduly increasing the distance from the scale to the mirror. The advantage of a great distance with a plane mirror and a rectilinear scale consists especially in this, that the telescope can observe the ends of the scale as well as the centre without change of sighting. On the other hand, if the telescope has been constructed for long distances, a small displacement of the eye-piece will allow the scale to be read. These are minor points; the principal question is the smallest angle of rotation of the mirror which the telescope can estimate.

Suppose the mirror at a distance d from the scale, and at a distance D from the object glass of the telescope, and let β be the smallest angle which the telescope can distinguish. The image of the scale being at a distance $D+d$ from the object-glass, this limiting angle corresponds to a length of scale ϵ such that

$$\beta = \frac{\epsilon}{D+d},$$

the corresponding deflection in the mirror is

$$a = \frac{\epsilon}{2d},$$

* W. Weber and Zöllner. *Berichte der K. S. Gesellschaft*. Leipzig, 1880.

which gives

$$\alpha=\beta\frac{D+d}{2d}=\frac{\beta}{2}\left(1+\frac{D}{d}\right).$$

In ordinary conditions in which $d=D$, we get $\alpha=\beta$; the delicacy, whatever be the distance of the scale and the telescope, is exactly the same as if the telescope were mounted directly on the movable part. The apparent advantage obtained by doubling the displacement by reflection is compensated by the double distance of the image of the scale. The sensitiveness increases when the scale is placed further than the telescope; it would even be doubled if the ratio $\frac{D}{d}$ were very small, but in that case it would be necessary to have scales of great dimensions, for which it would be difficult to get a good illumination.

This reasoning always pre-supposes that the optical power of the telescope is entirely utilised—that is to say, that the pencil of rays which proceed from a point of the scale covers the whole of the object-glass, or, at any rate, the horizontal diameter at right angles to the lines. If A is the diameter of the object-glass, a the horizontal dimensions of the mirror, we must have

$$\frac{a}{d}>\frac{A}{D+d}.$$

The least size of the mirror is thus

$$a=\frac{Ad}{D+d};$$

it is half the diameter of the object-glass for the ordinary arrangement, and is equal to this diameter itself when the scale is at a great distance.

It must be added that the construction of plane surfaces presents the greatest difficulties; all defects in the mirror injure the purity of images, and contribute to diminish the accuracy of the observations.

When it is merely proposed to observe the divisions of a scale, it is sufficient if the mirror has the dimensions defined by the preceding equation, in the direction at right angles to the lines. The sharpness of the image may, moreover, be greatly improved by covering the object-glass with a diaphragm, in which is a

rectangular aperture equal to the diameter of the object-glass, its length at right angles to the direction of the lines, and as narrow as is compatible with the illumination of the scale. The images of the figures are not so good, but the lines seen far more pure.

668. If the distance of the lens does not increase the sensitiveness, it modifies the number of divisions of the scale which can be seen at the same time. We may define the practical limit of the field by the condition that the rays from a point shall cover at least half the object-glass. According to this, if l is the length of that portion of the scale of which the image is seen, we shall have

$$\frac{d}{d+D}=\frac{a}{D}.$$

With the ordinary arrangement in which $d=D$, we have $l=2a$; the visible length of the scale is then equal to double the breadth of the mirror; that is, is equal to the diameter of the object-glass, if the mirror is the minimum size.

We have implicitly assumed that the angular value $\frac{l}{d+D}$ of the field thus defined is smaller than the optical field of the lens, as is ordinarily the case; if the mirror were very large, the size of the field would only depend on the telescope.

669. The method of the mirror does not enable us to measure large deflections; it is not convenient to estimate more than 5° on each side of the position of rest, or a total angle greater than 10°. As the image has twice the displacement, the apparent angle of the scale as seen from the mirror should be at least 20°; the total length must be 0·3 of its distance, which makes 40 centimetres for a distance of 1 metre.

If, when the mean value of the deflections is 3°, we wish to estimate the ten-thousandth, the accuracy of the reading should be $\frac{3.60.60''}{10000}$ or about 1″; in order that the angle of penetration shall be below this limit, we must have an object-glass 16 cm. in diameter, and a mirror 8 cm. As to the reading of the divisions, it must be made to within $\frac{1}{200000}$ at the distance $d+D$; that is to say, to 0·1 mm. if $d=D=10$ metres.

With larger scales it would be difficult, without greatly increasing the distance, to observe simultaneously the centre and the extremities, without modifying the focussing of the telescope. In that

case the scale may be curved along a cylindrical surface, the axis of which coincides with the axis of suspension of the mirror. In this case the expression $\frac{x-p}{d}$ does not represent the tangent of the double of the deviation, but the double of this deviation itself.

The scale is ordinarily divided in millimetres. The scales on paper cannot be considered as sufficient for accurate measurements; scales on ivory or on metal are better, but metal scales are difficult to illuminate.

The divisions may be ruled on glass, and lighted from behind by a reflecting mirror. If the glass is transparent, the lines appear dark on a light ground; if the glass is silvered, and the division is traced on a layer of silver, the lines appear bright on a dark ground.

670. In certain circumstances (for instance, in observing phenomena which extend over a long period and which require

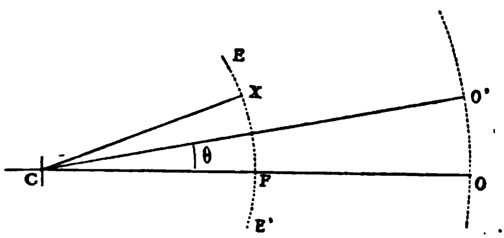


Fig. 131.

permanent installation, like variations of terrestrial magnetism), it is useful to have a mark in the field of the telescope independent of the cross wires. For this purpose we place, below the movable mirror, a perfectly similar fixed mirror. If the two mirrors were absolutely parallel, the two images of the scale, one fixed and the other movable, would appear superposed; but it is sufficient to slightly incline the fixed mirror to separate them.

The angle between the vertical planes passing through the perpendiculars to the two mirrors is given by the difference of the numbers of the divisions which coincide with the vertical cross wire; this difference remains the same when the telescope or the scale for any cause has been slightly displaced. The instrument carries thus its own fixed point, and the relative immobility of the

scale and of the telescope in respect of the mirrors may be verified at each instant by the position of the cross wire on the fixed mirror.

671. CONCAVE MIRRORS.—The methods of observation by mirrors may be modified in various ways. One of the most convenient is to use a concave mirror, placing the scale at the focal distance, and sighting the mirror through a telescope adjusted for parallel rays.

Let O (Fig. 131) be the centre of curvature of the mirror, F its focus in the initial position, f its focal length OF. The axis of the telescope is in the vertical plane OC, and directed towards the centre C of the mirror. The centre of a scale EE′ on the cylinder is at F, and its axis placed vertically passes through the centre C of the mirror. It is clear that if the mirror turns through an angle θ, we shall see in the telescope the image of the point X of the scale, such that the arc FX corresponds to the angle 2θ. All rays from the point X form after reflection a pencil parallel to the optic axis of the telescope, and the image is in the focal plane. The telescope, when once regulated for parallel rays, might then be placed at any given distance; the only effect of increasing the distance would be to diminish the field.

In fact, the field being always defined by the condition that at least one of the rays from a point passes through the centre of the object-glass, the visible length l of the scale, adopting the same notation as above, will be defined by the ratio

$$\frac{l}{f}=\frac{a}{\mathrm{D}}, \quad \text{or} \quad l=a\frac{f}{\mathrm{D}};$$

we have still $l=a$, for $f=\mathrm{D}$; but the value of l is inversely as the distance D of the telescope.

In this case the diameter a of the mirror should be equal to that of the object-glass, if we wish to utilise completely the optical power of the telescope.

We might, lastly, observe a deflection half that of the limiting angle of the telescope; for, retaining the same notation, we have

$$\beta=\frac{\epsilon}{f}, \text{ and } \alpha=\frac{\epsilon}{2f}=\frac{\beta}{2}.$$

The conditions are, in fine, the same as if we had a plane mirror with a scale at an infinite distance.

672. A concave mirror has some practical advantages. In the first place, it is easier to obtain a spherical surface than a plane one,

except that the focal distance has to be determined by experiment. So, too, if the surface is not spherical, the principal radii of curvature are not very different, and we may investigate the direction of the principal planes by means of the images, and make one of these parallel to the lines of the scale; the definition of the telescope is not diminished thereby.

On the other hand, the scales being smaller are easier to illuminate. It will, lastly, be at once seen that for the same illumination, images in the telescope are as bright as with a plane mirror.

Yet if somewhat large deflections are to be measured, the scale should be curved, and in any case the graduation must be empirical.

Another arrangement consists in placing the scale at the centre of curvature of the mirror. A reversed image of the scale is then produced of the same size, which can be observed with a lens or with a microscope. The precision only depends then on the diameter of the mirror, and a deflection can be observed equal to half the limiting angle which corresponds to this diameter.

673. We may, in the preceding cases, replace the concave mirror by a plane one, and a condensing lens at a small distance from each other.

Let f be the focal length of the lens and δ the distance of the mirror from the optical centre. In order that the scale be placed in the principal focal plane of the optical system, the rays from one point, after being refracted in the lens and then reflected from the mirror, must seem to start from the principal focus of the lens. It is readily seen that the scale should be placed at a distance d' from the lens, defined by the equation

$$d' = \frac{f}{2}\frac{f-2\delta}{f-d},$$

and that the angles a and β be in the ratio

$$a = \frac{\beta}{2}\frac{f}{f-\delta}.$$

The system is then virtually equivalent to a concave mirror. The hypothesis $\delta = 0$, would correspond to the case of a plano-convex lens of focal distance f silvered on its plane face.

With so complicated a construction, the graduation of the scale must necessarily be empirical by means of a divided circle.

The use of a lens is especially advantageous in apparatus for variation when we desire to keep as a mark the image furnished by a fixed mirror: in this case it is in fact more convenient to take a plane mirror.

674. PROJECTION OF IMAGES.—For routine experiments the method of reflection is greatly simplified by dispensing with the telescope, and projecting directly upon the scale the reflected image of a luminous object.

In the ordinary arrangement, the plane of the scale and that of the cross wire of the telescope are conjugate in reference to the system formed by the object-glass and the mirror. If then, the eye-piece being removed, the cross wire is illuminated, its image will be projected on the scale at the very division which would have been read on the telescope. The cross wire may then be replaced by a vertical slit illuminated from behind.

The use of a concave mirror gives a still more simple arrangement. The scale and the illuminated slit are placed in the same plane, which passes through the centre of curvature of the mirror—in other words, at twice the focal distance, and at the same distance on either side of this centre in a vertical direction. The image of the slit is then thrown on the scale in its proper size.

A plane mirror and lens, placed at a small distance, may be substituted for the concave mirror. The slit must then be placed so that its image is sharply formed on the scale.

In this method of reading by projection the external light must be shaded in order to see the image of the slit on the scale; the reading then becomes difficult. This inconvenience is obviated by taking a very large slit or circular aperture, across which wire is stretched vertically. The image of the aperture is then produced on the scale, and illuminates the space in which is the image of the wire.

Very good results are obtained with a scale on ground glass, which is observed from behind.

The method by projection, which allows a photographic impression of the image to be taken, is that used in registering apparatus for variations of terrestrial magnetism. By using two mirrors, one fixed and the other movable (670), we get on the sensitive surface two images of the same illuminated slit—one fixed, which determines the zero, and the other changing its position, which measures the variation. If the sensitive layer moves in a direction perpendicular to the displacement of the image, we obtain in the proof a right line corresponding to the fixed image, and a curve which represents the movable image.

CHAPTER II.

MEASUREMENT OF OSCILLATIONS.

675. Equations of the Oscillatory Motion.—We shall frequently have to consider the oscillatory motion of a solid about an axis, on each side of the position of equilibrium. Before touching on the methods of observation, we shall examine generally the mechanical conditions of this motion.

When the movable system is displaced from its position of equilibrium, from the very conditions of the experiment it is acted on by a couple, which tends to bring it back, and the moment of which is a function of the angle of deflection. This directive couple is due either to the elastic reaction of the system, or to an external force, or to a concurrence of several given causes. Besides this directive action, the value of which only depends on the position of the system, there are retarding forces analogous to friction, due to the motion itself, and which depend on the velocity.

Let us call—

K the moment of inertia of the movable system;

x the angle of deflection at the time t, counted from the position of equilibrium;

$f(x)$ the moment of the couple, which tends to bring the system into the position of rest;

$\omega = \dfrac{dx}{dt}$ the angular velocity of rotation;

$\phi(\omega)$ the moment of the retarding forces.

It is known that, in the rotation of a solid about an axis, the product of the moment of inertia by the angular acceleration is equal to the sum of the moments of the forces in reference to the axis. The equation of motion is then

$$(1) \qquad K\frac{d^2x}{dt^2} + \phi\left(\frac{dx}{dt}\right) + f(x) = 0.$$

We must first observe that if the function ϕ is zero, the movable body acquires the same velocity every time it passes the position of

equilibrium. The motion is *periodic*, and the system makes a series of oscillations which are perfectly identical.

In all cases, whether the successive oscillations are or are not identical, the position of the system when it makes the greatest angle with the position of equilibrium is called the ***elongation;*** the ***amplitude*** of oscillation is the sum of two successive angles of deflection—that is to say, the angle of two extreme positions; the ***duration of an oscillation*** is the time between two consecutive elongations.

The only cases useful for consideration from the experimental point of view, are those in which the directing couple is proportional to the angle of deflection, or to the sine of the angle. In that case we must replace the moment $f(x)$ by C x or C sin x, the factor C representing the couple which corresponds to an angle equal to unity in the first case, or to a rotation of 90° in the second.

Let us first suppose that the retarding actions are null, and put

$$n^2 = \frac{C}{K},$$

according to the case considered, equation (1) becomes,

$$(2) \qquad \frac{d^2x}{dt^2} + n^2x = 0.$$

or

$$(3) \qquad \frac{d^2x}{dt^2} + n^2 \sin x = 0.$$

The formulas (2) and (3) are the same for very small arcs.

676. Isochronous Oscillations.—The general integral of the equation (2) is

$$x = A \sin nt + A' \cos nt;$$

from which is deduced

$$\omega = \frac{dx}{dt} = n\,(A \cos nt - A' \sin nt).$$

If we count the time t from the moment at which the movable body passes through its position of equilibrium, and if we call ω_0 the initial angular velocity, we shall have for the constants A and A',

$$A' = 0, \qquad \omega_0 = nA;$$

from this we have

$$(4)\qquad \begin{aligned} x &= \frac{\omega_0}{n}\sin nt, \\ \frac{dx}{dt} &= \omega_0 \cos nt. \end{aligned}$$

The value of the angle of deviation a, which corresponds to $\frac{dx}{dt}=0$, in which $nt=\frac{\pi}{2}$, is

$$(5)\qquad a=\frac{\omega_0}{n}.$$

The first elongation is attained at the end of a time

$$nt=\frac{\pi}{2} \quad \text{or} \quad t=\frac{\pi}{2n};$$

the following ones at the times

$$\frac{3\pi}{2n}, \quad \frac{5\pi}{2n}, \quad \frac{7\pi}{2n} \cdots\cdots$$

The interval between two consecutive elongations is constant; the time of oscillation is constant, and its value is

$$(6)\qquad T=\frac{\pi}{n}=\pi\sqrt{\frac{K}{C}}.$$

This duration is independent of the amplitude; the oscillations are said to be *isochronous.*

The conditions of the problem—that is to say, the value of n, which gives the ratio of the directing couple to the moment of inertia, and the angular velocity of the system ω—may be determined by the investigation of the motion; that is, deduced from the *time of oscillation* T, and from the angle of deviation. We have, in fact,

$$(7)\qquad \begin{aligned} n &= \frac{\pi}{T}, \\ \omega_0 &= an = a\frac{\pi}{T}. \end{aligned}$$

The motion we are considering is susceptible of a very simple geometrical representation. If we imagine the arc $2a$ expressed as a straight line, and taken as the diameter of a circle, the position of the movable body on the diameter is given at each instant by the projection of a second movable body starting at the same time as the first from the end of the diameter, and moving uniformly along the circumference with the velocity

$$\omega_0 = na = a\sqrt{\frac{C}{K}}.$$

677. Pendulum Motion.—Equation (3) cannot be completely integrated in a finite form. If it be multiplied by $\frac{dx}{dt}dt$, and integrated for a first time, we get (observing that $x = a$, for $\frac{dx}{dt} = 0$),

$$(7) \qquad \left(\frac{dx}{dt}\right)^2 = 2n^2(\cos x - \cos a).$$

We deduce from this, for the time T of an oscillation,

$$T' = \frac{\sqrt{2}}{n}\int_0^a \frac{dx}{\sqrt{\cos x - \cos a}}.$$

Expanding the expression under the root, we get, by a well-known calculation,

$$(8) \qquad T' = \frac{\pi}{n}\left[1 + \left(\frac{1}{2}\right)^2 \sin^2\frac{a}{2} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \sin^4\frac{a}{2} + \ldots\ldots\right],$$

or, representing the series by $1 + \beta$, and still calling T the time $\frac{\pi}{n}$,

$$T' = T\,(1 + \beta).$$

The conditions of the problem will here be determined by the equations

$$(9) \qquad \begin{aligned} n &= \frac{\pi}{T'}(1 + \beta), \\ \omega_0^2 &= 2\,\frac{\pi^2}{T'^2}(1 + \beta)^2(1 - \cos a). \end{aligned}$$

If in expanding the value of T′ we only consider the first term—that is, if we suppose $\alpha = 0$—we find the time of oscillation obtained before; this is what is called the *time of an infinitely small oscillation.*

When the angle α is very small, we may restrict the series to its first two terms, and, neglecting the fourth power of the amplitude, may write

$$T' = T\left(1 + \frac{\alpha^2}{16}\right). \tag{10}$$

In all cases the time of oscillation increases with the amplitude, but at first very slowly. This motion is that of the pendulum in vacuo.

678. Damping of the Oscillations.—In practice the oscillations become more or less rapidly extinct, or they deaden themselves, owing to retarding actions, such as friction, the resistance of the medium, etc., and in the case of magnets in motion, owing to the effects of induction developed in adjacent masses of metal. All the forces of this kind are functions of the velocity. The simplest hypothesis is to consider them as proportional to a whole power m of the velocity. If C_1 is the moment of these forces for an angular velocity equal to unity, the equation of the oscillatory motion (1) in the case of a directive couple proportional to the angle of deflection becomes

$$K\frac{d^2x}{dt^2} + C_1\left(\frac{dx}{dt}\right)^m + Cx = 0;$$

or, putting

$$n^2 = \frac{C}{K}, \qquad 2\epsilon = \frac{C_1}{K},$$

$$\frac{d^2x}{dt^2} + 2\epsilon\left(\frac{dx}{dt}\right)^m + n^2x = 0. \tag{11}$$

If the principal couple was proportional to the sine of the deviation, the last term should be replaced by $n^2 \sin x$; but we shall more particularly consider equation (11), and therefore, in dealing with pendulum motions, the case of very small oscillations.

679. Resistance Proportional to the Square of the Velocity.—Poisson has examined the hypothesis $m = 2$.* He observes, first of all, that if the movable body is displaced through an angle α, from its position of rest, and then left to itself without initial velocity, the deflection at any given time is necessarily a

* Poisson. *Mecanique.* First edition, Vol. I., p. 405. 1811.

function of the initial deflection α. If we only consider small values of α, the arc may be developed in a very converging series, which contains no term independent of α; for as the initial velocity is null, x ought to vanish at the same time as α. Calling x_1, x_2.... functions of t, we may put

$$x = ax_1 + a^2x_2 + \ldots\ldots$$

Replacing this value in equation (11), and making the coefficients of the various powers of α equal to zero, we obtain a series of differential equations which will serve to determine the functions $x_1, x_2 \ldots$ Neglecting powers of α higher than the second, we have the two equations

$$\frac{d^2x_1}{dt^2} + n^2x_1 = 0,$$

$$\frac{d^2x_2}{dt^2} + 2\epsilon\left(\frac{dx_1}{dt}\right)^2 + n^2x_2 = 0.$$

The integral of the first is

$$x_1 = \mathrm{A}\cos(nt + \beta).$$

If we suppose that for $t = 0$, we have $x = -\alpha$ and $\frac{dx}{dt} = 0$, it follows that $x_1 = -1$ and $\frac{dx_1}{dt} = 0$, and therefore $\mathrm{A} = -1$ and $\beta = 0$; which gives

$$x_1 = -\cos nt.$$

Substituting this value of x_1 in the equation in x_2, and observing that $2\sin^2 nt = 1 - \cos 2nt$, this becomes

$$\frac{d^2x_2}{dt^2} + \epsilon n^2(1 - \cos 2nt) + n^2x_2 = 0;$$

the general integral of this latter equation is

$$x_2 = \mathrm{A}'\cos(nt + \beta') - \epsilon - \frac{\epsilon}{3}\cos 2nt.$$

The initial conditions give $\mathrm{A}' = \frac{4\epsilon}{3}$ and $\beta' = 0$; therefore

$$x_2 = -\epsilon + \frac{4\epsilon}{3}\cos nt - \frac{\epsilon}{3}\cos 2nt.$$

By means of the two values of x_1 and x_2, we get

$$(12) \qquad x = -a^2\epsilon - a\left(1 - \frac{4a\epsilon}{3}\right)\cos nt - \frac{a^2\epsilon}{3}\cos 2nt,$$

and

$$\frac{dx}{dt} = an\left(1 - \frac{4a\epsilon}{3}\right)\sin nt + \frac{2a^2n\epsilon}{3}\sin 2nt.$$

It is seen that the first value of t, after zero, which annuls $\frac{dx}{dt}$, is $t = \frac{\pi}{n} = \mathrm{T}$; from which it follows that the duration of the oscillation is the same as if the resistance did not exist.

The deflection of the ascending half oscillation which succeeds the initial half oscillation, will be obtained by making $t = \frac{\pi}{n}$ in the expression for x; its value is

$$a_1 = a - \frac{8a^2\epsilon}{3}.$$

The following deflection will be the same in absolute value

$$a_2 = a_1 - \frac{8a_1{}^2\epsilon}{3},$$

as to the degree of approximation adopted

$$a_2 = a - \frac{16a^2\epsilon}{3}.$$

The oscillations decrease therefore in arithmetical progression; they will vanish after a number p of oscillations given by the greatest solution by a whole number, of the inequality

$$p < \frac{3}{8\epsilon a}.$$

We shall have the time t_1 of the descending half oscillation by making $x = 0$, in the equation (12), which gives

$$0 = -a\epsilon - \left(1 - \frac{4a\epsilon}{3}\right)\cos nt_1 - \frac{a\epsilon}{3}\cos 2nt_1.$$

If the angle α were null, this equation would give $t_1 = \frac{\pi}{2n} = \frac{T}{2}$; we may put $t_1 = \frac{\pi}{2n} + \theta$, θ being a very small quantity, and, neglecting terms of the second order in α and in θ, we obtain

$$\theta = \frac{2\alpha\epsilon}{3n},$$

and therefore

$$t_1 = \frac{T}{2} + \frac{2\alpha\epsilon}{3n}.$$

It will be seen that the time of oscillation is no longer divided into two equal parts by the period of passing through zero, the time of the descending half oscillation being increased by as much as the ascending half oscillation is diminished—for the time of the entire oscillation has not changed.

These various conditions are not those which we most frequently meet with. Experiment shows indeed that the resistance of the medium acts as the square of the velocity only for greater velocities than we shall most frequently have to consider.

680. RESISTANCE PROPORTIONAL TO THE VELOCITY.—The hypothesis of $m = 1$ is that which best corresponds to the practical case of the resistance of the air, and it is always realised for the effects of induction; this hypothesis has been investigated by Gauss.*

Equation (11) becomes

$$\frac{d^2x}{dt^2} + 2\epsilon\frac{dx}{dt} + n^2x = 0; \tag{13}$$

its general integral, as we have seen, is

$$x = Ae^{\rho t} + A'e^{\rho' t}, \tag{14}$$

A and A′ being constants, and ρ and ρ' roots of the equation of the second degree.

$$\rho^2 \times 2\epsilon\rho + n^2 = 0. \tag{15}$$

These roots are real or imaginary according as we are dealing with one or the other of the two conditions, $\epsilon^2 - n^2 > 0$, or $\epsilon^2 - n^2 < 0$.

* GAUSS. *Resultate aus den Beob. des Magn. Vereins.* 1837. *Œuvres*, v., p. 374.

681. Isochronous Oscillations.—Let us first consider the case of the imaginary roots. Putting $\gamma^2 = n^2 - \epsilon^2$, equation (14) becomes

$$x = e^{-\epsilon t}[A_1 \cos \gamma t + A_2 \sin \gamma t].$$

We may write it in the equivalent form

$$x = Ae^{-\epsilon t} \sin \gamma (t - t_0).$$

The constant t_0 represents the period at which the movable body passes through its position of equilibrium. If we count the time starting from this passage, we have $t_0 = 0$, and the equation reduces to

$$x = Ae^{-\epsilon t} \sin \gamma t; \tag{16}$$

from this follows

$$\frac{dx}{dt} = Ae^{-\epsilon t} (\gamma \cos \gamma t - \epsilon \sin \gamma t),$$

which gives for the initial angular velocity

$$\omega_0 = A\gamma.$$

The deflections correspond to the times for which we have $\frac{dx}{dt} = 0$, that is to say

$$\tan \gamma t = \frac{\gamma}{\epsilon}. \tag{17}$$

The first takes place at the time t_1, given by the smallest root of this equation; the next one at the time t_2, such that

$$\gamma t_2 = \pi + \gamma t_1;$$

that is to say, that it follows the first after an interval $t_2 - t_1$, having the value

$$\tau = \frac{\pi}{\gamma}. \tag{18}$$

The same is the case with all the others. The oscillations are then still isochronous, but the time of an oscillation τ is greater than the

time T which would correspond to oscillations without any retarding force. The ratio of the two times is

$$\frac{\tau}{T}=\frac{n}{\gamma}.$$

The period t of any elongation defined by the equation satisfies the ratio

$$\sin \gamma t=\frac{\gamma}{\sqrt{\gamma^2+\epsilon^2}}=\frac{\gamma}{n}.$$

Substituting this value in equation (16), we obtain for the angle of deviation

$$\alpha=A\frac{\gamma}{n}e^{-\epsilon t}=\frac{\omega_0}{n}e^{-\epsilon t}.$$

As the times of the elongations increase in arithmetical progression, it follows that the deflections diminish as the terms of a geometrical progression, the rule of which is $e^{-\epsilon\tau}$. If we call the successive deflections α_1, $\alpha_2 \ldots, \alpha_n$, we have

$$\frac{\alpha_2}{\alpha_1}=\frac{\alpha_3}{\alpha_2}\ldots\ldots=\frac{\alpha_n}{\alpha_{n-1}}=e^{-\epsilon\tau}.$$

The amplitudes, each of which is equal to the arithmetical sum of two successive angles of deflection, vary according to the same law. If we represent by a_1, $a_2 \ldots$, a_n the successive amplitudes, and put

$$\epsilon\tau=\lambda$$

we shall have

$$(19)\qquad \lambda=l.\frac{a_1}{a_2}=\frac{1}{2}l.\frac{a_1}{a_3}=\ldots\ldots=\frac{1}{n-1}l.\frac{a_1}{a_n}.$$

The number λ is called by Gauss the *logarithmic decrement* of the oscillations.

Observing that the first angle of deflection α_1 corresponds to the time t_1, given by the least root of equation (17),

$$\gamma t_1=\text{arc tan}\frac{\gamma}{\epsilon} \qquad \text{or} \qquad \epsilon t_1=\frac{\epsilon}{\gamma}\text{arc tan}\frac{\gamma}{\epsilon};$$

we get from this

$$\alpha_1 = \frac{\omega_0}{n} e^{-\epsilon t_1} = \frac{\omega_0}{n} e^{-\frac{\epsilon}{\gamma} \arc\tan \frac{\gamma}{\epsilon}}.$$

If we express the constants of the motion as a function of the time of oscillations τ, of the logarithmic decrement λ, and of the initial angle of elongation α_1, which are quantities directly observable, we get

$$(20) \quad \left\{ \begin{aligned} &\gamma = \frac{\pi}{\tau}, \quad \epsilon = \frac{\lambda}{\tau}, \\ &n = \sqrt{\gamma^2 + \epsilon^2} = \sqrt{\frac{\pi^2 + \lambda^2}{\tau^2}} = \frac{\pi}{\tau} \sqrt{1 + \frac{\lambda^2}{\pi^2}}, \\ &T = \tau \frac{\gamma}{n} = \frac{\tau}{\sqrt{1 + \frac{\lambda^2}{\pi^2}}}, \\ &\omega_0 = \alpha_1 \frac{\pi}{\tau} \sqrt{1 + \frac{\lambda^2}{\pi^2}}\, e^{\frac{\lambda}{\pi} \arc\tan \frac{\pi}{\lambda}}. \end{aligned} \right.$$

When the damping is not rapid, the ratio $\frac{\lambda}{\pi}$ is small, and we replace the exponential in the value of ω_0 by the first terms of its development in a series. We obtain thus a simpler expression, which could be arrived at directly, observing that between the first angle of deviation α_1 and the third α_3 there have been four semi-oscillations, during each of which the deviation has undergone virtually the same loss as during the first. This loss is therefore $\frac{\alpha_1 - \alpha_3}{4}$. By adding it to the first deflection we shall have virtually the deflection which would have been obtained without damping. We may then write

$$(21) \qquad \omega_0 = \frac{\pi}{\tau}\left(\alpha_1 + \frac{\alpha_1 - \alpha_3}{4}\right).$$

Conversely, if we find experimentally that the successive amplitudes of a vibratory motion decrease in geometrical progression, we may conclude that some retarding force is at work proportional to the velocity.

Borda, for instance, had found that the amplitudes of the small oscillations of a pendulum in air decrease slowly in geometrical

progression. In his experiments on the measurement of the intensity of gravity,* the amplitude becomes $\frac{2}{3}$ of its value in the course of an hour—that is, after 1800 oscillations, which gives 0·0001 for the logarithmic decrement.

In these conditions the square of λ_1 may be neglected and the time of the oscillations is to within 0·000000001, the same as if there were no damping.

682. We observe that if, besides the term proportional to the velocity, we introduce a constant p_1 into equation (11) like that

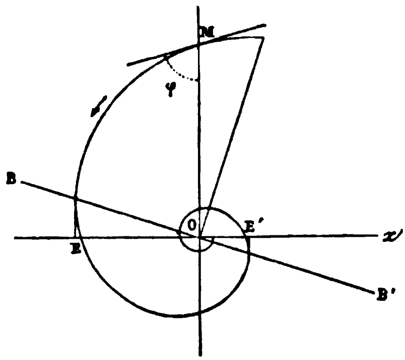


Fig. 132.

arising from the friction of a cap against a pivot, the equation acquires the form

$$\frac{d^2x}{dt^2} + 2\epsilon\frac{dx}{dt} + n^2x + p = 0.$$

In order to solve this fresh problem, it is sufficient to replace x in the preceding solution by $x + \frac{p}{n^2}$. The time of the oscillation is not modified, and therefore the isochronism is not altered. In this case the amplitudes, increased by a constant $\frac{2p}{n^2}$, still vary in geometrical progression. The damping is more rapid, and the motion soon stops.

* BORDA. *Base du Systeme Metrique*, Vol. III., p. 345.

683. The motion represented by equation (16) is susceptible of a simple geometrical representation. The position of a point on its trajectory is at any moment the projection of a movable body which would traverse, with a constant angular velocity γ, a logarithmic spiral (Fig. 132) in which the constant angle of the tangent with the radius vector will be defined by the equation

$$\cot \phi = \frac{\epsilon}{\gamma}.$$

The equation of this curve is, in fact,

$$\rho = Ae^{-\gamma t \cot \phi} = Ae^{-\epsilon t}.$$

If we count the time from the moment at which the radius vector OM is vertical, the projection of the movable body on the horizontal is at the time t_1 at a distance from the pole defined by the equation

$$x = Ae^{-\epsilon t} \sin \gamma t.$$

The deflections E and E′ correspond to the points of the curve where the tangent is vertical. These points are given by the intersection of the curve by a right line BB′ passing through the pole O, and making the angle ϕ with the vertical.

684. In the most general case in which the directing couple, owing to disturbing actions, is no longer proportional to the deflection, we may develop the function $f(x)$ by Taylor's theorem. We shall have then, observing that $f(x_0) = 0$,

$$f(x) = xf'(x_0) + \frac{x^2}{1 \cdot 2} f''(x_0) + \ldots = Cx + C' x^2 + \ldots.$$

If we assume that the resistance is proportional to the velocity, the equation of motion is of the form

$$(22) \qquad \frac{d^2x}{dt} + 2\epsilon \frac{dx}{dt} + n^2x + n'^2x^2 + \ldots = 0.$$

When the deflections are so small that the terms of an order higher than x^2 may be neglected, we may take as integral of equation (22)

$$x = Ae^{-\epsilon t} \sin \gamma (t - t_0) + A^2 \frac{n'^2}{3n^2} e^{-2\epsilon t} \left[1 + \cos^2\gamma \; (t - t_0) \right].$$

This expression shows that, to the degree of approximation, the disturbed motion, for a very small force proportional to the square of the deflection, may be regarded as the superposition of two motions: one, represented by the first term, is identical with that which we have investigated, and takes place about a certain position of equilibrium; the other, represented by the second term, is a periodical displacement of this very position of equilibrium itself considered as a moving body by the fact of the perturbation. This latter attains its first elongation at the time $t=t_0$, and the following at the time $t=t_0+\frac{\pi}{\gamma}$. The period of this motion is therefore less than half that of the first, so that at each half oscillation the effect of the disturbance again becomes the same. The extreme values of the bracket are 1 and 2. It follows therefore that the minimum and the maximum are to each other as 1 : 2. They correspond very nearly to the periods of the oscillation and to those of the medium of the oscillation.*

685. APERIODIC MOTION.—If the roots of equation (15) are real—that is to say, if we have $\epsilon^2-n^2>0$, the motion represented by equation (14) is no longer periodic. The system, moved out of its position of rest and left to itself, returns to it gradually, only attaining it after an infinitely long time.

M. Dubois-Raymond has investigated this motion, which he calls *aperiodic.*† We may first of all study the conditions in which the system reverts practically to its position of equilibrium, without oscillation, and in a relatively short time.

Putting now $\gamma^2=\epsilon^2-n^2$; equation (14) becomes

$$(23) \qquad x=e^{-\epsilon t}\left[\mathrm{A}e^{\gamma t}+\mathrm{A}'e^{-\gamma t}\right],$$

which gives

$$(24) \qquad \frac{dx}{dt}=-e^{-\epsilon t}\left[\mathrm{A}(\epsilon-\gamma)e^{\gamma t}+\mathrm{A}'(\epsilon+\gamma)e^{-\gamma t}\right].$$

To determine the constants, we will assume that at the time $t=0$ we leave the system to itself at a distance a from its position of

* CORNU and BAILLE. *Comptes rendus de l'Academie des Sciences.* Vol. LXXXVI., p. 1001. 1878.

† DUBOIS-RAYMOND. *Monatsberichte der König. Preuss. Akad. der Wissen., 1869, 1870, 1873.*

equilibrium ; we have then for $t=0$, $x=a$ and $\frac{dx}{dt}=0$. We get from this,

$$A = a\frac{\epsilon+\gamma}{2\gamma},$$

$$A' = -a\frac{\epsilon-\gamma}{2\gamma},$$

and therefore

(25) $$x=\frac{a}{2\gamma}e^{-\epsilon t}\left[(\epsilon+\gamma)e^{\gamma t}-(\epsilon-\gamma)e^{-\gamma t}\right].$$

In this case, the deflection represented in rectilinear co-ordinates as a function of the time, is equal to the difference of the ordinates of two experimental curves which approach asymptotically the axis of the abscissæ. The value of x only becomes null, for $t=\infty$.

686. Instead of leaving the system to itself when it makes an angle a, let us suppose that we give it an angular velocity $-\omega$ directed towards the position of equilibrium.

In this case the constants A and A′ of the equation become

$$A=\frac{a(\epsilon+\gamma)-\omega}{2\gamma}, \qquad A'=\frac{-a(\epsilon-\gamma)+\omega}{2\gamma};$$

which gives

(26) $$x=\frac{e^{-\epsilon t}}{2\gamma}\left\{\left[a(\epsilon+\gamma)-\omega\right]e^{\gamma t}-\left[a(\epsilon-\gamma)-\omega\right]e^{-\gamma t}\right\}.$$

The motion is still aperiodic; but if the original velocity is suitable the system may pass beyond its position of equilibrium. The time of the passage through this position is defined by the equation

$$t_1=\frac{1}{2\gamma}\,l\,\frac{\omega-a(\epsilon-\gamma)}{\omega-a(\epsilon+\gamma)};$$

the greatest deflection beyond the position of equilibrium takes place at the time

$$t_2 = \frac{1}{2\gamma} l. \frac{\omega - a(\epsilon - \gamma)}{\omega - a(\epsilon + \gamma)} \cdot \frac{\epsilon + \gamma}{\epsilon - \gamma},$$

which corresponds to the condition $\frac{dx}{dt} = 0$. The system returns to its position of rest after an infinite time.

In order that the value of t_1 be real, the two terms of the fraction, the logarithm of which is taken, must be positive; that is to say, that

$$\omega > a(\epsilon + \gamma).$$

687. Let us assume, finally, that for $t = 0$, $x = 0$, and $\frac{dx}{dt} = \omega_0$; we find then, for the constants,

$$A = \frac{\omega_0}{2\gamma}, \qquad A' = -\frac{\omega_0}{2\gamma},$$

and the equation of the motion becomes

$$x = \frac{\omega_0}{2\gamma} e^{-\epsilon t} \left[e^{\gamma t} - e^{-\gamma t} \right]; \tag{27}$$

The greatest deflection is reached at the time t_1, given by the equation

$$t_1 = \frac{1}{2\gamma} l. \frac{\epsilon + \gamma}{\epsilon - \gamma}.$$

The value of the elongation is not capable of a simple expression; but the time t_1 being independent of the initial velocity, equation (27) shows that the elongation is proportional to this velocity.

688. The case in which $\epsilon = n$, and therefore $\gamma = 0$, is particularly interesting. It will be sufficient in the formulas (25), (26), and (29), to develop in a series the exponentials $e^{\gamma t}$, stopping at the two first terms, and to suppose $\gamma = 0$.

If the movable body is left to itself at the time $t=0$, at a distance a from its position of equilibrium, equation (25) gives then

$$(28) \qquad x=a(1+\epsilon t)e^{-\epsilon t}, \qquad \frac{dx}{dt}=-a\epsilon^2 te^{-\epsilon t}.$$

The movable body approaches its position of equilibrium asymptotically; the velocity, which starts from zero, reaches its maximum at the time $t=\frac{1}{\epsilon}$, and then decreases to zero.

For the case of an initial velocity $-\omega$ directed towards the position of equilibrium, equation (26) gives

$$(29) \qquad x=e^{-\epsilon t}\left[a-(\omega-a\epsilon)t\right], \qquad \frac{dx}{dt}=-e^{-\epsilon t}\left[\omega-\epsilon(\omega-a\epsilon)t\right];$$

it will be seen, that the movable body will only pass beyond its position of equilibrium if we have $\omega>a\epsilon$.

Lastly, when the movable body is thrown from its position of equilibrium with the velocity ω, we get by equation (27)

$$(30) \qquad x=\omega_0 te^{-\epsilon t}, \qquad \frac{dx}{dt}=\omega_0 e^{-\epsilon t}(1-\epsilon t).$$

The elongation is reached at the time $t=\frac{1}{\epsilon}$, and its value is

$$a_1=\frac{\omega_0}{\epsilon e}.$$

689. Observation of Oscillations.—It follows from the preceding discussion that all the circumstances of oscillating motion will be defined by the time of the successive elongations, and by the magnitude of the angle of deflection.

In order to investigate this motion, we observe either the passage of an index in front of a divided scale, or the image of a line on a micrometer in the focus of a telescope, or the image of a division on the cross-wire of the telescope, whether this division has been traced on the movable system, or whether it arises from the reflection of a scale in a mirror. In all cases there should be a means of noting the passage of the system through a definite position, and of estimating the angles of deflection.

The time of an elongation could not be determined with exactitude owing to the appreciable time in which the movable body is in apparent rest; we can, on the contrary, exactly determine the time at which the movable body passes through any given position in the region in which its velocity is greatest—that is to say, near the position of equilibrium.

Whatever be the position which serves as marks, the difference between the time of the two successive passages *in the same direction* represents exactly twice the time of what is ordinarily called a single oscillation.

In order to get this time accurately, it ought to be deduced from a great number of transits. When the oscillations are maintained for a very long time, like those of the pendulum, Borda's method of *coincidences* enables us to solve the problem with all the accuracy desirable; but this method would lend itself with difficulty to the observations we have here in view. And, moreover, the mode of observing would not be the same according as the damping is more or less rapid.

690. If the motion can be observed for some time (a quarter of an hour, for instance), we may determine the time of oscillation as follows:

Let us assume, as is the case with a magnetised needle, that the time of the oscillation is from 3 to 5 seconds, and that the divisions of a scale pass the cross-wire of a telescope. A division near the centre of the amplitude is taken as mark—that is to say, near that which corresponds to equilibrium. By means of a counter, the time t_0 is noted at which this division passes over the wire in a certain direction, either from right or left—this is the initial passage of order zero; the successive vibrations are then counted, and the time t_1 of the 20th transit, which is in the same direction as the first. At the same time the extreme divisions, corresponding to the initial vibrations, and those of the 20th are observed, from which are deduced the corresponding angles of deflection α_0 and α_1. It is unnecessary to count the following vibrations, for we know that the

40th transit will take place near the time $t_1+(t_1-t_0)$. We get ready for the observation a little beforehand, and note exactly the time t_2 of the 40th transit; we observe the corresponding amplitude which would give the angle of deflection a_2. In like manner we determine the times t_3, t_4, t_5, and the deflections a_3, a_4, a_5, of the 60th, 80th, and 100th transit.

From these observations are deduced the values t_1-t_0, t_2-t_1.... of the period of 20 vibrations during successive series, and the corresponding mean angles of deflection $\frac{a_0+a_1}{2}$, $\frac{a_1+a_2}{2}$..... etc.

If the successive periods do not appreciably differ from each other for large variations in the angles of deflection, we conclude that the directing couple is proportional to the deflection; the time of vibration will then be given by $\frac{1}{100}$ of the total difference.

691. Most frequently the directing couple is proportional to the sine of the deflection; the time of vibration varies then with the amplitude, according to the law of pendulum motion, and it is given by formula (8). To simplify the operations, tables have been calculated which give the value in the bracket $1+\beta$ for various angles of deflection. The following, for instance, are the values of this angle up to 30 degrees.*

Table for the Reduction of Pendulum Vibrations to Infinitely Small Angles.

Deflection. °	$1+\beta$	Deflection. °	$1+\beta$	Deflection. °	$1+\beta$
1	1·0000	11	1·0023	21	1·0085
2	1·0001	12	1·0027	22	1·0093
3	1·0002	13	1·0032	23	1·0102
4	1·0003	14	1·0037	24	1·0111
5	1·0005	15	1·0043	25	1·0120
6	1·0007	16	1·0049	26	1·0130
7	1·0009	17	1·0056	27	1·0141
8	1·0011	18	1·0062	28	1·0151
9	1·0015	19	1·0070	29	1·0162
10	1·0019	20	1·0077	30	1·0174

* DURANDEAU et CHEVALIER. *Voyage de la Bonite. Observations Magnetiques,* Vol. II., p. 9.

Each partial time $t_1 - t_0$, $t_2 - t_1 \ldots.$ is divided by the value $1 + \beta_0$, $1 + \beta_1 \ldots..$ for the corresponding deflection $\frac{a_0 + a_1}{2}$, $\frac{a_1 + a_2}{2}$, and we get an approximate value $t'_0\ t'_1 \ldots..$ of twenty infinitely small vibrations. If the values thus calculated do not systematically differ, the phenomenon is regular and the correction exact. The mean of all these values is finally taken as the time of a vibration.

The table of observations and of successive reductions is arranged as follows:—

Order of Transit.	Time of Transit.	Angle of Deflection.	Duration of Vibration.	Mean Deflection.	Reduced Time.
			$t_1 - t_0$	$\frac{a_0 + a_1}{2}$	$t'_0 = \frac{t_1 - t_0}{1 + \beta_0}$
20	t_1	a_1			
			$t_2 - t_1$	$\frac{a_1 + a_2}{2}$	$t'_1 = \frac{t_2 - t_1}{1 + \beta_1}$
40	t_2	a_2			
			$t_3 - t_2$	$\frac{a_2 + a_3}{2}$	$t'_2 = \frac{t_3 - t_2}{1 + \beta_3}$
60	t_3	a_3			
			$t_4 - t_3$	$\frac{a_3 + a_4}{2}$	$t'_3 = \frac{t_4 - t_3}{1 + \beta_3}$
80	t_4	a_4			
			$t_5 - t_4$	$\frac{a_5 + a_4}{2}$	$t'_4 = \frac{t_5 - t_4}{1 + \beta_4}$
100	t_5	a_5			

We shall have, finally, for the time τ of infinitely small vibrations,

$$\tau = \frac{t'_0 + t'_1 + t'_2 + t'_3 + t'_4}{100}.$$

The errors made in the intermediate transits do not appreciably affect the value of τ, and the accuracy depends specially on the exactitude with which the initial and the final transit has been noted.

If the observations are made to within $\frac{1}{10}$ of a second, we shall have for a quarter of an hour an approximation of about $\frac{1}{5000}$.

If there is no interest in verifying the exactitude of the correction of the deflections separately for different series, the reduction may be made more rapidly. Let $p_1, p_2 \ldots . p_5$ be the numbers of the infinitesimally small vibrations of times τ which are made during the intervals $t_1 - t_0$, $t_2 - t_1$, $t_3 - t_2 \ldots . t_5 - t_4$ of the twenty vibrations which form the successive series; we have

$$\begin{aligned} t_1 - t_0 &= p_1\tau = 20\tau(1 + \beta_1), \\ t_2 - t_1 &= p_2\tau = 20\tau(1 + \beta_2), \\ &\vdots \\ t_5 - t_4 &= p_5\tau = 20\tau(1 + \beta_5). \end{aligned}$$

P being the total number of infinitely small vibrations which would correspond to the whole time $t_5 - t_0$ of the five series, it follows that

$$P = 20[5 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5] = 100\left[1 + \frac{\beta_1 + \beta_2 + \ldots \beta_5}{5}\right],$$

$$\tau = \frac{t_5 - t_0}{P}.$$

More generally, if during the time θ we have observed m series of n vibrations—that is, a total of $nm = N$ vibrations—and that the corrections of the mean amplitudes are $\beta_1, \beta_2 \ldots . \beta_m$, we have

$$(31) \qquad P = N\left[1 + \frac{\beta_1 + \beta_2 + \ldots + \beta_m}{m}\right], \quad \text{and} \quad \tau = \frac{\theta}{P}.$$

692. When the deflections are so small that the value of β may be reduced to its first term $\frac{a^2}{16}$, and that the amplitudes are in geometric progression, the correction may be simply made by means of the logarithmic decrement. The successive times of n vibrations will be

$$\tau\left(1 + \beta_1\right), \quad \tau\left(1 + \beta_1 e^{-2\lambda}\right), \ldots . \quad \tau\left(1 + \beta_1 e^{-2(n-1)\lambda}\right),$$

and the whole time θ,

$$\theta = \tau\left[n + \beta_1\left(1 + e^{-2\lambda} + \ldots + e^{-2(n-1)\lambda}\right)\right] = \tau\left[n + \beta_1 \frac{1 - e^{-2n\lambda}}{1 - e^{-2\lambda}}\right].$$

Putting

$$\gamma = \beta_1 \frac{1 - e^{-2n\lambda}}{1 - e^{-2\lambda}},$$

we get

$$(32) \qquad \theta = \tau(n + \gamma) = n\tau\left(1 + \frac{\gamma}{n}\right).$$

The correction $\frac{\phi}{n}$ being of the order of thousandths, we have, with sufficient approximation,

$$\tau = \frac{\theta}{n}\left(1 - \frac{\gamma}{n}\right) = \frac{\theta}{n} - \frac{\theta\gamma}{n^2}.$$

The term $\frac{\theta\gamma}{n}$ is the correction of time for the total duration of n oscillations; $\frac{\theta\gamma}{n^2}$ is the correction for the mean time $\frac{\theta}{n}$ of a vibration.

The logarithmic decrement λ only depends on the extreme deflections a and a_n, or on the corresponding values β_1 and β_n. We have

$$2\lambda = \frac{2}{n-1} l . \frac{a_1}{a_n} = \frac{1}{n-1} l . \frac{\beta_1}{\beta_n},$$

and may put

$$\gamma = \frac{\beta_1 - \beta_n e^{-2\lambda}}{1 - e^{-2\lambda}} = \frac{\beta_1 e^{2\lambda} - \beta_n}{e^{2\lambda} - 1} = \beta_1 + \frac{\beta_1 - \beta_n}{e^{2\lambda} - 1}.$$

When the logarithmic decrement is also very small, of the order of thousandths, so that its square may be neglected, the development of $e^{2\lambda}$ in series gives simply

$$e^{2\lambda} - 1 = 2\lambda.$$

From this follows

$$\gamma=\beta_1+\frac{\beta_1-\beta_n}{2\lambda}=\frac{\beta_1(1+2\lambda)-\beta_n}{2\lambda},$$

or again, to the same degree of approximation,

$$(33) \qquad \gamma=\frac{\beta_1-\beta_n}{2\lambda}=\frac{1}{16}\frac{\alpha_1^2-\alpha_n^2}{2\lambda}.$$

If the amplitudes of the vibrations are given by the readings a of a scale seen by reflection, and at a distance D from the axis of rotation, we have

$$\alpha=\frac{a}{4D}, \qquad \alpha^2=\frac{a^2}{16D^2};$$

we thus obtain

$$(34) \qquad \gamma=\frac{1}{(16D)^2}\frac{a_1^2-a_n^2}{2\lambda}=\frac{1}{512D^2}\frac{a_1^2-a_n^2}{\lambda}.$$

Damping does not in general much alter the time of vibration. Suppose, for instance, that the angle of deflection has diminished from 10° in the first to 2° in the hundredth vibration; the value of the logarithmic decrement is

$$\lambda=\frac{1}{99}\,l.\frac{10}{2}=0·0163.$$

The correction $\frac{\gamma}{n}$, which should be introduced in the mean time of an oscillation calculated by equation (32) does not then exceed 0·0005.

693. We shall take as an example the observation of a compass of horizontal intensity made at Pondicherry, June 7th, 1837, by Durandeau and Chevalier.* The observers had marked 1400 oscillations in series of 50; but, for the sake of brevity, we shall only give the readings relating to the hundredths.

* DURANDEAU and CHEVALIER. *Voyage de la Bonite. Observations Magnetiques*, Vol. II., p. 131.

Order of Transit.	Time of Transit.	Deflection.	Time of 100 Vibrations.	Mean Deflection.	Reduced Time.
	h m s	°	s	°	s
1	1 35 31·7	35·0			
			369·3	31·5	362·33
100	1 41 41·0	28·0			
			366·0	25·0	361·66
200	1 47 47·0	22·0			
			364·7	19·5	362·04
300	1 53 51·7	17·0			
			363·3	15·5	361·64
400	1 59 55·0	14·0			
			363·0	12·5	361·94
500	2 5 58·0	11·0			
			362·3	9·75	361·65
600	2 12 0·3	8·5			
			362·0	7·75	361·70
700	2 18 2·3	7·0			
			362·0	6·0	361·75
800	2 24 4·3	5·0			
			362·4	4·5	362·26
900	2 30 6·7	4·0			
			362·0	3·75	361·89
1000	2 36 8·7	3·5			
			362·0	3·25	361·93
1100	2 42 10·7	3·0			
			362·3	2·75	362·23
1200	2 48 13·0	2·5			
			362·0	2·25	361·96
1300	2 54 15·0	2·0			
			362·3	1·75	362·27
1400	3 0 17·0	1·5			

$$\text{mean} = 361{\cdot}94^{s},$$

$$\tau = 3{\cdot}6194^{s}.$$

A calculation made by equation (31) would give

$$P = 1400\,[1{\cdot}00363] = 1405{\cdot}08,$$

and

$$\tau = \frac{1^{h}24^{m}54{\cdot}6^{s}}{1405{\cdot}08} = \frac{5085{\cdot}6^{s}}{1405{\cdot}08} = 3{\cdot}6194^{s}.$$

The observations are made by a counter, the rate of which should be known, and it only then remains to make the corrections for this rate of the counter.

The two methods lead, of course, to the same results; but the table of successive reductions, spite of the differences due to errors of observation, has the advantage of showing that the reduced length of observations tends manifestly to increase with the time. In the particular case the increase of duration was due to a rise in temperature of the vibrating needle.

694. If the zero is fixed, or if it is merely displaced in consequence of a periodic motion like that above referred to (684), we may note the passages over any given division, since we always count a whole number. This is not the case when the zero is displaced without reference to any law, as is most frequently the case with

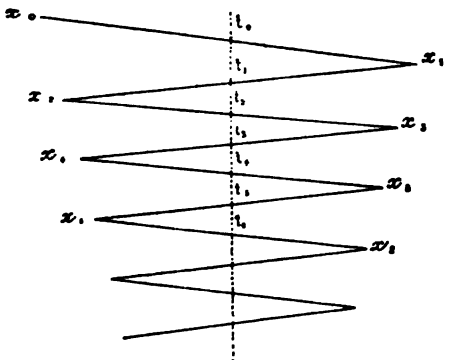


Fig. 133.

magnetised bars, owing to variations of the declination. If at the moment the transit is noted, the position of equilibrium is not the same as in the previous observation, an error is committed equal to the time which the moving point takes to traverse the arc the zero of which is displaced, and this error can only be avoided by observing each time the passage on the division which corresponds to the actual zero. With this object, a little before the observation the values of two successive elongations are called out to an assistant, who, taking into account the damping, would immediately deduce from them the division corresponding to zero, and the reading would be made on this division.

This method of correction obviously implies that the position of equilibrium is not appreciably displaced during several vibrations.

695. Gauss's Method. — The following method, used by Gauss,* may be applied to all kinds of vibrations, whatever be the damping; but it is especially convenient when the vibrations are very slow.

Suppose that we observe the image of a scale, the divisions of which are numbered from one end. We shall call x_0, x_1, x_2.... the divisions which correspond to the successive elongations (Fig. 133), the even indices representing the oscillations to the left, and the odd ones to the right.

We must first determine the position of equilibrium by the oscillations themselves, without waiting for the system to come to rest. If p is the division which represents equilibrium, we may take the value

$$(35) \qquad p = \frac{x_0 + 2x_1 + x_2}{4},$$

which represents the mean position $\frac{1}{2}\left(\frac{x_0 + x_1}{2} + \frac{x_1 + x_2}{2}\right)$ of the middles of two successive amplitudes.

If this first approximation is insufficient, by writing that the three successive deflections are in geometrical progression, and that

$$x_0 - p = (p - x_1)\, e^{\lambda} = (x_2 - p)\, e^{2\lambda};$$

from which follows

$$(p - x_1)^2 = (x_0 - p)\,(x_2 - p),$$

or

$$(36) \qquad p = \frac{x_0 x_2 - x_1^2}{x_0 + x_2 - 2x_1}.$$

The division p thus defined divides the oscillation into two parts, which are as $1 = e^{-\lambda}$. This point moreover has the property that successive transits in opposite directions take place in equal intervals.

This would not be the case with the perturbed motion described above (684), in which disturbing forces come into play proportional to the squares of the deflections. If p_1 be the division which corresponds to equal times of oscillations on either side, the divisions p and

* Gauss. *Resultate aus den Beob. des Mag. Vereins.* 1837. *Œuvres*, v., p. 374.

p_1 no longer coincide, and neither represents the true position of equilibrium under the action of the same forces. The value of p differs from p_0 by the quantity $A^2 \frac{n'^2}{3n^2} e^{-2\epsilon t}$; that of p_1 differs from it by twice this quantity, and in the same direction. It follows that the difference $p_1 - p$ gives the value of $A^2 \frac{n'^2}{3n^2} e^{-2\epsilon t}$, and that the position of p_0 would be obtained by taking the symmetric point of p_1 in reference to p.

696. To determine the time of the oscillations, seven successive elongations are noted, x_0, x_2, x_4, x_6 on one side, and x_1, x_3 and x_5 on the other, as well as the times $t_0, t_1, \ldots . t_5$, at which a division near that corresponding to equilibrium passes over the cross-wire; then at the end of a given time, which it is useless to note, a series of seven new elongations, x_0^1, x_1^1, x_2^1, are noted, and the corresponding times of passage t_0^1, $t_1^1 \ldots .$

The first series gives rise to the following table :—

Elongation.	Time of Passage.	Time of the Elongation.	Position of Zero.	Amplitude of each Oscillation.
x_0				
	t_0			
x_1		$\frac{t_0+t_1}{2}$	$\frac{x_0+2x_1+x_2}{4}$	$\frac{x_0+x_2-2x_1}{2}=a_1$
	t_1			
x_2		$\frac{t_1+t_2}{2}$	$\frac{x_1+2x_2+x_3}{4}$	$\frac{2x_2-(x_1+x_3)}{2}=a_2$
	t_2			
x_3		$\frac{t_2+t_3}{2}$	$\frac{x_2+2x_3+x_4}{4}$	$\frac{x_2+x_4-2x_3}{2}=a_3$
	t_3			
x_4		$\frac{t_3+t_4}{2}$	$\frac{x_3+2x_4+x_5}{4}$	$\frac{2x_4-(x_3+x_5)}{2}=a_4$
	t_4			
x_5		$\frac{t_4+t_5}{2}$	$\frac{x_4+2x_5+x_6}{4}$	$\frac{x_4+x_6-x_5}{2}=a_5.$
	t_5			
x_6				

Let n be the order of the initial elongation in the second series, or the difference of the orders of two corresponding elongations. We know whether this number is even or odd, and the approximate value of the time of oscillations deduced from the first series enables us to determine it without possible error.

The difference of the times of corresponding elongations in the two series is successively

$$\frac{t'_0+t'_1}{2}-\frac{t_0+t_1}{2}, \qquad \frac{t'_1+t'_2}{2}-\frac{t_1+t_2}{2}, \ldots\ldots$$

We obtain thus five values

$$\frac{1}{n}\left(\frac{t'_0+t_1}{2}-\frac{t_0+t_1}{2}\right), \qquad \frac{1}{n}\left(\frac{t'_1+t'_2}{2}-\frac{t_1+t_2}{2}\right), \ldots$$

of the time of oscillation relative to the mean period comprised between the two series. This time might be reduced to that of infinitely small vibrations by making the correction for the angle of deviation in the times $\frac{t_2-t_0}{2}$, $\frac{t_3-t_1}{2}$ in each series, or by introducing into the mean duration $\frac{1}{n}\left(\frac{t'_0+t'_1}{2}-\frac{t_0+t_1}{2}\right)$ that of the mean of the elongations which correspond to each other in the two series.

The reduction of each time of vibration to infinitely small angles makes the calculations longer, but it has the advantage of furnishing a continuous control of the observations.

697. The logarithmic decrement deduced from the initial amplitude a, and the amplitude b of the nth oscillation, which follows the preceding one, is $\lambda = \frac{1}{n} l. \frac{a}{b}$.

If we consider the first amplitude a as known exactly, and that the probable absolute error is the same for all the following, we may choose that number n of oscillations which gives the value of λ with the closest approximation. If $d\lambda$ is the error corresponding to an error db in the last amplitude b, we have

$$\frac{d\lambda}{\lambda} = -\frac{db}{bl.\frac{a}{b}}.$$

The first member of this equation which represents the relative error made in the determination of λ has the least value when the product $\frac{l.\frac{a}{b}}{?}$, considered as a function of b, is a maximum—that is to say, when

$$l.\frac{a}{b}=1, \quad \text{or} \quad n=\frac{1}{\lambda}.$$

With the series of observations of Gauss, the logarithmic decrement λ is determined according to the case, either by amplitudes of the same series,

$$\lambda=l.\frac{a_1}{a_2}=l.\frac{a_2}{a_3}=l.\frac{a_3}{a_4}=l.\frac{a_4}{a_5}=\frac{1}{4}l.\frac{a_1}{a_5},$$

or by corresponding amplitudes of two consecutive series separated by $n-1$ oscillations.

$$\lambda=\frac{1}{n}l.\frac{a_1}{a_1'}=\frac{1}{n}l.\frac{a_2}{a_2'}=\ldots\ldots=\frac{1}{5n}\left[l.\frac{a_1}{a_1'}+l.\frac{a_2}{a_2'}+\ldots\ldots+l.\frac{a_5}{a_5'}\right].$$

and finally and more simply, by the mean amplitudes a_1, a', a'', of the different series.

By calculating λ from the successive oscillations of the same series, or from corresponding oscillations of two successive series, we retain the advantage of verifying whether the amplitudes do vary in geometrical progression.

698. We shall reproduce here the example given by Gauss himself, with some change in the arrangement of the columns, and suppressing the last column of decimals which may be considered illusory. This observation is on the oscillations of a magnet of twenty-five pounds under the influence of the earth alone.

The times observed are those at which the line 1,000 of the scale passes over the cross-wires. The values of the position of zero deduced from formula (35) have been added to the table. These values do not enter into the calculation, but it is necessary to be certain that the zero has not varied sensibly in the course of the same series, and therefore in the interval of two consecutive passages, for a calculation of the time of elongation is only exact as long as this latter condition is fulfilled.

Elongations 0 and 4 give 42·20^s as the approximate value of the time of oscillation; dividing this time by the interval between the fourth elongation and the first of the following series, we get the number 142·98; this latter elongation, which is of the odd order, will have the number 147. We proceed in the same way for the following series:—

Order of Elongation.	Elongation.	Time of Passage.	Time of Elongation.	Position of Zero.	Amplitude of Oscillation.
		h. m. s.	h. m. s.		
	1755·1	21 55 26·9			
0	266·0	21 56 8·4	21 55 47·65	1009·72	1487·45
1	1731·8	21 56 51·2	21 56 29·83	1009·52	1484·55
2	268·5	21 57 33·0	21 57 12·10	1009·42	1481·85
3	1748·9	21 58 15·5	21 57 54·25	1009·47	1478·85
4	271·6	21 58 57·4	21 58 36·45	1009·07	1474·95
	1744·2				
	497·8	23 38 49·2			
147	1502·2	23 39 31·5	23 39 10·35	1000·57	1003·15
148	500·1	23 40 13·6	23 39 52·55	1000·37	1000·55
149	1499·1	23 40 56·0	23 40 34·80	1000·22	997·75
150	502·6	23 41 38·1	23 41 17·05	1000·20	995·20
151	1496·5	23 42 20·3	23 41 59·20	1000·40	992·20
	506·1				
	645·9	1 10 12·6			
277	1341·5	1 10 54·2	1 10 33·40	994·05	694·90
278	647·3	1 11 37·0	1 11 15·60	993·87	693·15
279	1339·4	1 12 18·4	1 11 57·70	993·70	691·40
280	648·7	1 13 1·3	1 12 39·85	993·45	689·50
281	1337·0	1 13 43·0	1 13 22·15	993·35	687·30
	650·7				
	1232·1	2 49 19·7			
418	775·9	2 50 1·5	2 49 40·60	1003·72	455·65
419	1231·0	2 50 44·1	2 50 22·80	1003·57	454·85
420	776·4	2 51 25·8	2 51 4·95	1003·12	453·45
421	1228·7	2 52 8·5	2 51 47·15	1002·95	451·50
422	778·0	2 52 0·05	2 52 29·25	1002·92	449·85
	1227·0				

The corresponding elongations in the two first series give for the time of 147 oscillations:

					h. m. s.
Between the elongations		0	and	147......	1 43 22·70
"	"	1	and	148......	1 43 22·75
"	"	2	and	149......	1 43 22·70
"	"	3	and	150......	1 43 22·80
"	"	4	and	151......	1 43 22.75

The mean $1^h 43^m 22{\cdot}74^s$ gives $42{\cdot}195^s$ for the mean time of an oscillation in the first interval: we obtain in the same way $42{\cdot}176^s$ for the second interval, and $42{\cdot}179^s$ for the third. The general mean is $42{\cdot}1834^s$.

Between the successive series 1 and 2, 2 and 3, 3 and 4, we find 0·002689, 0·002813, 0.002996 for the logarithmic decrement. These values are slightly increasing; the amplitudes diminish then a little more rapidly than is required by the law of geometrical progression.

The distance of the scale was 4775·9 mm. The reduction to infinitely small oscillations calculated by formula (34), gives for the interval which separates the middle of two consecutive series—

Elongations.	$\frac{\theta\gamma}{n}$	$\frac{\theta\gamma}{n^2}$	τ
	s.	s.	s.
2 and 149	1·611	0·0109	42·184
149 and 279	0·622	0·0051	42·171
279 and 420	0·328	0·0023	42·176

The mean $42{\cdot}177^s$ represents the value of t with a probable error less than the five-thousandth of a second.

CHAPTER III.

MEASUREMENT OF COUPLES.

699. The torsion couple, which tends to bring an apparatus movable about its axis towards its position of equilibrium, might be rigorously determined by the action of an antagonistic couple, such as is obtained by a weight attached to the end of the arm of a lever of known length; this direct method was used by Wertheim* in studying the elasticity of torsion of metal bars; but in most cases it would not be applicable, and the value of a torsion couple would most frequently be deduced from the study of oscillation.

The time T of infinitely small oscillations, when the damping can be neglected, enables us to determine the ratio of the coefficient C (**675**) to the moment of inertia K of the system, by the formula

$$\text{(1)} \qquad \frac{C}{K} = n^2 = \frac{\pi^2}{T^2}.$$

700. Moments of Inertia.—The determination of the moment of inertia K may be made either directly, when the body is homogeneous, and of geometric shape, from its weight and dimensions, or indirectly by comparison with a body the moment of inertia of which can be directly calculated.

We shall collate here the principal formula relative to moments of inertia of which we shall have to make use.

In the C.G.S. System (**612**), the mass of a body is given directly by the numerical value of its weight p in grammes. When the gramme is taken as unit of force the mass is equal to $\frac{p}{g}$, g being the acceleration of gravity expressed in centimetres.

* Wertheim. *Ann. de Chim. et de Phys.* [3], Vol. L., p. 195, pt. 385. 1857.

The radius of gyration of a body in reference to an axis is the distance to which the entire mass of the body must be transported, so that its moment of inertia in reference to this axis is not modified. If ρ is the radius of gyration measured in centimetres, the moment of inertia is expressed in C.G.S. units by

$$K = p\rho^2;$$

and, with the gramme as unit of force,

$$K = \frac{p}{g}\rho^2.$$

We may here mention a theorem very useful in practice.

The moment of inertia K′ *of a body in respect to a given axis, is equal to the moment of inertia* K *of the body referred to an axis parallel to the first, and passing through the centre of gravity, plus the product of the mass by the square of the distance* r *of the two axes.*

The radius of gyration ρ' in respect of the new axis satisfies the equation

$$\rho'^2 = \rho^2 + r^2.$$

701. We shall give the value of the radius of gyration of a certain number of homogeneous bodies of simple form.

Rectangular Parallelopipedon.—If the dimensions of the parallelopipedon are $2a$, $2b$, $2c$, we have for an axis passing through the centre parallel to the dimension c,

$$\rho^2 = \frac{a^2 + b^2}{3}.$$

The expression $\sqrt{a^2 + b^2}$ represents half the diagonal of the face perpendicular to the axis. When one of the transverse dimensions, b for instance, is very small, as in the case of a thin plate, this expression, put in the form

$$\rho^2 = \frac{a^2}{3}\left(1 + \frac{b^2}{a^2}\right),$$

shows that we may take as the value of ρ, the approximate value $\frac{a}{\sqrt{3}}$, when $\frac{b^2}{a^2}$ is a negligable quantity.

Solid Cylinder.—For a right circular cylinder with parallel bases, of radius a, and of any given length, the radius of gyration in respect of the axis is given by the expression

$$\rho^2 = \frac{a^2}{2}.$$

In respect of a right line passing through the centre, and perpendicular to the axis, if $2l$ is the length of the cylinder, we have

$$\rho^2 = \frac{l^2}{3} + \frac{a^2}{4} = \frac{l^2}{3}\left(1 + \frac{3}{4}\frac{a^2}{l^2}\right).$$

Hollow Cylinder.—If a and a' are the radii of two right circular concentric cylinders, the radius of gyration, in respect of the common axis of the volume between them, is given by the formula

$$\rho^2 = \frac{a^2 + a'^2}{2};$$

we may write

$$\rho^2 = \frac{(a+a')^2 + (a'-a)^2}{4} = \left(\frac{a+a'}{2}\right)^2 \left[1 + \left(\frac{a'-a}{a+a'}\right)^2\right].$$

If the thickness $a' - a = e$ is very small, we have, disregarding terms of $\frac{e}{a}$ of the second order,

$$\rho^2 = a^2\left(1 + \frac{e}{a}\right).$$

Sphere.—For a solid sphere of radius a, we have

$$\rho^2 = \frac{2}{5}a^2.$$

702. In accurate experiments, we must employ bodies of easy construction, and the dimensions of which can be precisely measured; it is further preferable, in order to load the oscillating apparatus as little as possible, to choose those which have the greatest moment of

inertia for a given weight, that is to say—those which have the greatest radius of gyration.

Coulomb used a *solid cylinder* movable about its axis; Sir W. Thomson recommends the use of a *hollow cylinder* resting on a thin square or circular plate, so that it can be accurately centred in reference to the axis of rotation. These two shapes have the advantage that they can be readily turned, and made true, and have only a very slight friction against the external medium. The hollow cylinder is less easy to regulate, but any defects of homogenity in the substance are of less importance, and do not perceptibly alter the radius of gyration.

A *rectangular parallelopipedon*, in the shape of a very elongated *thin plate*, may also be advantageously employed. The dimensions of the plate may be more easily measured than those of the cylinder, and the thickness only necessitates a very small term of correction; other things being equal, a plate gives a greater radius of gyration than the annular cylinder employed by Sir W. Thomson; for if we suppose the cylinder slit, and stretched out straight, its moment of inertia in the shape of plate is $\frac{\pi^2}{3}$ times as great as in the form of ring.

703. Experimental Determination of Moments of Inertia, and of Couples.—When the shape of the moving system is complicated, or its structure is not homogeneous, its moment of inertia can only in general be determined by experiment, and by comparison with that of a homogeneous body of simple form.

Two distinct cases present themselves, according as the directing couple, which tends to restore the system to its position of rest, is, or is not, independent of the weight of the system; if the damping cannot be neglected, the value T is deduced from the true time τ of infinitely small oscillations, correcting it for the effect of damping. An auxiliary body of known moment of inertia K′ is added to the system, and the new time T′ of oscillations is determined in the same way. We have then

$$\frac{C}{\pi^2} = \frac{K}{T^2} = \frac{K+K'}{T'^2} = \frac{K'}{T'^2 - T^2};$$

from which is deduced

$$K = \frac{T^2}{T'^2 - T^2} K',$$

$$C = \frac{\pi^2}{T'^2 - T^2} K'.$$

In his experiments on Terrestrial Magnetism, Gauss* fixed to the movable arrangement a horizontal divided scale, and as auxiliary body used *two spheres* which he suspended to the scale by hooks, at equal distances on either side of the axis. The distance d of the points of attachment of the two hooks is accurately measured. If q' is the weight of each hook, ρ' its radius of gyration in respect of an axis passing through the centre of gravity of the sphere, a the radius of the spheres, and q the weight of each of them, the value of the moment of inertia of the additional system is

$$K' = 2\left\{(q+q')\frac{d^2}{4} + \frac{2}{5}qa^2 + q'\rho'^2\right\};$$

the term $q'\rho'^2$ may usually be neglected.

If we desire to eliminate the moment of inertia K_1 of the scale, the system is made to oscillate in three different conditions; 1st, without the scale; 2nd, with the scale and without the spheres; 3rd, with the scale and the spheres. The equations

$$\frac{C}{\pi^2} = \frac{K}{T^2} = \frac{K+K_1}{T_1^2} = \frac{K+K_1+K'}{T'^2}$$

enable us to determine the three quantities, K, K_1, and C, as a function of the moment of inertia K′, and of the three times of oscillation, T, T_1, and T_2. It is moreover useful to repeat the third determination with different values of d, and to calculate the unknown quantities by suitable combinations of all the equations.

A *rectangular plate*, a *sphere* suspended to an axis, a solid or hollow *cylinder*, would perform the same office with more or less of advantage.

704. There is, in practice, always reason to fear that, whatever be the mode of suspension of the movable system, the directive couple is not independent of the weight.

The experiments are then made in such a way as to modify the moment of inertia of the system without altering the total weight. The oscillating arrangement must be composed of a fixed part, whose moment of inertia is K, and of a movable part, the moment of inertia of which acquires known values K′ and K″, in two successive

* GAUSS. *Intensitas vis Magnet*, etc. *Comm. Göttin.*, VIII. 1841. *Œuvres*, V., p. 95.

positions; the corresponding times of oscillation T′ and T″ will then give

$$\frac{C}{\pi^2}=\frac{K+K'}{T'^2}=\frac{K+K''}{T''^2}=\frac{K'-K''}{T'^2-T''^2}.$$

If we merely propose to determine the torsion couple, we shall have

$$C=\frac{\pi^2}{T'^2-T''^2}(K'-K'');$$

it is sufficient therefore to know the difference $K'-K''$ of the moments of inertia of the movable system in the two positions.

Suppose, for instance, that the movable system consists of the two spheres of Gauss, or of two weights q of any given form, placed successively at the distances d' and d'', and symmetrically in respect of the angle of rotation; we shall have

$$K'-K''=\frac{q}{2}(d'^2-d''^2).$$

We may also employ a *rectangular rule* of weight q, arranged so that two of its dimensions, a and b, might be made alternately perpendicular to the axis, by a simple rotation of 90° about a parallel to the edge c.

In this case the absolute values of the moments of inertia K′ and K″ depend on the distance from the centre of gravity of the scale to the axis of rotation, but their difference is

$$K'-K''=q\left(\frac{a^2+c^2}{3}-\frac{b^2+c^2}{3}\right)=q\frac{a^2}{3}\left(1-\frac{b^2}{a^2}\right).$$

705. Suppose, finally, that the moment of the directive couple is in a known ratio to the weight of the system; that, for instance, it is proportional, as we shall see is the case with the bifilar suspension. If P is the weight, and K the moment of inertia of the original system, P′ and K′ are the same quantities relative to the additional bodies, and h is a constant coefficient, we shall have

$$C=hP,$$
$$C'=h(P+P'),$$

and therefore for the times of oscillation

$$hPT^2 = \pi^2 K,$$
$$h(P+P')T'^2 = \pi^2(K+K').$$

From this is deduced

$$K = \frac{PT^2}{(P+P')T'^2 - PT^2} K',$$
$$h = \frac{\pi^2 K'}{(P+P')T'^2 - PT^2}.$$

706. Unifilar Balance.—The unifilar balance, or *torsion balance*, was devised by Coulomb, who showed the advantage that could be drawn from it in the measurement of small forces;* it consists essentially of an elastic wire, ordinarily of metal, fixed at the top, and carrying a movable system—for instance, a needle with an electrified sphere, a magnetised bar, etc.

The wire is attached at the top to a graduated drumhead, which measures the rotation imparted to the end of the wire by being movable against a mark or vernier. When the system is free, and has not of itself any directive force, its position of equilibrium is that in which the wire is without torsion. If the system is exposed to any extraneous directive action—if it supports, for instance, a magnet under the action of the earth, then by suitably turning the drumhead, we may bring the system into that direction which it would acquire under the directive force of the earth alone, the wire being without torsion. This condition is clearly realised if the position of equilibrium does not change when the external directive force is suppressed—when, for instance, the magnetised bar is replaced by a copper one of the same weight. Any angular displacement α of the body from this position produces a torsion of the same angle in the wire if the top is fixed; if, on the contrary, the upper micrometer is turned through an angle A in the contrary direction, the total torsion of the wire is equal to the sum of the two angles, and therefore equal to $A+\alpha$.

707. Laws of Torsion.—When a system suspended to a metal wire is moved from its position of rest and left to itself, it makes oscillations on each side of this position. The experiments of

* Coulomb. *Mem. des savants étrangers*, Vol. x., 1777; and *Mem. de l'Acad. pour 1784*.

Coulomb have shown that within pretty wide limits these oscillations are virtually isochronous. It follows that the moment of the couple which tends to bring the twisted wire to its position of equilibrium is proportional to the angle of deflection.

If the total torsion is θ, the expression for this couple is $C\theta$. The constant C represents the moment of the couple which would be necessary to twist the wire through unit angle, if the law of proportionality held for all angles; we shall call it the *coefficient of torsion of the wire.* Experiment shows that it is virtually independent of the weight of the suspended body—in other words, of the tension of the wire.

If the wire is cylindrical and circular in section, of length l and diameter d, we have, according to Coulomb,

$$C = \mu \frac{d^4}{l}. \tag{2}$$

The coefficient of a wire is therefore inversely as its length, and proportional to the fourth power of its diameter, or to the square of its section, the factor μ only depending on the nature of the wire, and on the temperature.

708. If in this formula we make $d = 1$ and $l = 1$, we have $C = \mu$. The coefficient μ is then the numerical expression of a couple which can twist through unit angle a cylinder the length of which is a centimetre, and the base a centimetre in diameter, and which would act upon one of the bases, the other being rigidly fixed. The coefficient μ is not really the moment of a couple; it represents the quotient of a force by a surface, and only differs by a numerical factor from what is usually called the *rigidity* or *second modulus of elasticity.*

Let us consider, in fact, a straight cylinder of section S, and suppose that while one of the bases is rigidly fixed the other is acted upon at each point by a tangential force, constant in magnitude and direction, equal to F for unit surface; each section parallel to the bases is displaced parallel to itself without deformation, and by a quantity proportional to its distance from the fixed base; the cylinder is then turned through an angle α, independent of its height, and proportional to the force F, and we may write

$$\alpha = \frac{1}{\phi} F, \qquad \text{or} \qquad F = \alpha\phi.$$

The coefficient ϕ is the second modulus of elasticity; it represents physically the force which would be necessary to incline through an

angle equal to unity, a cylinder the section of which is equal to unity, and one of the bases fixed.

Supposing now that a circular cylinder of length l, and diameter d is twisted through an angle θ; each of the primitive generating lines becomes a helix. The lower base being fixed, the displacement of a point of the upper base, at a distance r from the axis, is equal to $r\theta$, and the corresponding inclinations of the helix to the axis is $\alpha = \frac{r\theta}{l}$; the elastic reaction for unit surface is

$$F = \alpha\phi = \theta \frac{\phi}{l} r,$$

and its moment, in reference to the axis,

$$Fr = \theta \frac{\phi}{l} r^2.$$

The moment in respect to the axis of the elastic reactions relative to the surface of a circular corona comprised between the radii r and $r + dr$ has the value

$$Fr \,.\, 2\pi r dr = \theta \frac{2\pi\phi}{l} r^3 dr.$$

The torsion couple $C\theta$ is equal to the integral of this expression extended to the entire base of the cylinder; we have then

$$C = \frac{2\pi\phi}{l} \int_0^{\frac{d}{2}} r^3 dr = \frac{\pi}{32} \phi \frac{d^4}{l}.$$

Comparing this value with that of Coulomb given by equation (2), we deduce

$$\mu = \frac{\pi}{32} \phi.$$

709. Determination of the Coefficient of Torsion.—The application of the methods we have been discussing, to the determination of the torsional coefficient of a wire is not without serious difficulties.

If the wire used is to have a fixed moment of torsion, it must be reheated with the greatest care, so as to deprive it of any trace of previous torsion. If there were any residues of torsion, the zero would become displaced in the course of time, and moreover the elastic reaction would not be the same on either side of the position of equilibrium.

Coulomb's experiments tend to show that the coefficient C is independent of the tension; but this law is only approximate, and more accurate researches have shown that the elasticity of a wire diminishes as the tension increases, apart from the variation arising from the change of length and of diameter.*

The oscillations should always be comprised within narrow limits, for permanent deformations are not long in showing themselves. According to Wiedemann,† these deformations would be produced even with the smallest angles, and the vibratory motion of a wire about its axis would really be far more complicated than the simple pendulum motion, the position of equilibrium being displaced at every instant by the oscillations.

Experiment seems to show, in fact, that the coefficient of rigidity deduced from very small oscillations is greater than that furnished by greater oscillations. To the same cause may be attributed part at least of the differences found between the values of the coefficient found by the method of oscillations and by the statical method of simple torsion; the latter, which are the smallest, have always been deduced from more considerable torsions.

A permanent torsion diminishes the rigidity. In one experiment cited by Sir W. Thomson, a permanent torsion of 10 turns in a copper wire 3·5 metres in length and 0.154 cm. in diameter, caused it to lose the one-twentieth of its rigidity; for 100 turns the diminution exceeded a tenth, and it went on increasing until the wire broke. A permanent elongation produces the same effect.

710. Another curious observation due to Sir W. Thomson is that the rigidity of a wire always diminishes after a long period of oscillations. The *fatigue* of the wire tends to diminish its elasticity of torsion.

This influence of fatigue appears still more markedly in another property to which Sir W. Thomson has given the name of *viscosity*, and which is analogous to molecular friction in liquids. A body cannot change its shape, even if it were absolutely elastic, without an expenditure or dissipation of energy, and to this cause is due the greater part of the gradual damping of the oscillations. Experiment shows that the viscosity increases with the velocity of the oscillations, with the tension and fatigue of the wire; the time necessary for the oscillations to decrease from 2 to 1 is, in fact, less as the initial amplitude is greater; and, all other things being equal, the oscillations are more rapidly extinguished with a wire kept for a long time

* SIR W. THOMSON. *Encyc. Brit.* Article, *Elasticity*, § 81.

† WIEDEMANN. *Wiedemann's Annalen*, Vol. VI., p. 485. 1879.

in vibration, than for an identical wire which has been at rest for the same time.

The viscosity of glass is much greater than that of silver, of iron, or of aluminum; that of zinc is exceptionally large, and only a very small number of oscillations can be counted with this metal.

711. The following table gives the values of the coefficients ϕ and μ for the ordinary metals. The gramme is taken as unit of force, the centimetre as unit of length.

Some other useful coefficients have been added. The *tenacity* of a wire is the greatest weight it can support without breaking; this weight is evidently independent of the length of the wire, and proportional to the section. The *coefficient of tenacity* T, is the tenacity of a wire the section of which is equal to a square centimetre.

When a wire is stretched by a weight, it experiences an elongation λ proportional to its length l, to the stretching weight, and inversely as the section; we may therefore write

$$\lambda = \frac{lP}{ES}.$$

The coefficient E, which is called the *first modulus of elasticity*, or Young's modulus, would represent the weight necessary to double the length of a wire of unit section. The various coefficients μ, ϕ, T, and E have the same dimensions; they represent the quotient of a force by a surface. We shall have their values in C.G.S. units if we multiply the numbers of the table by the value of g, expressed in centimetres—that is to say, by 981.

Nature of the Metal.	Rigidity. Second Modulus of Elasticity. ϕ	Coulomb's Coefficient. μ	Coefficient of Tenacity. T	First Modulus of Elasticity. E	$\frac{\phi}{T}$	$\frac{E}{T}$	$\sqrt{\frac{E}{T}}$
Aluminum	$265 \cdot 2.10^6$	$260 \cdot 1.10^5$	—	673.10^6	—	—	—
Silver	$271 \cdot 8.10^6$	$266 \cdot 5.10^5$	$2 \cdot 96.10^6$	742.10^6	91·8	250	15·8
Gold	$281 \cdot 0.10^6$	$274 \cdot 8.10^5$	$2 \cdot 70.10^6$	813.10^6	104·1	301	17·3
Zinc	$338 \cdot 4.10^6$	$331 \cdot 9.10^5$	$1 \cdot 58.10^6$	767.10^6	214·0	485	22·0
Brass	$350 \cdot 5.10^6$	$343 \cdot 8.10^5$	$3 \cdot 43.10^6$	988.10^6	102·0	288	17·0
Pl.-Sil.*	$369 \cdot 9.10^6$	$362 \cdot 7.10^5$	$3 \cdot 43.10^6$	1050.10^6	102·0	288	—
Copper	$440 \cdot 6.10^6$	$432 \cdot 8.10^5$	$4 \cdot 22.10^6$	1200.10^6	104·4	284	16·8
Maillechort	$493 \cdot 7.10^6$	$483 \cdot 3.10^5$	$4 \cdot 22.10^6$	1300.10^6	104·4	284	—
Platinum	$692 \cdot 7.10^6$	$679 \cdot 5.10^5$	$3 \cdot 50.10^6$	1490.10^6	189·0	329	20·9
Iron	$773 \cdot 1.10^6$	758·9 †	$6 \cdot 40.10^6$	2000.10^6	136	312	17·6

* Alloy of platinum and silver, consisting of 2 silver and 1 of platinum.

† Coulomb's experiments led to the number 744.10^5. *Memoires de l'Academie pour 1784.*

Other things being equal, a wire will be more sensitive the greater the charge it can bear for a given couple—that is to say, the smaller the ratio of the coefficients ϕ and T; this is the case with silver. This metal has also other qualities which have often led to its use in torsion apparatus.

712. Influence of Temperature.—The coefficient of torsion varies greatly with the temperature. According to Kohlrausch's experiments* on iron, copper, and brass, the elasticity of torsion μ at any given temperature, may be represented as a function of its value μ_0 at zero by an expression of the form

$$\mu = \mu_0(1 - at - \beta t^2).$$

For these metals the values of the coefficients a and β are

	a	β
Iron	$4{\cdot}47.10^{-4}$	$5{\cdot}2.10^{-7}$
Copper	$5{\cdot}20.10^{-4}$	$2{\cdot}8.10^{-7}$
Brass	$4{\cdot}28.10^{-4}$	$13{\cdot}4.10^{-7}$

As the rise of temperature produces an expansion also, the coefficient of torsion for the temperature t of a wire of which l and d are the length and diameter at zero, and δ the coefficient of linear expansion, will be given as a function of the coefficient of torsion C_0 for the temperature of zero, by the formula

$$C = C_0(1 - at - \beta t^2)(1 + 3\delta t),$$

or more simply

$$C = C_0[1 - (a - 3\delta)t].$$

For copper we have $3\delta = 0{\cdot}51.10^{-4}$; the variation arising from the change of elasticity is accordingly ten times as great as that due to expansion.

713. Silk Threads.—These threads as they proceed from the cocoon are cylindrical; they present great regularity and are very strong. According to Coulomb's† numbers, a silk thread can support 10 grammes without breaking. Weber‡ found that at the

* Kohlrausch. *Pogg. Ann.* [3], Vol. CXLI., p. 181. 1870.

† Coulomb. *Mem. de l'Academie pour 1777.*

‡ Weber. *Comm. Göttin.*, VIII. *Pogg. Ann.*, Vol. XXXV. and LV. 1834–44.

moment of breaking the elongation of a silk thread is $\frac{1}{7}$, of which two-thirds are permanent elongation when the stretching weight is lessened. The influence of temperature and of the hygrometric state would be about half as much as with hairs.*

Coulomb cites an experiment in which a small copper cylinder an inch long (2·707 cm.), weighing 6 grains (0·3187 gm.), and suspended to a cocoon fibre an inch long, made its oscillation in 40 seconds. From these numbers we get for the torsion coefficient of a single silk fibre 1 cm. in length, in C.G.S. units,

$$C = 0{\cdot}003254.$$

A silver wire, to have the same coefficient of torsion, must have a diameter of 0·00595 mm., and would only support without breaking a weight of 0·818 gr.; while the cocoon thread can carry 10 grammes, or 13 times as much. In order to support the same weight, at least 13 of such wires must be used, or a wire whose section is 13 times as great. In the second case the coefficient would be 169 times that of the cocoon thread.

These numbers show the advantage of using cocoon threads in all suspensions like those of magnets, in which the movable system has of itself a directive force, and where it is desirable to diminish that of the suspension as much as possible. By joining these threads parallel to each other, so that they are all under the same tension, a bundle can be obtained which is as strong as necessary, and the torsion couple of which is almost the sum of the couples for each of the threads. When we seek to diminish the torsion couple, a bundle of parallel threads is always better than a single thread of the same total section. If we could neglect the effect arising from the distance of the points of support, the resulting couple with a single thread would be proportional to the square of the section—that is, to the square n^2 of the number of equivalent threads, and therefore n times greater than the couple constituted by these n threads.

714. BIFILAR BALANCE.—We may, on the other hand, take advantage of this very distance of the wires to produce a torsion couple. This is the principle of the *bifilar balance*, first used by

* *Ann. de Chim. et de Phys.* [2], Vol. XXVI., p. 367. 1824. Report of FRESNEL on Babinet's Hygrometer.

Snow Harris,* but really introduced into practice by the researches of Gauss and Weber.†

In the bifilar balance the movable system is suspended to two adjacent wires, which are usually fixed at the same height above and below. When the apparatus is displaced from its position of equilibrium, the weight of the system gives rise to a couple which tends to restore it to its original direction.

We shall first assume that the two threads are equal, the two points of attachment B and B′ (Fig. 134) at the same height, and

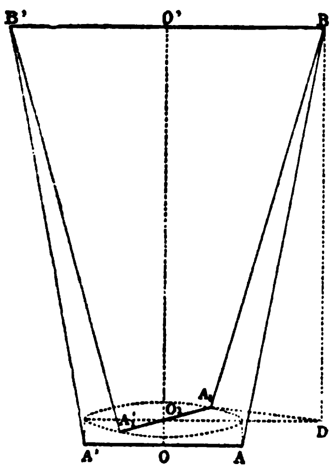


Fig. 134.

that the centre of gravity of the suspended body is in the vertical line which passes through the middle O of the distance of the two points of attachment A and A′ of the lower ends of these wires. In these conditions, equilibrium exists when the four points of attachment are in the same vertical plane, the line AA′ being parallel to the line BB′, and that therefore the two wires are symmetrical in respect of the vertical which passes through the centre of gravity of the system.

Let $AA' = 2a$, $BB' = 2b$, $AB = l$.

If the diameter AA′ be turned through the angle θ about the right line OO′, each of the wires is inclined to the vertical while still

* SNOW HARRIS. *Phil. Trans.*, 1836, p. 417.

† GAUSS et WEBER. *Resultate aus den Beob. des Magn. Vereins.* 1837.

rectilinear; the middle O of the cross-piece AA′ rests at O_1, and the wires take the positions BA_1 and $B'A'_1$. If z is the distance $O'O_1 = BD$, we have

$$(1)\qquad \begin{aligned} z^2 &= l^2 - \overline{A_1D}^2 = l^2 - a^2 - b^2 + 2ab\cos\theta \\ &= l^2 - (b-a)^2 - 4ab\sin^2\frac{\theta}{2}. \end{aligned}$$

From which we have

$$(2)\qquad zdz = -ab\sin\theta\, d\theta.$$

Let Q be the moment of the horizontal couple necessary to keep the system in its new position. The work of this couple, for an infinitely small rotation, is equal and opposite to the work of gravity on the weight O of the system; we have then

$$Qd\theta = -Pdz = \frac{Pab\sin\theta d\theta}{z},$$

or, taking into account equations (1) and (2),

$$(3)\qquad Q = \frac{Pab}{z}\sin\theta = P\frac{ab}{l}\frac{\sin\theta}{\sqrt{1 - \frac{(b-a)^2}{l^2} - \frac{4ab}{l^2}\sin^2\frac{\theta}{2}}}.$$

If the condition $a=b$ is fulfilled, that is to say, if the two wires are parallel in the position of equilibrium, we get

$$(4)\qquad Q = P\frac{ab}{l}\frac{\sin\theta}{\sqrt{1 - \frac{4a^2}{l^2}\sin^2\frac{\theta}{2}}}.$$

The ratio $\frac{a}{l}$ and $\frac{b}{l}$ are usually very small; in this case the denominator only differs from unity by an infinitely small quantity of the second degree, and we may write

$$(5)\qquad Q = \frac{Pab}{l}\sin\theta = C'\sin\theta,$$

putting

$$C' = \frac{Pab}{l}.$$

The quantity C′ may be called the *coefficient of torsion* of the bifilar; it is the moment of the couple which would produce a deflection of 90°. It will be seen that this moment is proportional to the suspended weight, to the product of the distances of the points of support, and inversely as the length of the wires.

The *sensitiveness* of suspension, being inversely as this couple, may be measured by the ratio

$$\frac{l}{Pab}.$$

The torsion cannot exceed 90°, for the couple would then diminish. Snow Harris avoided this restriction by using a long bifilar, the wires of which he connected at different heights by several cross-pieces; the apparatus formed thus a series of bifilars joined end to end. For n identical bifilars, and a total deflection θ, the value of the torsion couple is

$$nC' \sin\frac{\theta}{n}.$$

715. We may also allow for the rigidity of wires which cannot always be neglected. When the system is twisted through the angle θ, each of the wires is twisted to the same extent. If then they were without torsion in the original position, the two torsion couples $C\theta$, which act in the same direction, must be added to the couple $C' \sin\theta$; the equation of equilibrium becomes then

$$Q = C' \sin\theta + 2C\theta = C' \sin\theta \left[1 + 2\frac{C}{C'}\frac{\theta}{\sin\theta}\right].$$

As the angle θ is less than $\frac{\pi}{2}$, if the ratio $\frac{C}{C'}$ is very small we may look upon the terms within the bracket as constant; putting

$$(6) \qquad C_1 = C'\left(1 + 2\frac{C}{C'}\frac{\theta}{\sin\theta}\right),$$

we get

$$Q = C_1 \sin\theta;$$

that is to say, the couple is still virtually proportional to the sine of the angle of elevation.

716. The conditions of symmetry assumed in the preceding calculation can never be absolutely fulfilled, and it is necessary to investigate the consequences of a want of adjustment. Whatever be the length of the wires, and the arrangement of the points of support, there is no position of equilibrium other than that in which the wires are in the same plane containing the centre of gravity of the system. If the system is moved from this position, and kept at an angle θ, under the action of a horizontal couple, the only forces which are acting independent of the external couple and the torsion couple of the wire are the weights of the system and the tensions of the two wires. The vertical components of the tensions counterbalance the weights; the horizontal components form a couple which with the torsion couples are in equilibrium with the external couple; these components are therefore equal and parallel.

It follows from this, that if we project the system on a horizontal plane (Fig. 135) the projections of the two wires A_1D and $A_1'D'$ are

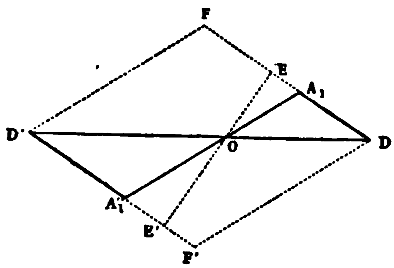


Fig. 135.

always parallel; the projections DD' and A_1A_1' of the lines which join the points of attachment, will cut at O, on the vertical which passes through the centre of gravity of the system, and are divided by this point into parts which are inversely proportional to the vertical components of the tensions.

If p and p' are these vertical components, q the common value of the horizontal components, h and h' the vertical projections of the two wires, λ and λ_1 their horizontal projections, and, finally, $2a$ and $2b$ the projections of the distances of the points of attachment; we have the ratios

$$P = p + p',$$

$$q = p\frac{\lambda}{h} = p'\frac{\lambda'}{h'}.$$

On the other hand, if k is the common perpendicular EE′ to the projection of the two wires, we have also

$$(\lambda+\lambda')k=4ab\sin\theta,$$

for the two numbers of this equation represent the surface of the parallelogram DFD′F′.

The moment Q of the couple of the forces q has the value

$$(7)\qquad Q=qk=\frac{4ab}{\frac{h}{p}+\frac{h'}{p'}}\sin\theta\,;$$

this formula reduces to the expression (5) found above, if we suppose $h=h'$ and $p'=p=\frac{P}{2}$, that is to say when the conditions of symmetry are fulfilled.

If z is the vertical distance of the middle of the two lines which join the points of support, we may put

$$h=z(1-\epsilon),$$
$$h'=z(1+\epsilon).$$

In like manner, remembering that the greatest value of p corresponds to the smallest value of h, if we put

$$p=\frac{P}{2}(1+\delta),$$
$$p'=\frac{P}{2}(1-\delta)\,;$$

it follows that

$$\frac{h}{p}+\frac{h'}{p'}=\frac{4z}{P}\,\frac{1+\epsilon\delta}{1-\delta^2},$$

and therefore

$$(8)\qquad Q=P\frac{ab}{z}\,\frac{1-\delta^2}{1+\epsilon\delta}\sin\theta.$$

As δ is positive, it will be seen that the maximum of Q corresponds to $\delta=0$. The conditions of symmetry are therefore those

for which the apparatus has the least sensitiveness—that is to say, those which give the most rapid oscillations.

The points of attachment are in general at the same height, and on the other hand, the lever AA_1 is by construction perpendicular to the straight line which joins its middle to the centre of gravity of the movable system, and it is only then necessary to make the wires of the same length. All difficulty is got rid of if only two points of support are retained, either above or below, for the two ends of a single wire, which passes over a pulley. When the two wires are independent, use is made of the property of the minimum duration of oscillations, and the adjustment is more easily made the lower is the centre of gravity of the movable system.

When the apparatus has been once adjusted, the same condition serves to bring it into position after it has been displaced; the levelling screws are adjusted until the minimum sensitiveness is obtained.

717. If the wire is rigidly fixed to the points of support so that the first element at each end is kept vertical, it is no longer rectilinear.* In these conditions the system, under the action of a given horizontal couple, undergoes a less displacement than if the wire were quite flexible. For a symmetrical suspension the lower end of a perfectly flexible wire, the top of which was fixed, would be displaced by a quantity $\delta = \frac{q}{p} l$. When the two extreme elements are kept vertical, it is found that the value of the displacement is very nearly

$$\delta_1 = \frac{q}{p}\left(l - 2\sqrt{\frac{\text{EK}}{p}}\right), \tag{9}$$

E being the first modulus of elasticity, and K the moment of inertia of the section of the wire in respect of an axis passing through the centre of gravity, and perpendicular to the plane of flexure.

In the case of a circular wire of radius r, we have $\text{K} = \frac{\pi r^4}{4}$, and therefore

$$\delta_1 = \frac{q}{p}\left[l - r^2\sqrt{\frac{\text{E}\pi}{p}}\right]; \tag{10}$$

* KOHLRAUSCH. *Wiedemann's Annalen*, Vol. XVII., p. 741. 1882.

the correction amounts then to shortening the wire by a quantity independent of its length.

If T′ be the practical tenacity of the metal—that is to say, the load which a wire a square centimetre in cross section can support without inconvenience—we may put

$$p = 2\pi r^2 T' = 2\pi r^2 \alpha T,$$

α being a coefficient less than unity. The correction is then expressed by $r\sqrt{\frac{E}{2\alpha T}}$ or $r\sqrt{\frac{E}{T}}$ assuming $\alpha = \frac{1}{2}$. The value of the root is found in the last column of the table in (711).

Assuming $r = 0{\cdot}005$ and $p = 100$, we find that the correction amounts to 0·28 cm. for iron, to 0·22 cm. for copper, to 0·17 cm. for silver, 0·25 cm. for platinum; it is, then, far from being negligable.

The torsion couple of the bifilar will most frequently be determined by methods of oscillation, and the experimental value will thus at once contain all the corrections. The preceding formulæ will only be useful when it is desired to calculate the absolute value of the coefficient of the bifilar by means of its dimensions.

718. Influence of Temperature.—The bifilar suspension has the advantage over the unifilar of being far less sensitive to changes of temperature; the only effect resulting from rise of temperature is the expansion, which alters the length of the wires, and the distances of the points of support. The sensitiveness being proportional to the first of these quantities, and inversely as the product of the two others, it would be possible by a suitable choice of the nature of the bodies used, to produce a complete compensation, but this precaution is superfluous. If the wires and the cross-pieces are of the same metal, the change of sensitiveness is proportional to the coefficient of expansion, from the numbers cited above (712), and the mean values of the coefficients of expansion (656), this variation is not the thirtieth of that which would be produced in the unifilar by the change of elasticity corresponding to the same change of temperature.

719. Comparison of the Unifilar with the Bifilar.—Let us assume that we suspend a body of p grammes, giving it a definite directive force.

For a bifilar with parallel wires of diameter d, and at a distance of $2a$, we shall have

$$C' = \frac{pa^2}{l} + 2\mu\frac{d^4}{l}.$$

The smallest diameter of the wire is defined as above by the equation

$$p = \frac{\pi d^2 T'}{2},$$

which gives

$$C' = \frac{d^4}{l}\left[\frac{\pi T'}{2}\frac{a^2}{d^2} + 2\mu\right].$$

With a unifilar suspension the wire should have twice the section, that is, a diameter equal to $d\sqrt{2}$; if the length is the same, the moment of torsion is

$$C = 4\mu\frac{d^4}{l}.$$

We deduce from these expressions the ratio of the coefficients

$$\frac{C'}{C} = \frac{\pi T'}{8\mu}\frac{a^2}{d^2} = \frac{1}{2}. \tag{11}$$

This ratio is equal to unity if

$$\frac{a^2}{d^2} = \frac{4}{\pi}\frac{\mu}{T'} = \frac{\phi}{8T'};$$

making $T' = \frac{T}{2}$, we get

$$\frac{a}{d} = \frac{1}{2}\sqrt{\frac{\phi}{T}}.$$

The numbers in (711) show that for most ordinary metals the value of the ratio $\sqrt{\frac{\phi}{T}}$ is near 10; it follows from this that for the same metal the sensitiveness of the bifilar becomes equal to that of the unifilar when the two wires are removed five times their diameter. A very different result would be arrived at with non-metallic threads, such as cocoon fibres.

720. Notwithstanding the advantages of the bifilar as regards changes of temperature, this mode of suspension seems to have been

always considered as less suitable for absolute measurements than the unifilar balance, on account of its more complicated construction.

It is however the only apparatus in which the torsion couple can be directly determined from the dimensions themselves—that is, the length and distance of the wires—without bringing into play the moment of inertia of a body, or determining the times of oscillation. It is true that the distances a and b of the points of support must be known with great accuracy. As the thickness of the wire itself introduces a cause of uncertainty into the measurement of these distances, they should be rather large, which greatly diminishes the sensitiveness. Prof. Kohlrausch has thus constructed a bifilar suspension of exceptional dimensions, with wires two metres in length, and in which the distance of the points of support, which is several centimetres, may be directly measured by the aid of a micrometric division.

721. Various Positions of Equilibrium of a Magnet which is Supported by a System of Wires.—If the suspended body has of itself a directive force, as in the case of a magnet under the action of the Earth, equilibrium may be obtained in several ways.

Gauss says, that the magnet is in its *natural* position when its magnetic axis is in the magnetic meridian, with the pole N turned towards the north, and is in its inverse position when the pole N is turned towards the south.

Let M be the magnetic moment of the bar, and H the horizontal component of the Earth's magnetism. When the magnet is in its natural position, the moment MH is added to the torsion couple of suspension, so that the value of the directive couple of the system is

$$Q = C + MH,$$

and the equilibrium is always stable.

If the magnet is in the inverse position, the directive couple of the system becomes

$$Q = C - MH.$$

In this case the equilibrium of the system is stable or unstable according as

$$C \gtrless MH.$$

With the bifilar the distance of the two wires can always be regulated, so that the condition $C > MH$ is realised. If then the system is made to oscillate alternately in the two positions, the

numbers N and N′ of infinitely small oscillations, or of equal amplitude made during the same time, give the ratio

$$\frac{N^2}{N'^2}=\frac{C+MH}{C-MH},$$

from which we deduce the ratio of the two couples

$$(12)\qquad \frac{C}{MH}=\frac{N^2+N'^2}{N^2+N'^2}.$$

Suppose that while the bar is in its natural position and the wire without torsion, the upper micrometer is turned through the angle ω. To attain the new state of equilibrium, the vertical plane passing through the magnetic axis of the bar turns through an angle θ, and with a unifilar suspension we have

$$MH \sin\theta = C\,(\omega-\theta).$$

A bifilar would give in like manner

$$MH \sin\theta = C' \sin(\omega-\theta).$$

The direction of the magnet is then *oblique* to the magnetic meridian. If the coefficients C and C′ are known, the angles ω and θ would enable us to determine the product HM.

The magnet is *transverse* when its magnetic axis is perpendicular to the meridian. In this case the two previous equations become

$$(13)\qquad \begin{aligned} MH &= C\left(\omega+\frac{\pi}{2}\right).\\ MH &= C' \cos\omega. \end{aligned}$$

If the value of M remains constant, as well as the couple relative to the suspension, the variations of the component H might be determined by the changes of direction of the system in the vicinity of its transverse position.*

* GAUSS. *Resultate aus den Beob. des Mag. Vereins.* 1837. *Œuvres*, p. 137.

CHAPTER IV.

PROPERTIES OF CIRCULAR CURRENTS.

722. Before entering on the investigation of electromagnetic instruments, it will be desirable to examine generally the properties of conducting circuits, and especially of circular coils.

We will consider principally cylindrical bobbins, or coils, which are formed of a metal wire enveloped in an insulating substance, and coiled in a channel of rectangular section. These coils are most easy to construct; they lend themselves best to measurements and to calculation, and are therefore those which are most convenient for use in apparatus for absolute measures.

A cylindrical coil formed of equidistant windings which has an even number of layers, or which has a return wire if the number is odd, is equivalent to a system of equidistant circular currents perpendicular to the axis. The action of such a coil traversed by a current, is equal to the magnetic induction of a uniformly magnetised cylinder of the same form. We shall assume that the conditions of regularity of winding which the similarity presupposes are exactly fulfilled, and that when the coil is constructed of several successive layers, the radii of these layers vary in arithmetical progression.

The winding of a wire is *homogeneous* when the spirals of each layer are equidistant as well as the successive layers, and the winding is *uniform* when the distance of the successive layers is equal to that of the individual windings.

We shall suppose that in each wire the current is concentrated on the axis of the conductor. The results calculated on this hypothesis are sometimes exact, and are always closely approximate as long as the diameter of the wire is very small compared with the radius of curvature of the circuit, and that we do not consider the action at a point too near the wire.

723. The properties of a coil must be deduced from the various data, which are defined by its dimensions and the mode of its construction.

Assuming that the winding is uniform, we shall call

y, the radius of the wire.
$y' = y + z$, the external radius of the wire covered with its insulating coating.
l, the length of the wire measured along its axis.
d, the specific gravity of the metal.
a', the radius of the channel of the coil.
a'', the outer radius of the coil.
$2c$, the thickness of the channel along the radius $2c = a'' - a'$.
$2b$, the length of the channel along the axis.
ω, the section of the channel: $\omega = 2b\,(a'' - a') = 4bc$.
a, the distance from the axis to the centre of gravity of the section of the channel.
U, the volume of the channel: $U = 2\pi\,(a''^2 - a'^2)\,b = 2\pi a\omega$.
V, the volume of the metal.
V′, the total volume of the wire together with the insulator.
n, the total number of windings.
n_1, the number of windings for unit length along the axis of the coil.
n_2, the number of windings for unit length along the radius of the coil.

These data will enable us to determine the other elements, such as the length, the volume and the weight of the wire, the volume of the insulator, and lastly the electrical properties of the coil.

Neglecting for a moment the effects of induction, we shall have to calculate:—

1st. The total resistance R of the wire.

2nd. The total surface—that is to say, the sum of the surfaces bounded by the different windings, or the magnetic moment of the coil for unit current.

3rd. The electromagnetic action G of the coil, or the intensity of the magnetic field for unit current.

724. Length of Wire. Mean Radius.—The thread of each winding is equal to $\frac{1}{n_1}$; the length of a winding whose projection on a plane perpendicular to the axis is a circumference of radius r, is

$$2\pi r\sqrt{1 + \frac{1}{4\pi^2 n_1^2 r^2}},$$

or, if the thread is very small compared with the diameter of the coil, very nearly

$$2\pi r\left(1+\frac{1}{8\pi^2 n_1^2 r^2}\right).$$

As the factor within the bracket differs from unity by a quantity which may be neglected, the winding may be confounded with the projection of its circumference.

The *mean circumference* of the windings of a coil, is that whose length is the mean of the circumferences of all the windings; the *mean radius* is the mean of all the radii. This radius is evidently equal to the distance a of the centre of gravity of the section of the channel to the axis of the coil, and therefore is independent of the diameter of the wire. We get from this, for the total length of the wire,

$$(1) \qquad l = 2\pi a n;$$

or, allowing for the correction due to the obliquity of the windings,

$$(1') \qquad l = 2\pi a n\left(1+\frac{1}{8\pi^2 n_1^2 a^2}\right).$$

We have, moreover,

$$n = n_1 n_2 \omega = 2 n_1 n_2 b\,(a'' - a') = 4 n_1 n_2 bc.$$

If the coiling is uniform,

$$n_1 = n_2, \qquad \text{and} \qquad n = n_1^2 \omega = 4 n_1^2 bc;$$

the expression (1) for the length of the wire becomes thus

$$(2) \qquad l = 2\pi a \omega n_1^2 = n_1^2 \mathrm{U} = 2\pi n_1^2 b\,(a''^2 - a'^2) = 8\pi n_1^2 abc.$$

725. Volume and Weight of the Wire.—Most frequently we assume the coiling uniform; we may then consider each section of the wire as occupying, in the mean section of the channel, the centre of a square, the side of which is equal to $\frac{1}{n_1}$.

If we represent by δ the ratio $\frac{z}{y}$ of the thickness of the envelope to the radius of the wire, we see that the section of the bare wire

occupies a fraction $n_1^2\pi y^2$ of the surface of the square, and the total section, including the envelope, a fraction $n_1^2\pi y_1^2$ or $n_1^2\pi y^2(1+\delta^2)$ of the same surface. Each of these fractions is a maximum when the wires are in contact; the side of the square is then equal to the diameter of the wire, and the value of the last fraction is $\frac{\pi}{4}$.

The ratio of the total volume of the wire to that of the channel is equal to that of the sections; when the wires are in contact we have

$$\frac{V'}{U}=\frac{V(1+\delta)^2}{U}=\frac{\pi}{4},$$

which gives for the weight of the wire

$$P=Vd=\frac{\pi}{4}\frac{d}{(1+\delta)^2}U.$$

It follows from this that if δ is constant—that is to say, if the thickness of the insulator is a constant fraction of the radius of the wire, the weight of the metal for a given volume of the channel is independent of the diameter of the wire; this weight is moreover proportional to the volume of the channel.

In the case of a uniform coiling we have

$$(4) \qquad l=n_1^2U;$$

and, when the wires are in contact,

$$l=\frac{U}{4y'^2}.$$

The volume of the channel U being given, the total length of the wire varies then inversely as the square of the diameter of the insulated wire; for a wire of a given diameter the length varies as the volume of the channel.

726. RESISTANCE.—The resistance of the coil is

$$(5) \qquad R=\frac{\rho l}{\pi y^2}.$$

When the wires touch, considering that $2n_1y' = 1$ and $l = n_1^2 U$,

$$\text{(6)} \qquad R = \frac{4}{\pi}(1+\delta)^2 \rho n_1^4 U = \frac{\rho U}{4\pi(1+\delta)^2}\frac{1}{y^4}.$$

From which it follows that if δ is constant, and if the volume of the channel, and therefore also the weight of metal, is given, the resistance varies inversely as the fourth power of the diameter of the wire. For a given wire the resistance varies as the volume of the channel.

This latter conclusion is manifest. We get the first result directly by observing that, if we make the diameter of the wire one-half, its section is one-fourth, and for the same volume of the channel the length is four times as great; the resistance is then sixteen times as great.

727. Surface.—Magnetic Moment.—The magnetic moment of a cylindrical coil (495) for unit current, is equal to the total surface comprised by the different windings. If the coil consists of a single layer comprising n turns of radius a, we shall have

$$S = n\pi a^2.$$

If the coil consists of several layers, the surfaces corresponding to each are added together; this calculation amounts to determining the sum of the squares of a series of terms which vary in arithmetical progression.

If r_0 is the smallest radius, the value of any given radius is

$$r = r_0 + \alpha,$$

which gives

$$r^2 = r_0^2 + 2r_0\alpha + \alpha^2.$$

Let p be the number of layers, the sum of the squares of the radii is

$$\Sigma r^2 = pr_0^2 + 2r_0\Sigma\alpha + \Sigma x^2.$$

The radius a_1 of the mean circle is given then by the equation

$$\text{(7)} \qquad a_1^2 = r_0^2 + 2r_0\frac{\Sigma\alpha}{p} + \frac{\Sigma\alpha^2}{p}.$$

On the other hand, the value of the mean radius a is

$$a = \frac{1}{p}\Sigma r = r_0 + \frac{\Sigma\alpha}{p},$$

which gives

$$a^2 = r_0^2 + 2r_0\frac{\Sigma\alpha}{p} + \left(\frac{\Sigma\alpha}{p}\right)^2.$$

Substituting in equation (7), we get

$$(8) \qquad a_1^2 = a^2 + \frac{\Sigma\alpha^2}{p} - \left(\frac{\Sigma\alpha}{p}\right)^2.$$

Calling h the distance $\frac{1}{n_2}$ of two successive windings, we shall give to α the successive values $o, h, 2h \ldots\ldots (p-1)h$; it follows that

$$\Sigma\alpha = h\frac{(p-1)p}{2},$$

$$\Sigma\alpha^2 = h^2\frac{(p-1)p(2p-1)}{2.3},$$

and therefore

$$\left(\frac{\Sigma\alpha}{p}\right)^2 = \frac{p^2h^2}{4}\left(1 - \frac{1}{p}\right)^2;$$

$$\frac{\Sigma\alpha^2}{p} = \frac{p^2h^2}{3}\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{2p}\right).$$

As the thickness $2c$ of the channel is equal to ph, we get from this

$$\frac{\Sigma\alpha^2}{p} - \left(\frac{\Sigma\alpha}{p}\right)^2 = \frac{c^2}{3}\left(1 - \frac{1}{p^2}\right) = \frac{c^2}{3}\left(1 - \frac{1}{4n_1^2c^2}\right),$$

$$(9) \qquad a_1^2 = a^2 + \frac{c^2}{3}\left[1 - \frac{1}{4n_1^2c^2}\right].$$

The radius a_1 of the *mean circle* differs thus appreciably from the radius a of the mean circumference.

If the windings are so close that the last term of correction may be neglected, the radius a only depends on the dimensions of the channel.

In this case, for a given channel the surface of the coil is proportional to the number of windings, and therefore inversely as the square of the diameter of the wire.

For a given wire, and similar volumes of the channel, the surface is on the one hand proportional to the number of windings—that is to say, to the section of the channel—and on the other to the surface of the mean section; it varies therefore as the fourth power of the ratio of the homologous dimensions.

728. ELECTROMAGNETIC ACTION.—The law of the distribution of force in the field of a coil only depends on the shape of the coil, and the strength of the field at each point is proportional to the strength of the current. We need then only calculate the action for unit current.

Let us first suppose that the channel is filled by a homogenous mass of metal, and the current uniformly distributed in its meridian section; the intensity for unit surface may be called *the density of the current.*

Let us now suppose the meridian section divided into equal squares by a series of lines, one set parallel and the other perpendicular to the axis, there being n_1^2 squares for each unit of surface. Let us also suppose the mass of metal divided into a series of concentric rings insulated from each other, and corresponding to the squares of the meridian section; this operation will produce no change either in the distribution of the current or in its external action. If we suppose that each of these rings is traversed by unit current, the density of the current is n_1^2, and the electromagnetic action is proportional to n_1^2—that is to say, it varies inversely as the dimensions of each elementary square.

If we replace each ring by a cylindrical wire which occupies the central part of the corresponding square, we form a coil uniformly wound. If each wire is traversed by the quantity of electricity which previously traversed the square, the density of the current is no longer uniform; but its mean value remains the same, whatever be the diameter of the wire.

On the other hand, if the current is uniformly distributed in the section of the wire, its external action is appreciably the same, whether we suppose the wire reduced to its axis, or whether its diameter is that of the side of the square.

The difference between the actions exerted by the coil, when we replace the cylindrical wire by a wire with a square section, may be neglected for any point which is at a great distance compared with the diameter of the wire (722).

It follows from this that for a given channel, and therefore for a given weight of metal, if the ratio is constant, the intensity of the field at each point is inversely as the square of the diameter of the wire; or again (726), is proportional to the square root of the resistance.

This consequence suggests a curious remark: if R is the resistance of the coil, and I the strength of the current, the thermal energy W developed in each unit of time is equal to RI^2. As the resistance R is proportional to n_1^2, or to the square G^2 of the magnetic action for unit current, we see that the thermal energy W is proportional to the product G^2I^2—that is to say, to the square of the magnetic action of the coil for the current I. It follows that if, varying the diameter of the wire, we modify the intensity so that the magnetic action remains constant, the thermal action will also be constant.*

729. Action on the Axis.—The potential of a current is the same as that of a uniform magnetic shell having the same contour, and the strength of which is equal to the strength of the current. The potential of a circular current of radius r, and strength equal to unity, at a point P on the axis, at a distance x from the centre, is expressed by

$$V = 2\pi\left(1 - \frac{x}{u}\right),$$

putting

$$u^2 = x^2 + r^2.$$

The magnetic force is

$$F = -\frac{\partial V}{\partial x} = 2\pi\frac{u^2 - x^2}{u^3} = 2\pi\frac{r^2}{u^3}.$$

Dividing this expression by the length $2\pi r$ of the circumference, we get the action of the current for unit length,

$$(10) \qquad f = \frac{F}{2\pi r} = \frac{r}{u^3},$$

which we may call *the specific action* of the current.

The action of a coil of length $2b$ and radius r, covered by a layer of wires containing n_1 turns for unit length, is equal for unit current

* Marcel Deprez. *Comptes rendus de l'Acad.*, Vol. xciv., p. 431. 1882.

to the magnetic induction of a uniformly magnetised cylinder (495), the intensity of magnetisation of which is n_1.

If α and β are the angles under which we see the extreme windings of the coil, from the point P, and if x is the distance from the point P to the median plane (373), the value of the action G of the bobbin at the point P is

$$(11)\quad G = 2\pi n_1(\cos\beta - \cos\alpha) = 2\pi n_1\left[\frac{x+b}{\sqrt{r^2+(x+b)^2}} - \frac{x-b}{\sqrt{r^2+(x-b)^2}}\right].$$

Suppose now that the coil is made up of several layers comprised within a channel of length $2b$ and thickness $2c = a'' - a'$, the coiling being uniform.

A layer of thickness dr contains a number $n_1^2 dr$ of turns for unit length. Replacing n_1 by this value in the preceding expression, and integrating between the limits a' and a'', we shall have for the action of the coil

$$G = 2\pi n_1^2 \int_{a'}^{a''}\left[\frac{x+b}{\sqrt{r^2+(x+b)^2}} - \frac{x-b}{\sqrt{r^2+(x-b)^2}}\right]dr,$$

which gives

$$(12)\qquad G = 2\pi n_1^2\left\{\begin{aligned} &(x+b)\,l.\frac{a''+\sqrt{a''^2+(x+b)^2}}{a'+\sqrt{a'^2+(x+b)^2}} \\ &-(x-b)\,l.\frac{a''+\sqrt{a''^2+(x-b)^2}}{a'+\sqrt{a'^2+(x-b)^2}}\end{aligned}\right\}.$$

At the centre of the bobbin, where $x = 0$, the action reduces to

$$(13)\quad G_0 = 4\pi n_1^2 b\,l.\frac{a''+\sqrt{a''^2+b^2}}{a'+\sqrt{a'^2+b^2}} = \frac{n\pi}{c}\,l.\frac{a''+\sqrt{a''^2+b^2}}{a'+\sqrt{a'^2+b^2}}.$$

We may write

$$G_0 = n\frac{2\pi}{a_2},$$

putting

$$(14)\qquad \frac{1}{a_2} = \frac{1}{2c}\,l.\frac{a''+\sqrt{a''^2+b^2}}{a'+\sqrt{a'^2+b^2}}.$$

The quantity a_2 represents the *radius of the winding of mean action* in respect of the centre.

Neglecting fourth powers of the ratios $\frac{b}{a}$ and $\frac{c}{a}$, we have simply

$$a_2 = a\left[1+\frac{1}{a^2}\left(\frac{b^2}{2}-\frac{c^2}{3}\right)\right].$$

730. THE MOST SUITABLE DIMENSIONS FOR A RECTANGULAR CHANNEL.*—The length of the wire being given, as well as the radius a' of the core, we may investigate the dimensions of the rectangular section which gives the maximum action at the centre of the coil for a uniform coiling; the problem then is to choose a'' and b so that G_0 is a maximum, these two quantities satisfying equation (2).

Instead of solving this problem by the ordinary method of maxima, it may be arrived at more simply by the following considerations.

It is clear that if the maximum is obtained, the mean specific action of the outer layer is the same as the mean specific action of the lateral layer, and there is no advantage in transferring the turns from one region to the other.

But from equation (11) the value of the total action of the layer of radius a'' at the centre is

$$2\pi n_1 \frac{2b}{\sqrt{a''^2+b^2}},$$

the length of the wire which forms it being $2\pi a'' \times 2n_1 b = 4\pi a'' n_1 b$, the specific action is equal to

$$\frac{1}{a''\sqrt{a''^2+b^2}}.$$

A cylinder of the lateral layer comprised within the radii r, and $r+dr$, contains a number $n_1 dr$ of windings; the action of this layer at the centre is accordingly

$$\int_{a'}^{a''} 2\pi\frac{r^2}{u^3}n_1 dr = 2\pi n_1 \int_{a'}^{a''}\frac{r^2 dr}{(b^2+r^2)^{\frac{3}{2}}}.$$

* W. WEBER. *Galvanometrie. Abh. der K. Gesellsch der Wiss. zu Göttingen.* Vol. x. 1861–62.

As the length of the corresponding wire is

$$\int_{a'}^{a''} 2\pi r \,.\, n_1 dr = \pi n_1 (a''^2 - a'^2),$$

the specific action of this layer is

$$\frac{2}{a''^2 - a'^2}\int_{a'}^{a''} \frac{r^2 dr}{(b^2+r^2)^{\frac{3}{2}}} = \frac{2}{a''^2 - a'^2}\left\{\begin{array}{l} \dfrac{a'}{\sqrt{b^2+a'^2}} - \dfrac{a''}{\sqrt{b^2+a''^2}} \\ + l \,.\, \dfrac{a''+\sqrt{b^2+a''^2}}{a'+\sqrt{b^2+a'^2}} \end{array}\right\}.$$

Equating the two specific actions, we get, as a condition of the maximum,

$$(15) \qquad l \,.\, \frac{a''+\sqrt{a''^2+b^2}}{a'+\sqrt{a'^2+b^2}} = \frac{3a''^2 - a'^2}{2a''\sqrt{a''^2+b^2}} - \frac{a'}{\sqrt{a'^2+b^2}}.$$

If we add to this equation (2), which gives the value of l, we might determine the dimensions a'' and b of the coil.

731. Coil of Maximum Action for a given Source.—Without altering the dimensions of the channel, we may further investigate what should be the length and diameter of the wire, in order that with a given source the intensity of the magnetic field is a maximum at the centre of the coil. It is necessary to introduce here the elements of the source, which is defined by the electromotive force, and its own resistance R′ including the resistance of the connecting wires, without which the magnetic action would increase indefinitely with the number of turns, or, in other words, with the density of the current.

The intensity of the current is expressed by

$$I = \frac{E}{R+R'},$$

and the action at the centre of the coil is

$$(16) \qquad F_0 = G_0 I = G_0 \frac{E}{R+R'}.$$

The condition of maximum of this expression is easily obtained in the case of a uniform winding: if we suppose that the wire is bare,

or that the thickness of the insulator is proportional to the radius of the wire—that is to say, that δ is constant, and all the windings are in contact. For if we replace the original wire of the coil by another wire of m times smaller diameter, so as still to fill up the volume of the channel, the values of R and of G_0 become

$$R_1 = m^4 R, \qquad G_1 = m^2 G_0,$$

which gives, for the action at the centre,

$$(17) \qquad F_1 = \frac{EG_1}{R_1 + R'} = \frac{Em^2 G_0}{m^4 R + R'} = \frac{EG_0}{m^2 R + \dfrac{R'}{m^2}}.$$

The factor m being here the only variable, the condition of maximum is

$$m^2 R = \frac{R'}{m^2},$$

whence

$$m^4 R = R' = R_1.$$

Hence *the resistance of the coil must be equal to the resistance of the external circuit.*

This condition is independent of the form of the coil. It would still be true if the diameter of the wire, instead of being uniform, varied from one winding to another, provided that the ratio of the diameter of the two coils compared are the same at all points.

If the thickness z of the insulator is constant instead of being proportional to the diameter of the wire, we must take the general expression

$$R = \frac{\rho}{\pi} \frac{l}{y^2};$$

as the windings are in contact, and the coiling is supposed to be uniform, we have

$$(19) \qquad U = 4\,(y + z)^2\, l.$$

If g is a constant, we may put

$$(20) \qquad G_0 = gl,$$

for it is clear that in all cases the total magnetic action for a given

volume of the channel is proportional to the length of the wire. From this it follows that

$$F_0 = E\frac{gl}{\frac{\rho}{\pi}\frac{l}{y^2}+R'} = E\frac{g}{\frac{\rho}{\pi y^2}+\frac{R'}{l}}.$$

The problem is then to make the expression $\frac{\rho}{\pi y^2}+\frac{R'}{l}$ a minimum, which is given by the condition

$$2\frac{\rho}{\pi}\frac{dy}{y^3}+\frac{R'dl}{l^2}=0, \qquad \text{or} \qquad 2R\frac{dy}{y}+R'\frac{dl}{l}=0.$$

We have further, by equation (19), U being a constant,

$$\frac{2dy}{y+z}+\frac{dl}{l}=0;$$

and therefore

(21) $$\frac{R}{R'}=\frac{y}{y+z},$$

that is to say, that *the resistance of the coil should be to the external resistance as the diameter of the bare wire is to the diameter of the covered wire.**

This result is independent of the shape of the channel.

732. The total resistance of the wire being determined by one of the preceding conditions, we might calculate the length and diameter of the wire with which the coil must be covered.

In the first case we have

$$R=\frac{\rho l}{\pi y^2}=R',$$

$$U=4y^2(1+\delta)^2 l;$$

we deduce from this

$$y^4=\frac{U\rho}{4\pi R'(1+\delta)^2},$$

$$l=\frac{\pi R'}{\rho}y^2.$$

* SCHWENDLER. *Phil. Mag.*, Vol. XXIX. 1867.

In the second case the equations

$$R = \frac{\rho l}{\pi y^2} = \frac{y}{y+z} R',$$

$$U = 4(y+z)^2 l$$

give

$$y^4 \left(1 + \frac{z}{y}\right) = \frac{U\rho}{4\pi R'},$$

(22) $$l = \frac{4\pi^2 R'^2}{U\rho^2} y^6,$$

$$R = \frac{4\pi R'^2}{U\rho} y^4.$$

The former equation may be written

$$y = \sqrt[4]{\frac{U\rho}{4\pi R'}} \cdot \frac{1}{\sqrt[4]{1 + \frac{z}{y}}} = \frac{y_0}{\sqrt[4]{1 + \frac{z}{y}}}.$$

The approximate value of y_0 represents the radius of the wire, if the thickness z of the insulator were equal to zero. This equation would be solved by successive approximation, by giving to y under the root an arbitrary value, y_0 for instance.

733. Best Form of the Channel.—We have hitherto supposed that the section of the channel is rectangular, but it is readily seen that this shape is not the most advantageous. As the action of a winding, other things being equal, is inversely as its radius, it is manifestly advantageous to multiply the number of those which correspond to the smallest radii—that is, those whose effect predominates.

We are not able to diminish indefinitely the radius of the first windings; in the first place there must be a core to support the wires, and in coils which are to be used in galvanometers we must moreover arrange about the axis the place necessary for the movement of the magnet.

Reasoning similar to that given above (**730**) shows that the most advantageous form for the contour of the channel is that in which all the windings on the surface have the same specific action on the magnet, for otherwise, it would be better to displace some of the

wires and bring them to another point. We shall assume that the magnet is infinitely small, and placed at the centre of the coil.

That being admitted, let us consider the meridional section; let P (Fig. 136) be a point of the bounding curves, r the radius of the winding, and a the distance of its plane from the surface. As the specific action of the winding at the point O should be the same,

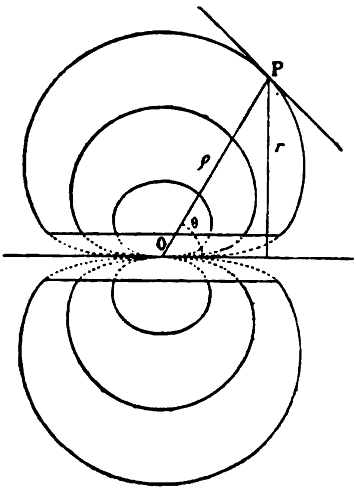


Fig. 136.

whatever be the position of the point P on the curve, this will be defined by equation

$$\frac{r}{(r^2+x^2)^{\frac{3}{2}}} = \text{const} = \frac{1}{c^2}.$$

If u is the radius vector OP of the point P, and θ the angle of this radius with the axis, the equation becomes

$$u^2 = c^2 \sin\theta. \quad (23)$$

The curve represented by equation (23) has the form of a circle flattened along the vertical diameter. The tangent is vertical at the point of intersection with the straight line, the angular coefficient of which is equal to $\frac{\sqrt{2}}{2}$. Figure 136 gives the successive curves for

values of c, which increase in arithmetical progression; the dotted parts of the curves correspond to the central space arranged for the magnet.

734. The parameter c will in general be determined by the resistance, and the dimensions which we give to the coil.

For a uniform winding with the wires in contact, we have for the relation between resistance and volume,

$$R = \frac{\rho U}{4\pi y^4 \left(1 + \frac{z}{y}\right)^2},$$

which determines y.

The expression for the volume U′ comprised within the surface produced by the revolution of the curve of the parameter c, is

$$U' = 2\pi \int_0^\pi \sin\theta d\theta \int_0^u u^2 du = \frac{2}{3}\pi \int_0^\pi u^3 \sin\theta d\theta = \frac{2}{3}\pi c^3 \int_0^\pi \sin^{\frac{5}{2}} \theta d\theta.$$

The integral $2\pi \int_0^\pi \sin^{\frac{5}{2}} \theta d\theta$ is a number; if we represent it by N, we shall have

$$U' = \frac{1}{3} N c^3.$$

The cavity inside the coil comprising the core is usually cylindrical, and the proportion which it deducts from the preceding volume may be expressed by $\frac{1}{3} N a^3$, a representing the parameter of the curve which would bound a volume equal to that of the cavity. The volume U of the channel is accordingly

$$U = \frac{N}{3}(c^3 - a^2), \tag{24}$$

an equation which would give c as a function of U.

Considering u as the radius vector of the curve which would correspond to the element dl of the current, and the parameter c as a variable, the expression for the action of the coil at the centre for unit current is

$$G_0 = \int \frac{dl}{u^2} \sin\theta = \int \frac{dl}{c^2}.$$

The parameter is thus a function of l. If, in like manner, we consider U as a function of c, or of l, equation (24) gives

$$dU = Nc^2dc = 4(y+z)^2dl;$$

we have, moreover,

$$dR = \frac{\rho dl}{\pi y^2}.$$

We get from these ratios

$$(25)\qquad \begin{aligned} dG_0 &= \frac{N}{4(y+z)^2}dc = n_1^2Ndc, \\ dl &= \frac{Nc^2}{4(y+z)^2}dc = n_1^2Nc^2dc, \\ dR &= \frac{\rho}{\pi}\frac{Nc^2dc}{4(y+z)^2y^2} = \frac{\rho n_1^2Nc^2}{\pi y^2}dc. \end{aligned}$$

These values of dG_0, dl, and dR are those which correspond to the layer bounded by two meridional curves, the parameters of which have the values c and $c+dc$. The quotient

$$\frac{dG_0}{dl} = \frac{1}{c^2}$$

represents the specific action of the layer; this action is inversely as the square of c; that is to say, as the square of the distance of the layer from the centre.

Taking into account the internal cavity, the quantities G_0, l, and R expressed as a function of c, have the values

$$(26)\qquad \begin{aligned} G_0 &= n_1^2N(c-a), \\ l &= n_1^2N\frac{c^3-a^3}{3}, \\ R &= \frac{\rho n_1^2N}{\pi y^2}\frac{c^3-a^3}{3}. \end{aligned}$$

735. Wire of Variable Diameter.—Instead of using a wire whose diameter was constant for the entire coil, it would be more advantageous, while retaining the same channel, and the same total resistance, to lessen the diameter of the wire in the first layers, and then increase it progressively in the succeeding wires; in this way

we should increase the number of windings of the first layers, which are especially efficacious, and diminish the number of the more distant ones which are less so.

Supposing that we fix the values R, R′, and U, we may propose to ourselves to determine the diameter $y+z$ of the wire for each of the similar layers of thickness dc so as to make the expression $F_0 = E\frac{G_0}{R+R'}$ a maximum.

$$F_0 = E\frac{G_0}{R+R'}.$$

Let us consider one of these layers corresponding to the two values c and $c+dc$ of the parameter. If we vary the diameter of the wire the constant of the coil will vary from dG_0, the resistance from dR, and the magnetic action at the centre will experience a variation

$$\delta F_0 = E\left(\frac{G_0+dG_0}{R+dR+R'} - \frac{G_0}{R+R'}\right),$$

from which, disregarding infinitely small quantities of the second order,

$$\delta F = E\frac{G_0}{R+R'}\left[\frac{dG_0}{G_0} - \frac{dR}{R+R'}\right].$$

The condition of the maximum of δF is evidently

$$\frac{\frac{d}{dy}\cdot dG_0}{\frac{d}{dy}\cdot dR} = \frac{G_0}{R+R'} = \text{const.}$$

If we refer to expressions (25), we find

$$\frac{d}{dy}dG_0 = -\frac{Ndc}{2}\frac{1}{(y+z)^3},$$

$$\frac{d}{dy}dR = -\frac{\rho}{\pi}\frac{Nc^2dc}{2}\frac{2y+z}{y^3(y+z)^3};$$

from this follows, for the desired condition,

$$c^2 \frac{2y+z}{y^3} = \text{const.} \tag{27}$$

If the thickness of the insulator is in a constant ratio with the diameter of the wire, the equation reduces to

$$\frac{c}{y} = \text{const};$$

that is to say that the diameter of the wire in each layer should be proportional to the linear dimensions of this layer. If the thickness z is constant, equation (27) shows that the diameter of the wire increases a little less rapidly than the parameter c.

In the first hypothesis, if we put

$$y = ac,$$

we shall have $y + z = a(1+\delta)c$; the expressions for dG_0 and dR (25) will acquire the form

$$dG_0 = \frac{N}{4a^2(1+\delta)^2} \cdot \frac{dc}{c^2}.$$

$$dR = \frac{\rho}{\pi} \cdot \frac{N}{4a^4(1+\delta)^2} \cdot \frac{dc}{c^2}.$$

We may allow for the central cavity if we assume that it is bounded by a curve of parameter a; we get then

$$G_0 = \frac{N}{4a^2(1+\delta)^2} \frac{1}{a}\left(1 - \frac{a}{c}\right).$$

$$R = \frac{\rho N}{4\pi a^4(1+\delta)^2} \frac{1}{a}\left(1 - \frac{a}{c}\right).$$

We see that the influence of the external layers diminishes very rapidly when the thickness of the coil increases, and that the maximum action is virtually inversely as the parameter of the curve which bounds the cavity.

By the same reasoning as above (731), we find that the maximum action corresponds to the case in which the resistance of the coil is equal to that of the external resistance.*

736. Action of a Circular Current Outside the Axis.—The potential of a circular magnetic layer of unit density (366) at a point at a distance y from the axis, may be expressed by a series developed in even powers of y, of the form

$$(28)\qquad P = 2\pi(f_0 + f_1 y^2 + f_2 y^4 + \ldots),$$

in which the coefficients $f_0, f_1, f_2, \ldots$ are functions of the distance x of the point in question from the plane of the current.

If a be the radius of the circle, and we put

$$u^2 = a^2 + x^2,$$

the series is always converging when $y < u$.

If $f_0', f_1', f_2', \ldots\ldots f_0'', f_1'' \ldots$ are the successive differentials of the coefficients in respect of x, the potential V at the same point of a uniform shell of the same contour whose magnetic power was equal to unity (368), or the potential of unit current along the circular contour, is then

$$V = -\frac{\delta P}{\delta x} = -2\pi(f_0' + f_1' y^2 + f_2' y^4 + \ldots).$$

The expressions for the components of the magnetic action of the current are

$$(29)\qquad \begin{aligned} X &= -\frac{\delta V}{\delta x} = 2\pi(f_0'' + f_1'' y^2 + f_2'' y^4 + \ldots), \\ Y &= -\frac{\delta V}{\delta y} = 4\pi y(f_1' + 2f_2' y^2 + 3f_3' y^4 + \ldots). \end{aligned}$$

The values of the coefficients are

$$(30)\qquad \begin{aligned} f_0 &= u - x, & f_0' &= \frac{du}{dx} - 1, \\ f_1 &= -\frac{1}{2^2}\cdot\frac{d^2u}{dx^2}, & f_1' &= -\frac{1}{2^2}\cdot\frac{d^3u}{dx^3}, \\ f_2 &= \frac{1}{(2.4)^2}\cdot\frac{d^4u}{dx^4}, & f_2' &= \frac{1}{(2.4)^2}\cdot\frac{d^5u}{dx^5}, \\ f_3 &= -\frac{1}{(2.4.6)^2}\cdot\frac{d^6u}{dx^6}, & f_3' &= -\frac{1}{(2.4.6)^2}\cdot\frac{d^7u}{dx^7}. \\ &\ldots\ldots\ldots\ldots & &\ldots\ldots\ldots\ldots; \end{aligned}$$

* Ayrton and Perry. *Journal of the Society of Telegraph Engineers*, Vol. VII., p. 297. 1878.

The successive derivatives satisfy the equations

$$
(31)\qquad
\begin{array}{lll}
f_0'' = -2^2 f_1, & f_0''' = -2^2 f_1', & \cdots\cdots \\
f_1'' = -4^2 f_2, & f_1''' = -4^2 f_2', & \cdots\cdots \\
f_2'' = -6^2 f_3, & f_2''' = -6^2 f_3', & \cdots\cdots \\
\cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots
\end{array}
$$

The coefficients to be used, when the powers of y are higher than the fourth, will be expressed by the aid of the following differentials:

$$
(32)\qquad
\begin{aligned}
\frac{du}{dx} &= \frac{x}{u}, \\
\frac{d^2u}{dx^2} &= \frac{a^2}{u^3}, \\
\frac{d^3u}{dx^3} &= -\frac{3a^2x}{u^5}, \\
\frac{d^4u}{dx^4} &= \frac{3a^2(4x^2-a^2)}{u^7}, \\
\frac{d^5u}{dx^5} &= \frac{3.5a^2x(3a^2-4x^2)}{u^9}, \\
\frac{d^6u}{dx^6} &= \frac{3^2.5a^2(a^4-12a^2x^2+8x^4)}{u^{11}}, \\
\frac{d^7u}{dx^7} &= \frac{3^2.5a^2x(-35a^4+140a^2x^2-56x^4)}{u^{13}}.
\end{aligned}
$$

If we limit the expansion to the fourth powers of the ordinate y, the components of the force are

$$
(33)\qquad
\begin{aligned}
X &= 2\pi\frac{a^3}{u^3}\left[1 - \frac{3}{2^2}\frac{4x^2-a^2}{u^2}\frac{y^2}{u^2} + \frac{3^2.5}{(2.4)^2}\frac{a^4-12a^2x^2+8x^4}{u^4}\frac{y^4}{u^4}\right]. \\
Y &= 3\pi\frac{a^2xy}{u^5}\left[1 + \frac{5}{2.4}\frac{(3a^2-4x^2)}{u^2}\frac{y^2}{u^2}\right].
\end{aligned}
$$

737. When the ratio $\frac{y}{u}$ is very small, the value of the component Y is an infinitely small quantity of the first order in respect of X.

If in the expression for the component X, we multiply the terms with y^2 by a^2, and the terms in y^4 by a^4, the factors of $\frac{y^2}{a^2}$ and of $\frac{y^4}{a^4}$ expressed as a function of $\frac{x}{a}$, are numbers whose variations are represented respectively by the curves A and B of Fig. 137; the curve A represents the variation of the factor of $\frac{y^2}{a^2}$, the curve B those of the factor of $\frac{y^4}{a^4}$. The second correction may in most cases be neglected.

The first term of correction is positive for $x=0$, which shows that the magnetic action at the centre of the circle is a minimum in

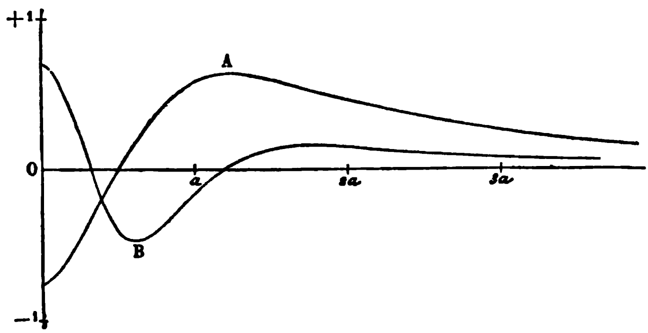


Fig. 137.

respect of points on the plane; we know, on the other hand, that it is a maximum relative to the axis. The correction vanishes for $x=\frac{a}{2}$ and is then always negative; the maximum takes place for $x=1{\cdot}22a$, and its value is then $0{\cdot}8$ of that which corresponds to zero.

The second term of correction is likewise positive for $x=0$; it disappears for $x=0{\cdot}3a$, and $x=1{\cdot}187a$ becomes negative between these two points, and remains positive for all other values of x. It attains its maximum for $x=0{\cdot}856a$, that is to say, almost at the point at which the first term disappears.

738. Disregarding terms above the second order, we find for the components X and Y, and their partial differentials in respect of x,

$$X = 2\pi(f_0'' + f_1''y^2) = 2\pi\frac{a^2}{u^3}\left[1 - \frac{3}{4}\frac{4x^2 - a^2}{u^2}\frac{y^2}{u^2}\right],$$

$$\frac{\partial X}{\partial x} = 2\pi(f_0''' + f_1'''y^2) = -6\pi\frac{a^2x}{u^5}\left[1 + \frac{5}{4}\frac{3a^2 - 4x^2}{u^2}\frac{y^2}{u^2}\right],$$

$$\frac{\partial^2 X}{\partial x^2} = 2\pi(f_0^{\text{iv}} + f_1^{\text{iv}}y^2) = -6\pi\frac{a^2}{u^5}\left[\frac{a^2 - 4x^2}{u^2} + \frac{15}{4}\frac{a^4 - 12a^2x^2 + 8x^4}{u^4}\frac{y^2}{u^2}\right].$$

$$Y = 4\pi f_1' y = 3\pi\frac{a^2x}{u^4}\frac{y}{u},$$

$$\frac{\partial Y}{\partial x} = 4\pi f_1'' y = -3\pi\frac{a^2}{u^4}\frac{4x^2 - a^2}{u^2}\frac{y}{u},$$

$$\frac{\partial^2 Y}{\partial x^2} = 4\pi f_1''' y = -15\pi\frac{a^2x}{u^6}\frac{3a^2 - 4x^2}{u^2}\frac{y}{u}.$$

The value of X decreases continuously from the plane of the circle to infinity. The first differential which is zero for $x=0$, is constantly negative; the position of the point, for which it is a maximum, varies with y, but is always near the plane of the circle.

The value of Y, which is at first zero for $x=0$, attains its maximum for $x=\frac{a}{2}$, and then decreases to zero. The second differential is zero for $x=0$, and for $4x^2=3a^2$. This latter value of x corresponds to the maximum negative of the first differential.

739. Action of a Coil Outside the Axis.—In order to obtain the action of a regular coil, containing n_1 windings for unit length, it will be sufficient to calculate that of a uniformly magnetised cylinder, the intensity of whose magnetisation is n_1, or that of two magnetic surfaces A and B of constant densities $+n_1$ and $-n_1$ which covered the bases.

After what has been said (736), the potential V of a circular layer of radius r and density n_1, at a point P, the co-ordinates of which in respect of the centre of the circle are x and y, has the value

$$V = 2\pi n_1(f_0 + f_1y^2 + f_2y^4 + \ldots).$$

The components of the action of the first surface A are then

$$X_a = -\frac{\partial V}{\partial x} = -2\pi n_1(f_0' + f_1' y^2 + f_2' y^4 + \ldots).$$

$$Y_a = -\frac{\partial V}{\partial y} = -4\pi n_1 y(f_1 + 2f_2 y^2 + 3f_3 y^4 + \ldots).$$

If $u^2 = r^2 + x^2$, the value of the coefficients found above (736) give

$$X_a = 2\pi n_1\left(1 - \frac{x}{u} - \frac{3}{2^2}\frac{r^2 x}{u^3}\frac{y^2}{u^2} - \frac{3\cdot5}{(2.4)^2}\frac{r^2 x}{u^3}\frac{3r^2 - 4x^2}{u^2}\frac{y^4}{u^4} + \ldots\right),$$

$$Y_a = \pi n_1 \frac{r^2 y}{u^3}\left(1 - \frac{3}{2.4}\frac{4x^2 - r^2}{u^2}\frac{y^2}{u^2} + \frac{3\cdot5}{(2.4)^2}\frac{r^4 - 12r^2x^2 + 8x^4}{u^4}\frac{y^4}{u^4} + \ldots\right).$$

If we take the centre of the coil as the origin of the coordinates, we may replace x by $x - b$ in the preceding expressions, which become

$$u^2 = r^2 + (x - b)^2,$$

$$X_a = 2\pi n_1\left(1 - \frac{x - b}{u} - \frac{3}{2^2}\frac{r^2(x - b)}{u^3}\frac{y^2}{u^2} - \ldots\right).$$

$$Y_a = \pi n_1 \frac{r^2 y}{u^3}\left(1 - \frac{3}{2.4} - \frac{4(x - b)^2 - r^2}{u^2}\frac{y^2}{u^2} + \ldots\right).$$

On the second base of the coil, the density is $-n_1$; we shall obtain the components X_b and Y_b of the action which this second surface produces at the point P, if we change the sign of n_1, and replace b by $-b$.

The components X and Y of the action of the two bases, are given by the terms $Xa + Xb$ and $Ya + Yb$.

Putting

$$v^2 = r^2 + (x + b)^2,$$

we thus get

$$X = 2\pi n_1\left[\frac{x + b}{v} - \frac{x - b}{u} + \frac{3}{2^2}\left(\frac{x + b}{v^5} - \frac{x - b}{u^5}\right) r^2 y^2 + \ldots\right],$$

$$Y = \pi n_1 r^2 y\left[\frac{1}{u^3} - \frac{1}{v^3} - \frac{3}{2.4}\left(\frac{4(x - b)^2 - r^2}{u^7} - \frac{4(x + b)^2 - r^2}{v^7}\right) y^2 + \ldots\right].$$

For a coil of thickness dr, the winding of which is uniform, n_1 may be replaced in these expressions by $n_1^2 dr$. Integrating then in respect of r, from a' to a'', we shall have the components X and Y of the magnetic action of a coil of thickness $a''-a'$, which gives

$$(34)\quad \begin{cases} X = 2\pi n_1^2 \int_{a'}^{a''} \left[\frac{x+b}{v} - \frac{x-b}{u} + \frac{3}{2^2}(\dots) r^2 y^2 + \dots \right] dr, \\ Y = \pi n_1^2 y \int_{a'}^{a''} \left[\left(\frac{1}{u^3} - \frac{1}{v^3} \right) r^2 - \frac{3}{2.4}(\dots) r^2 y^2 + \dots \right] dr. \end{cases}$$

The first term X_0 of the value X, obviously represents the action on the axis given by equation (12).

The value of the second term, X, is

$$X_1 = \frac{3}{2}\pi n_1^2 y^2 \int_{a'}^{a''} \left(\frac{x+b}{[r^2+(x+b)^2]^{\frac{5}{2}}} - \frac{x-b}{[r^2+(x-b)^2]^{\frac{5}{2}}} \right) r^2 dr.$$

As we have

$$\int \frac{r^2 dr}{(r^2+a^2)^{\frac{5}{2}}} = \frac{1}{3a^2} \cdot \frac{r^3}{(r^2+a^2)^{\frac{3}{2}}},$$

we get

$$X_1 = \frac{\pi n_1^2 y^2}{2} \left\{ \frac{1}{x+b} \left[\frac{r^3}{[r^2+(x+b)^2]^{\frac{3}{2}}} \right]_{a'}^{a''} - \frac{1}{x-b} \left[\frac{r^3}{[r^2+(x-b)^2]^{\frac{3}{2}}} \right]_{a'}^{a''} \right\}.$$

The first term of y is

$$Y = \pi n_1^2 y \int_{a'}^{a''} \left(\frac{1}{[r^2+(x-b)^2]^{\frac{3}{2}}} - \frac{1}{[r^2+(x+b)^2]^{\frac{3}{2}}} \right) r^2 dr.$$

A similar expression has already been previously integrated (730), and we have

$$Y_1 = \pi n_1^2 y \left\{ \begin{array}{l} \frac{a'}{\sqrt{a'^2+(x-b)^2}} - \frac{a'}{\sqrt{a'^2+(x+b)^2}} \\ -\frac{a''}{\sqrt{a''^2+(x-b)^2}} + \frac{a''}{\sqrt{a''^2+(x+b)^2}} \\ +l.\frac{a''+\sqrt{a''^2+(x-b)^2}}{a'+\sqrt{a'^2+(x-b)^2}} - l\frac{a''+\sqrt{a''^2+(x+b)^2}}{a'+\sqrt{a'^2+(x+b)^2}} \end{array} \right\}.$$

In the case of $x=0$, the component reduces to

$$X=G_0=\frac{2\pi n}{a_2}+\frac{\pi n_1^2}{b}\left[\frac{a''^3}{(a''^2+b^2)^{\frac{3}{2}}}-\frac{a'^3}{(a'^2+b^2)^{\frac{3}{2}}}\right]y^2+\ldots,$$

a_2 representing the radius of the winding of mean action in respect of the centre (729).

The value of Y_1 is then zero, and the component Y is of the third order of y.

740. Particular Cases.—Action of a Coil at a Small Distance.—For the applications it is useful to make the calculation in a more complete manner.

In order to determine the action of a coil on a point near the centre we might develop the expressions (34), but it is more advantageous to treat the problem directly.

We may in particular consider the component parallel to the axis. The value X_1 of this component for a circular current of radius r, at a point whose co-ordinates in respect of the centre are x and y is (736)

$$(35)\quad X_1-2\pi\left[\frac{d^2u}{dx^2}-\frac{y^2}{2^2}\frac{d^4u}{dx^4}+\frac{y^4}{(2.4)^2}\frac{d^6u}{dx^6}-\frac{y^6}{(2.4.6)^2}\frac{d^8u}{dx^8}+\ldots.\right].$$

We have further, with $u^2=r^2+x^2$,

$$\frac{d^2u}{dx^2}=\frac{r^2}{u^3}=\frac{1}{r}\left(1+\frac{x^2}{r^2}\right)^{-\frac{3}{2}},$$

$$(36)\quad r\frac{d^2u}{dx^2}=1-\frac{1.3}{2}\frac{x^2}{r^2}+\frac{1.3.5}{2.4}\frac{x^4}{r^4}-\frac{1.3.5.7}{2.4.6}\frac{x^6}{r^6}+\ldots.$$

From this we can deduce the successive differentials of u in respect of x, and it is easily seen that all the series thus obtained

converge more or less rapidly, so long as the ratio $\frac{x}{r}$ is less than unity. We thus obtain

$$
\begin{aligned}
X_1 = \frac{2\pi}{r}\Big\{ & 1 - \frac{1\cdot3}{2}\frac{x^2}{r^2} + \frac{1\cdot3\cdot5}{2\cdot4}\frac{x^4}{r^4} - \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6}\frac{x^6}{r^6} + \dots \\
& + \frac{1}{2^2}\frac{y^2}{r^2}\left[1^2\cdot3 - \frac{1\cdot3^2\cdot5}{2}\frac{x^2}{r^2} + \frac{1\cdot3\cdot5^2\cdot7}{2\cdot4}\frac{x^4}{r^4} - \frac{1\cdot3\cdot5\cdot7^2\cdot9}{2\cdot4\cdot6}\frac{x^6}{r^6} + \dots\right] \\
& + \frac{1}{(2\cdot4)^2}\frac{y^4}{r^4}\left[1^2\cdot3^2\cdot5 - \frac{1\cdot3^2\cdot5^2\cdot7}{2}\frac{x^2}{r^2} + \frac{1\cdot3\cdot5^2\cdot7^2\cdot9}{2\cdot4}\frac{x^4}{r^4} - \dots\right] \\
& + \frac{1}{(2\cdot4\cdot6)^2}\frac{y^6}{r^6}\left[1^2\cdot3^2\cdot5^2\cdot7 - \frac{1\cdot3^2\cdot5^2\cdot7^2\cdot9}{2}\frac{x^2}{r^2} + \frac{1\cdot3\cdot5^2\cdot7^2\cdot9^2\cdot11}{2\cdot4}\frac{x^4}{r^4} - \dots\right] \\
& + \dots\dots\dots\dots \Big\}.
\end{aligned} \tag{37}
$$

In order to obtain the component X in respect of a coil with a rectangular channel, the dimensions of which are $2b$, and $2c$, it is sufficient to multiply the preceding value of X_1 by $n_1 dx_1$ and $n_1 dr_1$ and then to integrate between the limits $x-b$ and $x+b$ on the one hand, and $r-c$ and $r+c$ on the other—that is to say, to calculate the integral

$$
X = n_1^2 \int_{r-c}^{r+c} dr \int_{x-b}^{x+b} X_1 dx, \tag{38}
$$

in which r would be replaced by the mean radius a of the coil.

If the conditions are such that the fourth powers of the ratios $\frac{c}{a}$, $\frac{b}{a}$ and $\frac{x}{a}$ may be disregarded, we thus find when all the reductions are made

$$
\begin{aligned}
X = \frac{2n\pi}{a}\Big\{ & 1 + \frac{c^2}{3a^2} - \frac{b^2}{2a^2} - \frac{3x^2}{2a^2} \\
& + \frac{3}{2^2}\frac{y^2}{a^2}\left[1 + \frac{3\cdot4}{2\cdot3}\frac{c^2}{a^2} - \frac{3\cdot5}{2\cdot3}\frac{b^2+3x^2}{a^2}\right] \\
& + \frac{3^2\cdot5}{(2\cdot4)^2}\frac{y^4}{a^4}\left[1 + \frac{5\cdot6}{2\cdot3}\frac{c^2}{a^2} - \frac{5\cdot7}{2\cdot3}\frac{b^2+3x^2}{a^2}\right] \\
& + \frac{3^2\cdot5^2\cdot7}{(2\cdot4\cdot6)^2}\frac{y^6}{a^6}\left[1 + \frac{7\cdot8}{2\cdot3}\frac{c^2}{a^2} - \frac{7\cdot9}{2\cdot3}\frac{b^2-3x^2}{a^2}\right] \\
& + \dots\dots\dots\dots \Big\}.
\end{aligned} \tag{39}
$$

The series converges then whenever the ratio $\frac{y}{a}$ is smaller than unity, for the factor which multiplies the successive powers of this ratio tends towards $\frac{2}{\pi}$.

741. Action of a Coil at a Great Distance.—We shall in like manner consider what the value X becomes when the distance x of the point P from the plane of the coil is very great compared with the radius. We may in this case, in the expression for the action of a circular current, expand the value of u in respect of increasing powers of the ratio $\frac{x}{r}$.

We have then

$$\frac{du}{dx}=\frac{x}{u}=\left(1+\frac{r^2}{x^2}\right)^{-\frac{1}{2}}.$$

$$\frac{du}{dx}=1-\frac{1}{2}\frac{r^2}{x^2}+\frac{1.3}{2.4}\frac{r^4}{x^4}-\frac{1.3.5}{2.4.6}\frac{r^6}{x^6}+\frac{1.3.5.7}{2.4.6.8}\frac{r^8}{x^8}-\ldots,$$

and consequently

$$\begin{aligned}
X_1=\frac{2\pi r^2}{x^3}\Bigg\{&1-\frac{3}{2}\frac{r^2}{x^2}+\frac{3.5}{2.4}\frac{r^4}{x^4}-\frac{3.5.7}{2.4.6}\frac{r^6}{x^6}+\ldots\\
&-\frac{1}{2^2}\frac{y^2}{x^2}\left[\frac{1}{2}2.3.4-\frac{1.3}{2.4}4.5.6\frac{r^2}{x^2}+\frac{1.3.5}{2.4.6}6.7.8\frac{r^4}{x^4}-\ldots\right]\\
(40)\quad&+\frac{1}{(2.4)^2}\frac{y^4}{x^4}\left[\frac{1}{2}2.3.4.5.6-\frac{1.3}{2.4}4.5.6.7.8\frac{r^2}{x^2}+\ldots\right]\\
&-\frac{1}{(2.4.6)^2}\frac{y^6}{x^6}\left[\frac{1}{2}2.3.4.5.6.7.8-\frac{1.3}{2.4}4.5.6.7.8.9.10\frac{r^2}{x^2}+\ldots\right]\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Bigg\}.
\end{aligned}$$

The action of a coil will be again obtained by a double integration. If we neglect the fourth powers of the ratios $\frac{a}{x}$, $\frac{b}{x}$, $\frac{c}{a}$, and consequently the ratio $\frac{c^2}{x^2}$, we get

$$
\begin{aligned}
X = \frac{2n\pi a^2}{x^3}\Bigg\{ & 1 + \frac{c^2}{3a^2} + \frac{2b^2}{x^2} - \frac{3}{2}\frac{a^2}{x^2} \\
& - \frac{3.4}{2^2}\frac{y^2}{x^2}\left[1 + \frac{c^2}{3a^2} + \frac{5.6}{2.3}\frac{b^2}{x^2} - \frac{5.6}{2.4}\frac{a^2}{x^2}\right] \\
& + \frac{3.4.5.6}{(2.4)^2}\frac{y^4}{x^4}\left[1 + \frac{c^2}{3a^2} + \frac{7.8}{2.3}\frac{b^2}{x^2} - \frac{7.8}{2.4}\frac{x^2}{a^2}\right] \\
& - \dots\dots\dots\dots \Bigg\}. \qquad (41)
\end{aligned}
$$

This action tends to become inversely as the cube of the distance, as could be seen directly by comparing the coil to a magnet.

742. Action of a Long Coil.—In the centre of a coil of a great length, the field in the interior is sensibly uniform (**495**); the examination of this problem presents accordingly a particular interest.

We may observe, in the first place, that the action exerted in the interior by a coil of length $2b$, which only contains one layer of wire, and therefore is equal to the induction of the equivalent magnetic cylinder, has a component parallel to the axis

$$X_1 = 4\pi n_1 - (X_a + X_b),$$

an expression in which X_a and X_b denote the absolute values of the components in respect of the two bases of the cylinder, where the density is n_1.

We have further (**739**)

$$X_a = 2\pi n_1\left[1 - \frac{du}{dx} + \frac{y^2}{2^2}\frac{d^3u}{dx^3} - \frac{y^4}{(2.4)^2}\frac{d^5u}{dx^5} + \dots\right],$$

with

$$u^2 = r^2 + (b + x)^2.$$

If we put $p = \frac{r}{b+x}$, which gives $\frac{dp}{dx} = \frac{-p^2}{r}$, we may write

$$\frac{du}{dx} = \frac{b+x}{u} = 1 - \frac{1}{2}p^2 + \frac{1.3}{2.4}p^4 - \frac{1.3.5}{2.4.6}p^6 + \ldots.$$

From this can be deduced the successive values of the differentials which enter into the expression for X_a; it is sufficient to change the sign of x, to have that of X_b.

Neglecting the fourth powers of the ratio $\frac{x}{b}$, we thus find

$$\begin{aligned}
X_1 = 4\pi n_1 \Bigg\{ & 1 - \frac{1}{2}\frac{r^2}{b^2}\left(1 + \frac{2.3}{1.2}\frac{x^2}{b^2}\right) + \frac{1.3}{2.4}\frac{r^4}{b^4}\left(1 + \frac{4.5}{1.2}\frac{x^2}{b^2}\right) - \ldots \\
& - \frac{y^2 r^2}{2^2 b^4}\left[\frac{1}{2}2.3\left(1 + \frac{4.5}{1.2}\frac{x^2}{b^2}\right) - \frac{1.3}{2.4}4.5\frac{r^2}{b^2}\left(1 + \frac{6.7}{1.2}\frac{x^2}{b^2}\right) + \ldots\right] \\
(42) \quad & + \frac{y^4 r^2}{(2.4)^2 b^6}\left[\frac{1}{2}2.3.4.5\left(1 + \frac{6.7}{1.2}\frac{x^2}{b^2}\right)\right. \\
& \qquad\qquad \left. - \frac{1.3}{2.4}4.5.6.7\frac{r^2}{b^2}\left(1 + \frac{8.9}{1.2}\frac{x^2}{b^2}\right) + \ldots\right] \\
& - \frac{y^6 r^2}{(2.4.6)^2 b^8}\left[\frac{1}{2}2.3.4.5.6.7\left(1 + \frac{8.9}{1.2}\frac{x^2}{b^2}\right) - \ldots\right] \\
& + \ldots\ldots\ldots\ldots \Bigg\}.
\end{aligned}$$

The component X in respect of the coil will be obtained by integrating this expression from $r-c$ to $r+c$, after having multiplied by $n_1 dr_1$ and replacing r by the mean radius a.

Disregarding again fourth powers of the ratio $\frac{c}{a}$, we get

$$\begin{aligned}
X = \frac{2n\pi}{b}\Bigg\{ & 1 - \frac{1}{2}\frac{a^2}{b^2}\left(1 + \frac{1}{3}\frac{c^2}{a^2} + 3\frac{x^2}{b^2}\right) + \frac{3}{8}\frac{a^4}{b^4}\left(1 + 2\frac{c^2}{a^2} + 10\frac{x^2}{b^2}\right) - \ldots \\
(43) \quad & - \frac{y^2 a^2}{2^2 b^4}\left[3\left(1 + \frac{1}{3}\frac{c^2}{a^2} + 10\frac{x^2}{b^2}\right) - \frac{15}{2}\frac{a^2}{b^2}\left(1 + 2\frac{c^2}{a^2} + 21\frac{x^2}{b^2}\right) + \ldots\right] \\
& + \frac{y^4 a^2}{(2.4)^2 b^6}\left[60\left(1 + \frac{1}{3}\frac{c^2}{a^2} + 21\frac{x^2}{b^2}\right) - \ldots\right] \\
& - \ldots\ldots\ldots\ldots \Bigg\}.
\end{aligned}$$

If the ratios $\frac{a}{c}$, $\frac{x}{b}$ and $\frac{a^2}{b^2}$ are of the same order, we get to the same degree of approximation.

$$
(44)\qquad X=\frac{2n\pi}{b}\left[1-\frac{1}{2}\frac{a^2}{b^2}+\frac{3}{8}\frac{a^2}{b^2}\left(\frac{a^2}{b^2}-\frac{4}{9}\frac{c^2}{a^2}-4\frac{x^2}{b^2}\right)\right.
\left.-\frac{3}{4}\frac{a^2y^2}{b^4}\cdot\left(1-\frac{5}{2}\frac{a^2}{b^2}\right)+\frac{15}{16}\frac{a^2y^4}{b^6}\right].
$$

743. Maxwell's Method.—In passing from the action of a single current to that of a coil, instead of using quadratures as in the preceding paragraph, it is sometimes more advantageous to use the following method employed by Maxwell.*

Let P be a function of two variables x and y; it is desired to calculate the function $\overline{P}$ defined by the equation

$$
(45)\qquad \overline{P}=\frac{1}{xy}\int_{-\frac{x}{2}}^{+\frac{x}{2}}\int_{-\frac{y}{2}}^{+\frac{y}{2}}P\,dx\,dy,
$$

that is to say, the mean value of P within the limits of the integration.

If P_0 is the value of the function P for $x=0$ and $y=0$, Taylor's theorem gives

$$
P=P_0+x\frac{\partial P_0}{\partial x}+y\frac{\partial P_0}{\partial y}+\frac{x^2}{2}\frac{\partial^2 P_0}{\partial x^2}+\ldots
$$

Replacing P by this value in the above expression, and integrating between the limits in question, all the uneven terms of x and y vanish; the final result, divided by xy, is the desired value of $\overline{P}$.

The term $\frac{x^2}{2}\frac{\partial^2 P_0}{\partial x^2}$ for instance, gives

$$
\frac{1}{2}\frac{\partial^2 P_0}{\partial x^2}\int_{-\frac{x}{2}}^{+\frac{x}{2}}\int_{-\frac{y}{2}}^{+\frac{y}{2}}x^2\,dx\,dy=\frac{x^3y}{3\cdot 2^3}\cdot\frac{\partial^2 P_0}{\partial x^2}.
$$

We have finally,

$$
\overline{P}=P_0+\frac{1}{3\cdot 2^3}\left(x^2\frac{\partial^2 P_0}{\partial x^2}+y^2\frac{\partial^2 P_0}{\partial y^2}\right)
+\frac{1}{3\cdot 4\cdot 5\cdot 2^5}\left[x^4\frac{\partial^4 P_0}{\partial x^4}+2x^2y^2\frac{\partial^4 P_0}{\partial x^2\partial y^2}+y^4\frac{\partial^4 P_0}{\partial y^4}\right]+\ldots
$$

* Maxwell. *Electricity and Magnetism*, Vol. II., p. 304.

744. If we apply this formula to the first term of the value of X, that which represents the action of a circular current of radius r on a point of the axis at the distance x, then denoting the mean radius by a, and the radial and axial dimensions of the coil by $2c$ and $2b$, we have

$$\overline{P}=\frac{a^2}{u^3}\left[1+\frac{1}{6}\frac{3a^2(4x^2-a^2)}{u^4}.\frac{b^2}{a^2}+\frac{1}{6}\frac{2x^4-11a^2x^2+2a^4}{u^4}.\frac{c^2}{a^2}\right],$$

and the first term of the value X is equal to $2n\pi\overline{P}$.

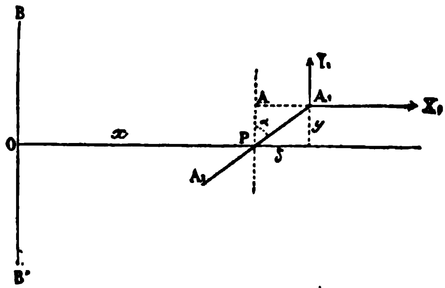


Fig. 138.

745. For a point beyond the axis, if we stop at terms of the second order in y, we may consider as constant the second value of X in equation (33), which gives

$$X=2\pi n\overline{P}-2n\pi\frac{a^2}{u^3}\frac{3}{4}\frac{4x^2-a^2}{u^2}\frac{y^2}{u^2}.$$

If we wish to push the approximation further, we shall apply Maxwell's method to the correction, considering the term in y^2 as constant.

746. Action of a Coil on a Magnetic Needle.—In the investigation of the galvanometer (particularly the tangent galvanometer), we shall have to consider the action which the current of a coil exerts on a magnetised needle.

If the field of the coil were uniform, or the needle infinitely small, the electromagnetic action would be reduced to a couple; but in general it will be equal to a single force and to a couple.

Let BB′ (Fig. 138) be a circular current of radius a, the centre of which is at O, Ox its axis, A_2 A_1 a needle the centre of which is on the axis of the coil at a distance x from the centre. Let $2l$ be the length

of the needle, y and $x+\delta$ the co-ordinates of the pole A_1, $-y$ and $x-\delta$ those of the pole A_2; we shall assume that these poles are equal to unity.

Let X, T_1, and X_2, Y_2 denote the components of the field of the frame in absolute values at the points A_1 and A_2. As the poles A_1 and A_2 are of opposite signs, the values of the components ξ and η of the force, and the moment D of the couple are

$$\xi = X_1 - X_2, \qquad \eta = Y_1 - Y_2;$$
$$D = (X_1 + X_2)y - (Y_1 - Y_2)\delta.$$

On the other hand, if X and Y are the components of the field at the point A, the co-ordinates of which are x and y, we have

$$X_1 = X + \delta\frac{\partial X}{\partial x} + \frac{\delta^2}{1.2}\frac{\partial^2 X}{\partial x^2} + \frac{\delta^3}{1.2.3}\frac{\partial^3 X}{\partial x^3} + \frac{\delta^4}{1.2.3.4}\frac{\partial^4 X}{\partial x^4} + \ldots,$$
$$Y_1 = Y + \delta\frac{\partial Y}{\partial x} + \frac{\delta^2}{1.2}\frac{\partial^2 Y}{\partial x^2} + \frac{\delta^3}{1.2.3}\frac{\partial^3 Y}{\partial x^3} + \frac{\delta^4}{1.2.3.4}\frac{\partial^4 Y}{\partial x^4} + \ldots$$

The values of X_2 and Y_2 will be deduced from the preceding by changing the sign δ; it follows that

$$X_1 + X_2 = 2\left[X + \frac{\delta^2}{2}\frac{\partial^2 X}{\partial x^2} + \frac{\delta^4}{24}\frac{\partial^4 X}{\partial x^4} + \ldots\right]$$
$$Y_1 - Y_2 = 2\delta\left[\frac{\partial Y}{\partial x} + \frac{\delta^2}{6}\frac{\partial^3 Y}{\partial x^3} + \ldots\right].$$

If as above we put (736)

$$X = 2\pi(f_0'' + f_1''y^2 + f_2''y^4 + \ldots),$$
$$Y = 4\pi y(f_1' + 2f_2'y^2 + \ldots),$$

limiting the expansion to terms of the fourth order, we shall have

$$D = 4\pi y\left\{\begin{array}{c} f_0'' + f_1''y^2 + (f_0^{\text{IV}} - 4f_1'')\dfrac{\delta^2}{2} \\ + f_2''y^4 + (f_1^{\text{IV}} - 8f_2'')\dfrac{y^2\delta^2}{2} + (f_0^{\text{VI}} - 8f_1^{\text{IV}})\dfrac{\delta^4}{24} \end{array}\right\}.$$

The properties (31) of the functions $f_0, f_1 \ldots$ give directly

$$D = 4\pi y[f_0'' + f^1(y^2 - 4\delta^2) + f_2''(y^4 - 12y^2\delta^2 + 8\delta^2)].$$

To within factors, the terms of correction are the same f_1'', f_2'' as those of the component X at the point A given by equation (29). The factors of f_1'' and f_2'' are moreover composed of y and δ, as these functions themselves are of a and x—that is to say, as functions in respect of a circular current of radius y, parallel to the former for a given point, the abscissa of which would be δ.

If α be the angle which the magnetic axis $A_2 A_1$ of the needle makes with the plane of the current, which gives

$$\delta = l \sin \alpha,$$
$$y = l \cos \alpha,$$

we get

$$D = 4\pi f_0'' \, l \cos \alpha \left[1 + \frac{f_1''}{f_0''}(1 - 5 \sin^2 \alpha) + \frac{f_2''}{f_0''}(1 - 14 \sin^2 \alpha + 21 \sin^4 \alpha) \right].$$

If we replace the functions f_0'', f_1'', f_2'' by their values (736), we get finally,

$$(47) \quad D = 2\pi \frac{a^2}{u^3} 2l \cos \alpha \left[1 + \frac{3}{2^2} \frac{a^2 - 4x^2}{u^2} (1 - 5 \sin^2 \alpha) \frac{l^2}{u^2} + \frac{3^2 5}{(2.4)^2} \frac{a^4 - 12a^2x^2 + 8x^4}{u^4} \left(1 - 14 \sin^2 \alpha + 21 \sin^4 \alpha \right) \frac{l^4}{u^4} \right].$$

When x is made $= 0$, that is, when the middle of the needle is at the centre of the frame, the formula reduces to

$$(48) \qquad D_0 = \frac{2\pi}{a} 2l \cos \alpha \left[1 + \frac{3}{2^2} (1 - 5 \sin^2 \alpha) \frac{l^2}{a^2} + \frac{3^2 5}{(2.4)^2} \left(1 - 14 \sin^2 \alpha + 21 \sin^4 \alpha \right) \frac{l^4}{a^4} \right].$$

The components ξ and η of the force ϕ will be calculated in the same way, but this force is very small, and has no interest in experiments; when a coil is made to act on a magnet suspended by a wire, the only effect of the force ϕ is to deflect the wire a little from the vertical.

747. When the coil has a frame with a rectangular section, the principal term $2\pi \frac{a^2}{a^3}$ must be replaced by the value of G (12). As to the terms of correction within the bracket, it will be sufficient to take for the quantities a and u, their values relative to the near circumference of the coil, unless the magnet has an appreciable length as compared with the diameter of the frame.

If the needle, which we assume is symmetrical, cannot be reduced to two poles, it is readily seen that we should replace the length l by $\frac{\Sigma\mu l}{\Sigma\mu}$, and l^2 by $\frac{\Sigma\mu l^3}{\Sigma\mu l}$, μ being the magnetic mass at a distance λ from the axis of rotation.

The principal term in the expression for D, represents the moment of the action which the needle would experience in a uniform field, the strength of which was the same as at the point P.

The first term of the correction within the bracket, is proportional to the factors $(1 - 5\sin^2\alpha.)$ This factor is positive and equal to unity, for $\alpha = 0$; it vanishes for $\sin^2 \alpha = \frac{1}{5}$, or $\tan \alpha = \frac{1}{2}$; that is to say, for an angle of 26° 4′. It then again becomes negative and attains the value 1·5, for $\alpha = 45°$.

The second term proportional to $(1 - 14\sin^2\alpha + 21\sin^4\alpha)$ is zero, for $\alpha = 4° 40'$ and $\alpha = 35° 49'$. This factor is positive and equal to unity for $\alpha = 0$, passes through a minimum equal to $-\frac{4}{3}$, near 19°28, and resumes the same positive value near 43°46′. In most cases this term can be neglected.

748. Gaugain's Coil.—Various devices may be made use of to get rid of the first term of correction of the couple D, so that the needle may be sensibly under the same conditions as if the field were uniform.

We might in the first case arrange the experiments in such a manner that the needle always made an angle of about 26° with the plane of the frame, which would make the factor $(1 - 5\sin^2\alpha)$ virtually zero.

Another plan consists in placing the centre of the needle at the distance $x = \frac{a}{2}$, a condition which would annul the factor $a^2 - 4x^2$.

Gaugain had found experimentally* that a needle whose centre is on the axis of a circular current, and at a distance from the centre equal to half the radius, experiences deflections the tangents of which are very exactly proportional to the strength of the current—that is to say, that the couple of the current is proportional to the cosine of the deflection. Starting from this observation, he had coils made in the form of the frustrum of a cone such that the tangent of the same angle at the summit was equal to 2. If the centre of the needle be placed at the apex of the cone the preceding condition is satisfied

* Gaugain. *Comptes rendus de l'Acad. de Science*, Vol. xxxvi., p. 191. 1853.

for all the windings, but this arrangement presents great practical difficulties in coiling the wire and centering the needle. It is more advantageous to arrange the coils in a rectangular section so as to produce a field which is very nearly uniform.

749. Von Helmholtz's Arrangement.—If we consider two circumferences, B_1 and B_2 (Fig. 139) of the same radius a, having the same axis O_1 O_2, and traversed by parallel currents, we readily see that at the point O, half way between O_1 O_2, the component X parallel to the axis is a minimum, or a maximum relative to points on the axis, or in a plane perpendicular to the axis; and that the component Y is zero in the plane perpendicular to the axis, which passes through the point O. The field is sensibly uniform near the

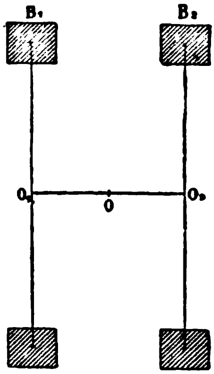


Fig. 139

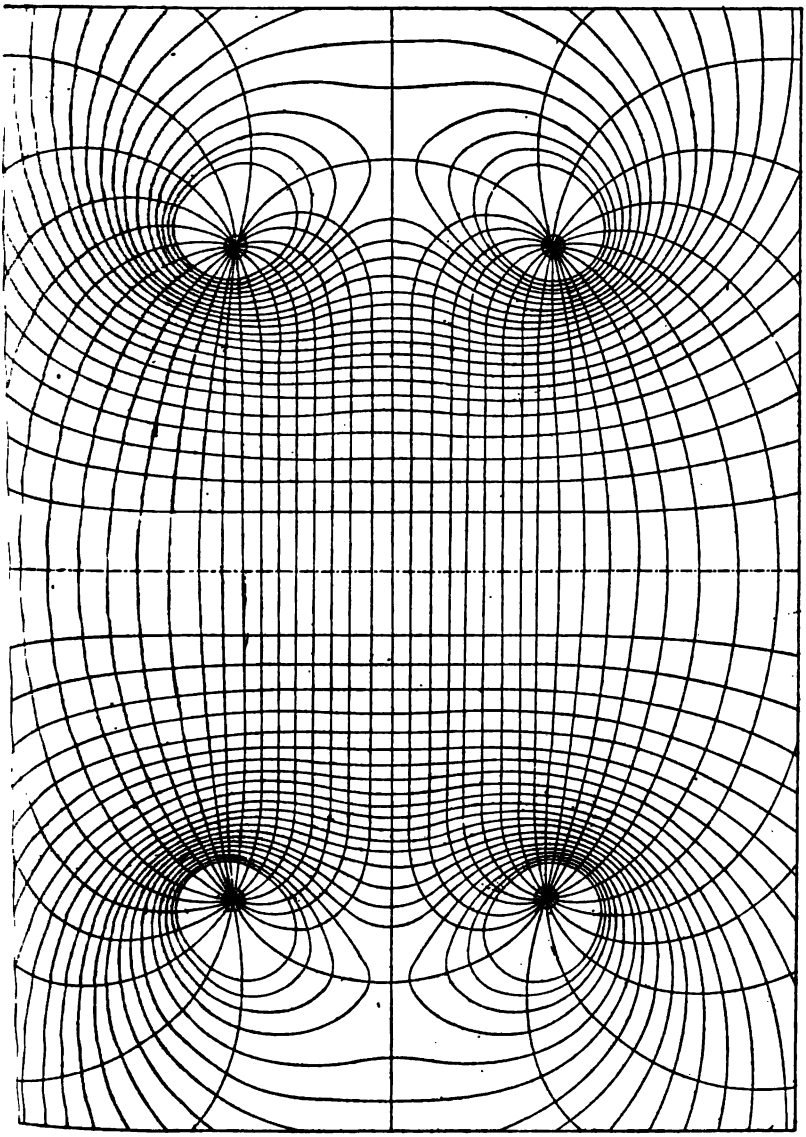
point O, and the moment of the action produced by these two currents on a magnetised needle situate in this region is sensibly the same as if the needle were exactly in the centre; a difficulty of adjustment is thus easily got rid of.

Moreover, if the distance of the frames is equal to the radius a, the first term of the correction in the value of X given by equation (33) vanishes, and we have

$$X = 4\pi\frac{a^2}{u^3}\left[1 - \frac{3}{2}\cdot\frac{45}{64}\frac{a^4}{u^4}\frac{y^4}{u^4}\right] = \frac{32\pi}{5a\sqrt{5}}\left[1 - \frac{54}{125}\frac{y^4}{a^4}\right].$$

This is the arrangement devised by Von Helmholtz.* The condition of the magnetic field is represented by Fig. 140, taken from

* Wiedemann. *Electricität*, Vol. III., p. 250.



MAGNETIC FIELD OF TWO CIRCULAR CURRENTS.

Fig. 140.

Maxwell;* one of the two systems of orthogonal lines represents the

* MAXWELL. *Electricity and Magnetism*, Vol. II., pl. XIX.

lines of force, the other the intersection of the plane of the figure by the equipotential surfaces.

The action at the point O is

$$G_0 = \frac{32\pi}{5a\sqrt{5}}.$$

If the circles are replaced by coils with rectangular section of dimensions $2b$ and $2c$, each containing n windings, and we confine ourselves to terms of the second order, we have

$$G_0 = \frac{32n\pi}{5a\sqrt{5}}\left[1 - \frac{1}{15}\frac{b^2}{a^2}\right],$$

the term with c^2 would disappear as having the factor $a^2 - 4x^2$.

If we push the approximation as far as terms of the fourth order, we find that the term in y^2 contains the factor $(31b^2 - 36c^2)$; this factor may be make to disappear if we choose the section so that

$$\frac{b}{c} = \frac{6}{\sqrt{31}} = 1{\cdot}079.$$

There only remains then for the fourth order the term in y^4, the value of which is

$$-\frac{54}{125}\frac{y^4}{a^4}.$$

750. Coil with Four Circles.—We obtain a more complete solution by means of four circular currents having the same axis, symmetrical in pairs in respect of a point of this axis. We shall assume that the currents are four parallel circles of the same sphere, the centre of which is the point in question. Let a be the radius of the sphere, r and r' the radii of the circles, x and x' their distances from the centre, p the ratio of the number of the windings of the great circles to that of the small. If the currents are in the same direction, the horizontal components add themselves. We may arrange two of the three indeterminates r, r' and p, so as to nullify the two first terms of correction in the value of x. As we have $u = a$ for all the currents, this condition is

$$pr^2(4x^2 - r^2) + r'^2(4x'^2 - x'^2) = 0.$$
$$pr^2(r^4 - 12r^2x^2 + 8x^4) + r'^2(r'^4 - 12r'^2x'^2 + 8x'^4) = 0.$$

Taking into account the ratios

$$a^2 = r^2 + x^2 = r'^2 + x'^2,$$

and putting

$$m = \frac{r}{a}, \qquad m' = \frac{r'}{a},$$

these equations may be written

$$pm^2(4 - 5m^2) + m'^2(4 - 5m'^2) = 0.$$
$$pm^2(21m^4 - 28m^2 + 8) + m'^2(21m'^4 - 28m'^2 + 8) = 0.$$

Subtracting here the first equation from the second, we may replace the latter by the simpler equation

$$pm^4(7m^2 - 6) + m'^4(7m'^2 - 6) = 0.$$

We get from this

$$p = -\frac{m'^2}{m^2}\frac{4 - 5m'^2}{4 - 5m^2} = -\frac{m'^4}{m^4}\frac{7m'^2 - 6}{7m^2 - 6},$$

which gives

$$m'^2 = \frac{4}{7}\frac{7m^2 - 6}{5m^2 - 4},$$

$$p = \frac{32}{49}\frac{7m^2 - 6}{m^2(5m^2 - 4)^3}.$$

We may observe that m and m' represent the sines of the angles a and a^1, under which the two radii r and r' are seen from the centre; their squares must always be between zero and unity; from this follows

$$m^2 \underset{>}{=} 0, \qquad m'^2 \underset{>}{=} \frac{6}{7}, \qquad p \underset{<}{=} \infty,$$

$$m^2 \underset{=}{<} \frac{4}{7}, \qquad m'^2 \underset{=}{<} 1, \qquad p \underset{=}{>} \frac{49}{32};$$

where

$$m^2 \underset{>}{=} \frac{6}{7}, \qquad m'^2 \underset{>}{=} 0, \qquad p \underset{>}{=} 0,$$

$$m^2 \underset{=}{<} 1, \qquad m'^2 \underset{=}{<} \frac{4}{7}, \qquad p \underset{=}{<} \frac{32}{49}.$$

751. Coil with Three Circles.—A remarkable particular case is that in which $m^2 = 1$, from which follows $m'^2 = \frac{4}{7}$ and $p = \frac{32}{49}$; the two circles of radius r, merge then in the great circle of the sphere, and form a single coil having 64 windings, if the two small circles have each 49, or 162 in all.

The radius of the small circles is equal to $a\sqrt{\frac{4}{7}}$, and their distance from the centre equal to $a\sqrt{\frac{3}{7}}$.

The magnetic action at the centre is then

$$G_0 = \frac{2\pi}{a}\left(n + \frac{8n'}{7}\right) = \frac{2\pi}{a} \cdot 120; \qquad (50)$$

that is to say, 120 times that of the great circle.

We know that a sphere, entirely covered with circular currents situate in parallel planes, the successive distances of which are equal to $\frac{1}{n}$, produces in the interior a uniform field, the strength of which is $\frac{8}{3}\pi n_1$ (497). This arrangement would be difficult to realise in practice, while that which we have investigated would give almost the equivalent in the case of a complete spherical coil; and in that of a coil with three circles we should have

$$n_1 = \frac{90}{a};$$

that is to say, the sphere must be covered with 180 windings.

For the sphere the total length of the wire will be

$$2\pi n_1 \int_{-a}^{+a} r dx = n_1 \pi^2 a^2 = 90\pi^2 a = 2\pi a \cdot 141{\cdot}4.$$

In the case of three circles we should have

$$64 \cdot 2\pi a + 98 \cdot 2\pi a \frac{2}{\sqrt{7}} = 2\pi a\left(64 + \frac{196}{\sqrt{7}}\right) = 2\pi a \cdot 138;$$

the length of the wire is then a little less, so that this latter combination is advantageous from all points of view.

752. Two Circles with Currents in Opposite Directions.—Let us suppose in two parallel frames the current passed in opposite directions. The components parallel to the axis of the actions of the two frames are at each point in opposite directions, and the components perpendicular to the axis remain in the same direction in the interval of the frames.

Let X_0 and Y_0 be the values of each of the partial components in the plane of symmetry of the two currents. In two planes on either side at the distance dx, and for the same distance from the axis, we shall have

$$X_1 = X_0 + \frac{\partial X_0}{\partial x}dx + \frac{1}{2}\frac{\partial^2 X_0}{\partial x^2}dx^2 + \ldots\ldots,$$

$$X_2 = X_0 - \frac{\partial X_0}{\partial x}dx + \frac{1}{2}\frac{\partial^2 X_0}{\partial x^2}dx^2 - \ldots\ldots,$$

and analogous expressions for the components Y_1 and Y_2.

Limiting ourselves to the first terms

$$X = 2\frac{\partial X_0}{dx}dx = -\frac{12\pi a^2 x}{u^5}\left[1 + \frac{5}{4}\frac{3a^2 - 4x^2}{u^2}\frac{y^2}{u^2}\right]dx,$$

$$Y = 2Y_0 + \frac{\partial^2 Y_0}{\partial x^2}dx^2 = \frac{6\pi a^2 x}{u^4}\frac{y}{u} - \frac{15\pi a^2 x}{u^6}\cdot\frac{3a^2 - 4x^2}{u^2}\frac{y}{u}dx^2.$$

In these expressions x represents the distance of the middle plane of the two circles. It will be seen that the component will be independent of y, and the component Y independent of dx, if we take $4x^2 = 3a^2$.

With this arrangement the field is symmetrical in respect of the centre of the system of the two frames, and the action of the current would be null on a needle exactly centred in this point.

753. Mean Actions.—Let us suppose a circle of radius y, covered with a uniform magnetic layer of density equal to unity, and situate in the field of a current or of a coil. If the circle has the same axis as the currents in question, the action of the field on this axis is parallel to the axis and equal to the integral of $X2\pi ydy$ taken between the limits 0 and y. If the value of X has an expression of the form

$$X = A + By^2 + Cy^4 + \ldots\ldots,$$

in which A, B, C, are functions of x, we shall have

$$\int_0^y X 2\pi y dy = \pi y^2 \left[A + B\frac{y^2}{2} + C\frac{y^4}{3} + \ldots\ldots \right].$$

If S is the surface πy^2 of the circle, and Fm the *mean action* of the field on the circle, it follows that

$$\int_0^y X 2\pi y dy = F_m S,$$

$$F_m = A + B\frac{y^4}{2} + C\frac{y^4}{3} + \ldots\ldots \tag{51}$$

The expression for the mean force is thus the same as that for the component X, with this single difference, that we should divide y^2 by 2, y^4 by 3, y^{2n} by $n+1$. If we apply this observation to the expressions calculated above, we see that the mean action of a circular current, from equation (33), is

$$F_m = 2\pi \frac{a^2}{u^3} \left[1 - \frac{3}{2^3} \frac{4x^2 - a^2}{u^2} \frac{y^2}{u^2} + \frac{3 \cdot 4}{(2.4)^2} \frac{a^4 - 12a^2x^2 + 8x^4}{u^4} \frac{y^4}{u^4} - \ldots \right]. \tag{52}$$

In like manner, we shall obtain the mean action of a coil by equations (34), (39), (41), (43), etc., for various particular cases.

We shall especially consider equation (39); it gives for the mean action of a coil on a circle at a small distance from the mean plane

$$\begin{aligned} F_m = \frac{2n\pi}{a} \Bigg\{ & 1 + \frac{c^2}{3a^2} - \frac{b^2}{2a^2} - \frac{3x^2}{2a^2} \\ & + \frac{3}{2.2^2} \frac{y^2}{a^2} \left[1 + \frac{3 \cdot 4}{2.3} \frac{c^2}{a^2} - \frac{3 \cdot 5}{2.3} \frac{b^2 + 3x^3}{a^2} \right] \\ & + \frac{3^2 5}{3\,(2.4)^2} \frac{y^4}{a^4} \left[1 + \frac{5 \cdot 6}{2.3} \frac{c^2}{a^2} - \frac{5 \cdot 7}{2.3} \frac{b^2 + 3x^2}{a^2} \right] \\ & + \frac{3^2 5^2 7}{4\,(2.4.6)^2} \frac{y^6}{a^6} \left[1 + \frac{7 \cdot 8}{2.3} \frac{c^2}{a^2} - \frac{7 \cdot 9}{2.3} \frac{b^2 + 3x^2}{a^2} \right] \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \Bigg\}. \end{aligned} \tag{53}$$

754. Annular Coil.—We have seen that in the interior of an annular coil (496) the flow of force across an element dS of the section, at a distance x from the axis, is equal for unit current to $4\pi n_1 \frac{dS}{x}$.

The mean action on a surface S is then

$$F_m = \frac{4\pi n_4}{S}\int\frac{dS}{x} = \frac{2n}{S}\int\frac{dS}{x}.$$

If the surface S is a circle of radius a, the centre of which is at a distance R from the axis of the coil, we have

$$\int\frac{dS}{x} = 2\pi\left(R - \sqrt{R^2 - a^2}\right),$$

and accordingly,

$$F_m = \frac{4n}{a^2}R\left(1 - \sqrt{1 - \frac{a^2}{R^2}}\right) = \frac{2n}{R}\left[1 + \frac{1}{4}\frac{a^2}{R^2} + \frac{1}{8}\frac{a^4}{R^4} + \ldots\right].$$

Let us suppose that the section S is an ellipse, one of the axes of which, $2a$, is perpendicular, and the other, $2b$, parallel to the axis of the ring; we have

$$\int\frac{dS}{x} = 2\pi\frac{b}{a}\left(R - \sqrt{R^2 - a^2}\right).$$

The mean action is the same as that for a circle of radius a; it is independent of the radius b.

In like manner we shall have for a rectangle, one of the sides of which, $2a$, is perpendicular, and the other, $2b$, parallel to the axis of the ring,

$$\int\frac{dS}{x} = 2bl.\frac{R+a}{R-a},$$

$$F_m = \frac{n}{a}l.\frac{R+a}{R-a} = \frac{2n}{R}\left[1 + \frac{1}{3}\frac{a^2}{R^2} + \frac{1}{5}\frac{a^4}{R^4} + \ldots\right].$$

The mean action is still independent of the dimension b; the same would be the case for any figure having an axis of symmetry parallel to the axis of the ring.

CHAPTER V.

COEFFICIENTS OF INDUCTION.

755. Induction in a Circuit.—We may still assume as a first approximation (**722**) that the current which traverses a wire is concentrated on the axis of the conductor.

The induction of a field in a circuit is then the flow of force across a surface S, bounded by the axis of the wire which constitutes the circuit, and the relative error made by this mode of calculation is very small as long as the dimensions of the surface are very great in comparison with the diameter of the wire. If the medium is magnetic, in order to have the flow of induction, we must multiply the force by the coefficient of induction or of magnetic permeability μ_0 (**393**) of the medium. Denoting by F_n the mean component of the force perpendicular to the surface S, the induction of the field in the circuit is $\mu_0 F_n S$.

If the field is produced by a current which traverses a second circuit, the flow of force is proportional to the intensity I of this current, and may be represented by MI; the flow of induction is then μ_0MI. The factor M is the *coefficient of mutual induction* (**519**) of the two circuits; it is that portion of the flow of force starting from one of the circuits for unit current which traverses the surface bounded by the second.

In like manner the coefficient of self-induction of a circuit is the flow of induction which it emits for unit current, and which traverses the surface bounded by the axis of the wire.

These simple considerations suffice in most cases, but if the section of the conducting wires cannot be neglected, it is necessary to have recourse to the general properties of the electromagnetic field.

756. Parallel Currents.—Let us consider, in the first case, a system of parallel rectilinear currents, so long that, in a plane perpendicular to the currents, the distribution of forces is independent of the manner in which the ends of the conductor are connected with each other and with the sources of electricity.

The action of a rectilinear and linear current of intensity I on a point P, a distance r (444 and 449), is equal to $2\frac{I}{r}$, and is perpendicular to the plane which passes through this point and the current.

If the section of the conductor is circular, and the current is distributed in homogeneous concentric layers, the force F, by symmetry, is the same for all points of a circumference which has the axis for a centre. The value of the work of the current on an external mass equal to unity, which is on the circumference of radius r, is $2\pi r$F. As this work, moreover, is equal to 4πI (452), we have

$$4\pi I = F \cdot 2\pi r, \qquad \text{or} \qquad F = \frac{2I}{r};$$

the force is therefore inversely as the distance of the point from the axis of the current.

It follows from this, that the action on an internal point only depends on the quantity of electricity which traverses the central core, the radius r of which is equal to the distance of this point from the axis. If the current is homogeneous and of density σ for unit surface, the radius of the cylinder being a, the external action is

$$(1) \qquad F_e = \frac{2I}{r} = 2\pi\sigma\frac{a^2}{r},$$

and the internal action

$$(2) \qquad F_i = \frac{2I_1}{\rho} = 2\pi\sigma\rho = 2I\frac{\rho}{a^2};$$

the action of the current which traverses the hollow cylinder $\pi(a^2 - r^2)$ is null in the cavity of the tube formed by the conductor.

Let us draw a plane through the axis, and consider in this plane a rectangle of height equal to unity, the base of which measured from the axis is equal to b. The expression for the flow of force which the current emits in the rectangle of base $(b - a)$ is

$$2I\int_a^b \frac{dr}{r} = 2Il.\frac{b}{a}.$$

Let us now suppose that a current I′ in the opposite direction traverses a second wire of radius a', the axis of which is at the distance b. The flow of force from this second current in the

rectangle of base $(b-a')$ is $2I(lb-la')$. If the second current is only the return of the first we have $I=I'$, and the flow of force comprised between the two conductors, for a height equal to unity, is

$$2I\left(l.\frac{b}{a}+l.\frac{b}{a'}\right)=2Il.\frac{b^2}{aa'}.$$

The expression

$$(3) \qquad L_1=2l.\frac{b^2}{aa'}$$

represents the induction relative to a current equal to unity; that is, the coefficient of self-induction of two parallel wires for unit length. The value thus obtained is, however, in any case only approximate; we have to take account of the action which each wire exerts in the space it occupies.

757. Let us suppose that the cylindrical conductor has any given section S, and that the density σ of the current is constant. The *magnetic* action which the filament of intensity σdS, corresponding to the element dS, exerts on a point P at a distance r, is equal to $\frac{\sigma dS}{r}$, and is perpendicular to the plane rdS.

If, at the point P, there is a linear current parallel to the first, and of unit strength, the action which the filament σdS would exert on unit length of the second, is directed along the straight line V, and is equal to $\frac{2\sigma dS}{r}$. The potential of this action, to within a constant, is $-2\sigma dSlr$.

This expression integrated with respect to the surface S would give for the potential V of the total action of the current σdS on unit length of the other

$$V=-2\sigma\int dSl.r.$$

Putting

$$(4) \qquad Sl.R_0=\int dSl.r,$$

we may write

$$V=-2\sigma Sl.R_0=-2Il.R_0.$$

758. GEOMETRICAL MEAN DISTANCES IN A PLANE.—The value of R_0 defined by equation (4) is the geometrical mean distance* of the point P from the different points of the surface S. Let us in like manner consider two surfaces S and S′ situate in the same plane, and let r be the distance of the two elements dS and dS'; if we put

$$(5) \qquad SS'l.R_1 = \int\int dSdS'l.r,$$

the integration extending to the surfaces S and S′, the length R_1 is the geometrical mean distance of the two surfaces.

Finally, if the two elements dS and dS' belong to the same surface, the value of R_2 given by the equation

$$(6) \qquad S^2l.R_2 = \int\int dSdS'l.r$$

is the geometrical mean distance of the surface S—that is to say, that of all the points taken in pairs.

The geometrical mean distances play an important part in calculating the coefficients of self-induction.

The geometrical mean distance from a point to a figure is manifestly comprised within the greatest and least of the distances from this point to the different elements of the figure. The same is the case for the geometrical mean distance of the two figures A and B.

The following property is a direct consequence of the definition.

If R_{ac} and R_{bc} are the geometrical mean distances of two figures A and B to a third C, and R_{a+bc} the geometrical mean distance of the sum of the two figures A and B from the third, we have

$$(A+B)l.R_{(a+b)c} = Al.R_{ac} + Bl.R_{bc}.$$

This equation enables us to determine the value of R for a complex figure, when we know it separately for different parts of the figure.

The calculation of R in different cases reduces to a question of analysis; we shall give a few examples.

1st. For a straight line of length a, and at a point at a distance x from the line, the geometrical mean distance is defined by the equation

$$al.R_0 = \int dsl.r = \int dsl.\sqrt{s^2+x^2},$$

* MAXWELL. *Trans. Roy. Soc. Edinburgh.* 1871–72. *Electricity and Magnetism.* Vol. II., § 691–693.

s being the distance of the element ds from the base of the perpendicular upon the line. From this we deduce

$$al.\,R_0 = \left[sl.\sqrt{s^2+x^2} - s + x \text{ arc tan } \frac{s}{x} \right]_{s_0}^{s_0+a}.$$

If the perpendicular x falls at the end of the line, we have $s_0 = 0$, and therefore

$$l.\,R_0 = l.\sqrt{a^2+x^2} - 1 + \frac{x}{a} \text{ arc tan } \frac{a}{x}.$$

2nd. Multiplying this expression by adx, we shall have the geometrical mean distance from a point to the rectangle adx, which will enable us to calculate the values of R_0 relative to the rectangle.

We get in this way for the summit of a rectangle ab,

$$2l.\,R_0 + 3 = 2l.\sqrt{a^2+b^2} + \frac{a}{b} \text{ arc tan } \frac{b}{a} + \frac{b}{a} \text{ arc tan } \frac{a}{b}.$$

If the point has any given position, we may decompose the rectangle into four others having a summit at this point.

3rd. The action of a uniform current which traverses a cylinder whose section is a hollow cylinder, being zero in the interior, and inversely as the distance from the axis for an external point, it follows (757) that:

The geometrical mean distance of a point to a circumference is constant; it is equal to the radius if the point is on the inside, or to the distance from the centre if the point is on the outside.

4th. The geometrical mean distance from a point to a corona bounded by the radii a_1 and a is equal to the distance from the centre if the point is on the outside.

If the point is inside the corona, the value of R_0 is the same as for the centre, which gives

$$\pi(a^2 - a_1^2)l.\,R_0 = \int_{a_1}^{a} 2\pi r dr l.\,r = 2\pi \left[\frac{r^2}{2} l.\,r - \frac{r^2}{4} \right]_{a_1}^{a},$$

$$l.\,R_0 = \frac{a^2 l.\,a - a_1^2 l.\,a_1}{a^2 - a_1^2} - \frac{1}{2}.$$

5th. The geometrical mean distance from a corona to any given figure is equal to the geometrical mean distance from the centre to the figure if the latter is entirely without the corona.

If the figure is within the interior of the corona, the geometrical mean distance being the same for all points, the mean distance is equal to that of the centre to the corona, and is independent of the shape of the figure.

6th. For a point situate in a circle of radius a, and at a distance r from the centre, the mean distance will be obtained by considering the point as external to the circle of radius r and inside the complementary corona. We thus find

$$7. R_0 = l.a - \frac{1}{2}\left(1 - \frac{r^2}{a^2}\right),$$

and for the centre

$$R_0 = ae^{-\frac{1}{4}} = 0{\cdot}60663\,a.$$

7th. For two circles external to each other the geometrical mean distance R is the distance of the centres, since the action of two currents parallel to circular sections (756) only depends on this distance R.

8th. The geometrical mean distance R_2 of a point to a straight line, enables us to determine the geometrical mean distance of all points of a right line of length a, which gives

$$l.R_2 = l.a - \frac{3}{2}, \qquad \text{or} \qquad R_2 = ae^{-\frac{3}{2}} = 0{\cdot}22313\,a.$$

9th. In like manner for a rectangle with sides a and b, we have

$$l.R_2 = l.\sqrt{a^2+b^2} - \frac{1}{6}\frac{a^2}{b^2}l.\sqrt{1+\frac{b^2}{a^2}} - \frac{1}{6}\frac{b^2}{a^2}l.\sqrt{1+\frac{a^2}{b^2}}$$

$$+\frac{2}{3}\frac{a}{b}\text{arc tan}\frac{b}{a} + \frac{2}{3}\frac{b}{a}\text{arc tan}\frac{a}{b} - \frac{25}{12},$$

and for a square of side a,

$$l.R_2 = l.a + \frac{1}{3}l.2 + \frac{\pi}{3} - \frac{25}{12} = l.a - 0{\cdot}80508,$$

where

$$R_2 = 0{\cdot}44705\,a.$$

10th. The geometrical mean distance of all points of a corona of radii a and a_1 will be deduced from the geometrical mean distance from a point to the corona, which gives

$$l.\,R_2 = l.\,a - \frac{a_1^2}{(a^2 - a_1^2)^2}\,l.\,\frac{a}{a_1} + \frac{1}{4}\,\frac{3a_1^2 - a^2}{a^2 - a_1^2}.$$

From this we get, for a circle of radius a,

$$l.\,R_2 = l.\,a - \frac{1}{4},$$

or

$$R_2 = ae^{-\frac{1}{4}} = 0{\cdot}7788a,$$

and for a circumference of radius a,

$$R_2 = a.$$

759. Self-Induction of Two Parallel Wires.—We have seen (566) that the properties of an electromagnetic field are defined by the electromotive force J at each point. The components of the induction are expressed (567) as a function of the components of the electromotive force, and the component of the current by the aid of those of induction.

If the field only contains parallel currents in cylindrical conductors, and we take the z-axis parallel to the currents, the electromotive force is also parallel to the z-axis from symmetry. All the phenomena depend then only on the component H.

Let us consider a linear current of intensity I at the origin of co-ordinates. At the distance r from the axis, the force F only depends on r; taking the point on the x-axis, we have

$$X = 0,$$

$$Y = F = \frac{2I}{r}.$$

Changing the direction of the y, to have the ordinary representation in a plane, equation (2) of (567) gives

$$\mu_0 X = \frac{\partial H}{\partial y} = 0,$$

$$\mu_0 Y = -\frac{\partial H}{\partial x} = -\frac{\partial H}{\partial r} = \mu\frac{2I}{r}.$$

From this follows, to within a constant depending on the position of the return current,

$$(7) \qquad H = -2\mu_0 I \int \frac{dr}{r} = -2\mu_0 I l . r.$$

For a series of parallel currents the value of H at each point is the sum of the values which result from the different currents.

Let us consider two parallel currents of constant densities σ and σ' in cylindrical conductors whose sections are S and S'. The potential energy of these currents reduces in the present case to

$$W = \frac{1}{2} \int\int\int Hw\,dx\,dy\,dz.$$

If we restrict the field to the space comprised between two parallel planes at unit distance, and perpendicular to the current, the energy relative to this portion of the field becomes

$$W = \frac{1}{2} \int\int Hw\,dx\,dy = \frac{1}{2} \int Hw\,dS.$$

The integral should be extended over the whole plane, but it only differs from zero in points in which is the current w—that is to say, for the sections S and S' of the conductor. If we denote by H and H' the values of the electromotive force which arise from the currents σS and $\sigma' S'$, and we denote by an index to the sign $\int$ the spaces which correspond to the different integrals, we may write

$$(8) \qquad 2W = \int_{S'} H\sigma' dS' + \int_{S} H'\sigma dS + \int_{S} H\sigma dS + \int_{S'} H'\sigma' dS'.$$

If the second conductor S' is only the return wire of the first current, we have $I + I' = 0$, or $\sigma S + \sigma' S' = 0$. In this case the common constant of the different integrations gives a zero term in the value of the energy. At the point P', where is the element dS', the portion SH of the electromotive force due to the elementary current σdS at the distance r, is, from equation (7),

$$\delta H = -2\mu_0 \sigma dS l . r.$$

The value of H at the point P′ is the integral of this expression extended to the whole surface S, which gives

$$H = -2\mu_0\sigma \int dSl \,.\, r.$$

We have then

$$\int_{S'} H\sigma' dS' = -2\mu_0\sigma\sigma' \int dS' \int dSl \,.\, r = -2\mu_0\sigma\sigma' SS'l \,.\, R_1 = 2\mu_0 I^2 l \,.\, R_1,$$

or

$$\int_{S'} H\sigma' dS' = I^2\mu_0 l \,.\, R_1^2,$$

R_1 being the geometrical mean distance of the sections S and S′ of the two conductors.

In like manner we have

$$\int_{S} H'\sigma dS = I^2\mu_0 l \,.\, R_1^2.$$

If μ is the coefficient of magnetic permeability of the first conductor, the value of H at the point P, where is the element dS, is

$$H = -2\mu\sigma \int dS_1 l \,.\, r,$$

an expression in which dS_1 denotes any other given element of the surface S. It follows that

$$\int_{S} H\sigma dS = -2\mu\sigma^2 \int dS \int dS_1 l \,.\, r = -2\mu I^2 l \,.\, R_2,$$

R_2 being the geometrical mean distance from the surface S.

We have, finally,

$$\int_{S'} H'\sigma' dS' = -2\mu' I^2 l \,.\, R_2'.$$

Substituting these values in the expression (8) of the energy, we get

$$\frac{2W}{I^2} = 2[\mu_0 l \,.\, R_1^2 - \mu l \,.\, R_2 - \mu' l \,.\, R_2'].$$

On the other hand, the intrinsic energy of a current (524) is equal to $\frac{1}{2}$ LI². Replacing W by this expression, we find that the coefficient of induction of two parallel wires traversed by the same current in opposite directions, is, for unit length of each,

$$(9) \qquad L = 2[\mu_0 l.\, R_1^2 - \mu l.\, R_2 - \mu' l.\, R_2'].$$

Unless the field contains iron, we might put $\mu_0 = \mu = \mu' = 1$, which gives

$$(10) \qquad L = 2[l.\, R_1^2 - l.\, R_2 - l.\, R_2'] = 2l.\, \frac{R_1^2}{R_2 R_2'}.$$

760. If the sections of the two conductors are circles of radii a and a', the centres of which are at the distance b, we have

$$R_1 = b, \qquad l.\, R_2 = l.\, a - \frac{1}{4},$$

$$(11) \qquad L = 2\left(l.\, \frac{b^2}{aa'} + \frac{1}{2}\right).$$

This expression only differs materially from that found above (756) when the wires are very close to each other. When the wires have the same diameter and are in contact, we have $b = a + a' = 2a$, and therefore

$$(12) \qquad L = 2\left(l.\, 4 + \frac{1}{2}\right) = 3{\cdot}7726.$$

This is the least value which the coefficient of self-induction could have for unit length in the case of a wire folded on itself, like those used in resistance boxes.

As the wires should be separated by an insulating substance, if, as above (723), y is the radius of the bare wire, and z the thickness of the insulating layer, we should make $b = 2(y + z)$, which gives for the minimum value

$$(13) \quad L = 2\left[l.\, \frac{4(y+z)^2}{y^2} + \frac{1}{2}\right] = 3{\cdot}7726 + 4l.\left(1 + \frac{z}{y}\right).$$

This coefficient could be considerably reduced if we replaced the wires by very thin plates of metal.

The geometrical mean distances R_π and R_q relative to the circle, of radius y, and to the square of the side $2(y+z)$ which circumscribes the insulating layer of the wire, gives the ratio

$$l.\frac{R_q}{R_\pi} = l.\frac{y+z}{y} + \frac{4}{3}l.2 + \frac{\pi}{3} - \frac{11}{6} = l.\frac{y+z}{y} + 0{\cdot}1380606.$$

The excess of the coefficient of self-induction of the circular wire over that of the square wire for unit length is then, from formula (10),

$$(14) \qquad 2l.\frac{R_q}{R_\pi} = 2\left[l.\frac{y+z}{z} + 0{\cdot}1380606\right].$$

761. MUTUAL ACTION OF TWO CIRCLES.—The formulæ given in the preceding chapter for expressing the magnetic field of a circular current will enable us to calculate the coefficient of mutual induction of two circles. We shall assume that the medium is not magnetic, and shall, in the first place, neglect the thickness of the wires. If one of the circles S has a smaller diameter in respect of the distances of the centres, we might take as mean value of the force that which the second circle would exert for unit current on the centre of the first. In the opposite case we should take into account the variations of the force.

Consider two circular currents S and S′ parallel and in the same direction, of radii a and a' having the same axis, at a distance x, and take the centre of the first as origin of the co-ordinates. The flow of force which the first circuit sends for unit current in the surface of the second—that is to say, the coefficient of mutual induction—is then

$$(15) \qquad M = F_m S'.$$

The expression of the mean force F_m is defined, as we have seen, by that of the component X, which gives

$$M = \frac{2SS'}{u^3}\left[1 - \frac{3}{2^3}\frac{4x^2 - a^2}{u^2}\frac{a'^2}{u^2} + \frac{3.5}{(2.4)^2}\frac{a^4 - 12a^2x^2 + 8x^4}{u^4}.\frac{a'^4}{u^4}\right].$$

The various expressions of X, which we have calculated for coils in particular cases, will give directly the coefficient of mutual induction of these coils, and of a circle with the same axis.

762. MUTUAL INDUCTION OF TWO COILS.—Knowing thus the mean action F_m of a coil on a circle of radius y, we can calculate the

coefficient of mutual induction of the two coils. For the flow of force Q issuing from the coil, and which traverses the circle, is

$$Q = \pi F_m y^2.$$

If y is the mean radius a' of a coil with a rectangular channel, the dimensions of which are $2^{b'}$ and $2^{c'}$, the total flow of force from the first coil, and which traverses the different windings of the second—that is to say, the coefficient of mutual induction of the two coils—will be obtained if we multiply the value of Q by $n_1'dy \times n_1'dx$, and then integrating between the values $y - c'$ and $y + c'$, $x - b'$ and $x + b'$.

$$M = n_1^2 \int_{y-c'}^{y+c'} dy \int_{x-b'}^{x+b'} Q dx.$$

Suppose, for instance, that the dimensions of the coils are such that we can neglect the fourth powers of the ratios $\frac{c}{a}$, $\frac{b}{a}$, $\frac{c'}{a'}$ and $\frac{x}{a}$, retaining the letters a, b and c to denote the dimensions of the larger of the two coils. Equation (39) of (740), which gives the value of x, will also give the expression for the mean action F_m by the ordinary rule (753) and it is sufficient to make the integrations, which present no difficulty.

We thus find if l and l' are the lengths of the wires, n and n_1 the numbers of windings for the two coils, and putting

$$(16) \qquad M_0 = \pi \frac{n^2 l'^2}{n' l},$$

$$\begin{aligned}
\frac{M}{M_0} = 1 &+ \frac{1}{3}\left(\frac{c^2}{a^2} + \frac{c'^2}{a'^2}\right) - \frac{b^2 + b'^2 + 3x^2}{2a^2} \\
&+ \frac{3}{2.2^2}\frac{a'^2}{a^2}\left[1 = \frac{3.4}{2.3}\left(\frac{c^2}{a^2} + \frac{c'^2}{a'^2}\right) - \frac{3.5}{2.3}\frac{b^2 + b'^2}{a^2}\right] \\
(17) \qquad &+ \frac{3^2 5}{3(2.4)^2}\frac{a'^4}{a^4}\left[1 + \frac{5.6}{2.3}\left(\frac{c^2}{a^2} + \frac{c'^2}{a'^2}\right) - \frac{5.7}{2.3}\frac{b^2 + b'^2}{a^2}\right] \\
&+ \frac{3^2 5^2 7}{4(2.4.6)^2}\frac{a'^6}{a^6}\left[1 + \frac{7.8}{2.3}\left(\frac{c^2}{a^2} + \frac{c'^2}{a'^2}\right) - \frac{7.9}{2.3}.\frac{b^2 + b'^2}{a^2}\right] \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

The series is always converging, for the ratio $\frac{a'}{a}$ is less than unity.

In order to abbreviate the calculations, we shall put, for example,

$$\alpha=\frac{a'^2}{a^2},$$

$$\beta=\frac{1}{2.3}\left[\frac{b^2}{a^2}+\frac{b'^2}{a^2}\right]=\frac{1}{2.3}\left[\frac{b^2}{a^2}+\alpha\frac{b'^2}{a'^2}\right],$$

$$\gamma=\frac{1}{2.3}\left(\frac{c^2}{a^2}+\frac{c'^2}{a'^2}\right).$$

We have then, for the first terms,

$$\begin{aligned}\frac{M}{M_0}=1+2\gamma-3\beta-\frac{3x}{2a^2}&\\ &+0\text{·}375000\alpha\,[1+12\gamma-15\beta]\\ &+0\text{·}234375\alpha^2[1+30\gamma-35\beta]\\ &+0\text{·}170900\alpha^3[1+56\gamma-63\beta]\\ &+0\text{·}134580\alpha^4[1+90\gamma-99\beta]\\ &+0\text{·}111103\alpha^5[1+132\gamma-143\beta]\\ &+0\text{·}094510\alpha^6[1+182\gamma-195\beta].\end{aligned} \tag{18}$$

An analogous expression appended to expression (41) of (**741**) will give the coefficient of mutual induction of two very distant coils; with the expression (44) of (**742**) we shall in like manner obtain the coefficient of induction of a long coil on another coil in the interior.

When the convergence of the expressions thus obtained is not sufficiently rapid, recourse should be had to the use of elliptic integrals, of which they are indeed only the development in series.

763. Determination of M by Elliptic Integrals.—If r be the distance of the two elements ds and ds' of the contours, ϵ the angle which these two elements make between them, we might calculate the coefficient of mutual induction by Neumann's formula

$$M=\int\int\frac{\cos\epsilon}{r}\,ds\,ds'.$$

If ϕ and ϕ' are the angles which the elements ds and ds' make respectively with a fixed plane passing through the axis, the distance r is given by equation.

$$r^2=x^2+a^2+a'^2-2aa'\cos(\phi-\phi').$$

Since we have

$$\epsilon=\phi-\phi',\qquad ds=a\,d\phi,\qquad ds'=a'\,d\phi',$$

the substitution of these values in Neumann's formula will give

$$M=\int_0^{2\pi}\int_0^{2\pi}\frac{aa'\cos(\phi-\phi')d\phi d\phi'}{\sqrt{a^2+a'^2+x^2-2aa'\cos(\phi-\phi')}}.$$

This integral is given by the expression

$$(19)\qquad M=4\pi\sqrt{aa'}\left[\left(k-\frac{2}{k}\right)F+\frac{2}{k}E\right],$$

in which

$$k=\frac{2\sqrt{aa'}}{\sqrt{(a+a')^2+x^2}}$$

and F and E denote complete elliptic integrals of the first and second kind with the modulus k.

If r_1 and r_2 are the extreme values of r, we have

$$r_1^2=(a+a')^2+x^2,\qquad r_2^2=(a-a')^2+x^2.$$

If we put

$$\cos\gamma=\frac{r_2}{r_1},\qquad\text{or}\qquad k=\sin\gamma,$$

formula (19) may be written

$$(20)\qquad M=4\pi\sqrt{aa'}\left[\left(\sin\gamma-\frac{2}{\sin\gamma}\right)F+\frac{2}{\sin\gamma}E\right].$$

Taking $k_1=\frac{r_1-r_2}{r_1+r_2}$, which gives $k=\frac{2\sqrt{k}}{1+k_1}$, we have, in the case of the complete integrals, the ratios

$$F(k)=(1+k_1)F(k_1),$$

$$E(k)=\frac{2}{1+k_1}E(k_1)-(1-k_1)F(k_1).$$

The substitution of these values in equation (19) gives, for the value of M, an expression which is sometimes more advantageous:

$$(21)\qquad M=8\pi\sqrt{aa'}\frac{1}{\sqrt{k_1}}\left[E(k_1)-F(k_1)\right].$$

Writing equation (20) in the form

$$(22)\qquad \frac{M}{4\pi\sqrt{aa'}}=\frac{1}{\sin\gamma}\left[2E-(1+\cos^2\gamma)F\right],$$

we see that the second expression is simply a function of the angle γ. The following tables, taken from the second edition of Maxwell's Treatise, give logarithms of the values of the first number for values of γ, varying from 6′ to 6′, from 60° to 90°.

Table of the values of log. $\frac{M}{4\pi\sqrt{aa'}}$.

γ	Log. $\frac{M}{4\pi\sqrt{aa'}}$.	γ	Log. $\frac{M}{4\pi\sqrt{aa'}}$.	γ	Log. $\frac{M}{4\pi\sqrt{aa'}}$.
60° 0′	$\bar{1}$·4994783	65° 0′	$\bar{1}$·6376629	70° 0′	$\bar{1}$·7758000
60 6	$\bar{1}$·5022651	65 6	$\bar{1}$·6404137	70 6	$\bar{1}$·7785903
60 12	$\bar{1}$·5050505	65 12	$\bar{1}$·6431645	70 12	$\bar{1}$·7813823
60 18	$\bar{1}$·5078345	65 18	$\bar{1}$·6459153	70 18	$\bar{1}$·7841762
60 24	$\bar{1}$·5106173	65 24	$\bar{1}$·6486660	70 24	$\bar{1}$·7869720
60 30	$\bar{1}$·5133989	65 30	$\bar{1}$·6514169	70 30	$\bar{1}$·7897696
60 36	$\bar{1}$·5161791	65 36	$\bar{1}$·6541678	70 36	$\bar{1}$·7925692
60 42	$\bar{1}$·5189582	65 42	$\bar{1}$·6569189	70 42	$\bar{1}$·7953709
60 48	$\bar{1}$·5217361	65 48	$\bar{1}$·6596701	70 48	$\bar{1}$·7981745
60 54	$\bar{1}$·5245128	66 54	$\bar{1}$·6624215	70 54	$\bar{1}$·8009803
61 0	$\bar{1}$·5272883	66 0	$\bar{1}$·6651732	71 0	$\bar{1}$·8037882
61 6	$\bar{1}$·5300628	66 6	$\bar{1}$·6679250	71 6	$\bar{1}$·8065983
61 12	$\bar{1}$·5328361	66 12	$\bar{1}$·6706772	71 12	$\bar{1}$·8094107
61 18	$\bar{1}$·5356084	66 18	$\bar{1}$·6734296	71 18	$\bar{1}$·8122253
61 24	$\bar{1}$·5383796	66 24	$\bar{1}$·6761824	71 24	$\bar{1}$·8150423
61 30	$\bar{1}$·5411498	66 30	$\bar{1}$·6789356	71 30	$\bar{1}$·8178617
61 36	$\bar{1}$·5439190	66 36	$\bar{1}$·6816891	71 36	$\bar{1}$·8206836
61 42	$\bar{1}$·5466872	66 42	$\bar{1}$·6844431	71 42	$\bar{1}$·8235080
61 48	$\bar{1}$·5494545	66 48	$\bar{1}$·6871976	71 48	$\bar{1}$·8263349
61 54	$\bar{1}$·5522209	66 54	$\bar{1}$·6899526	71 54	$\bar{1}$·8291645
62 0	$\bar{1}$·5549864	67 0	$\bar{1}$·6927081	72 0	$\bar{1}$·8319967
62 6	$\bar{1}$·5577510	67 6	$\bar{1}$·6954642	72 6	$\bar{1}$·8348316
62 12	$\bar{1}$·5605147	67 12	$\bar{1}$·6982209	72 12	$\bar{1}$·8376693
62 18	$\bar{1}$·5632776	67 18	$\bar{1}$·7009782	72 18	$\bar{1}$·8405099
62 24	$\bar{1}$·5660398	67 24	$\bar{1}$·7037362	72 24	$\bar{1}$·8433534
62 30	$\bar{1}$·5688011	67 30	$\bar{1}$·7064949	72 30	$\bar{1}$·8461998
62 36	$\bar{1}$·5715618	67 36	$\bar{1}$·7092544	72 36	$\bar{1}$·8490493
62 42	$\bar{1}$·5743217	67 42	$\bar{1}$·7120146	72 42	$\bar{1}$·8519018
62 48	$\bar{1}$·5770809	67 48	$\bar{1}$·7147756	72 48	$\bar{1}$·8547575
62 54	$\bar{1}$·5798394	67 54	$\bar{1}$·7175375	72 54	$\bar{1}$·8576164
63 0	$\bar{1}$·5825973	68 0	$\bar{1}$·7203003	73 0	$\bar{1}$·8604785
63 6	$\bar{1}$·5853546	68 6	$\bar{1}$·7230640	73 6	$\bar{1}$·8633440
63 12	$\bar{1}$·5881113	68 12	$\bar{1}$·7258286	73 12	$\bar{1}$·8662129
63 18	$\bar{1}$·5908675	68 18	$\bar{1}$·7285942	73 18	$\bar{1}$·8690852
63 24	$\bar{1}$·5936231	68 24	$\bar{1}$·7313609	73 24	$\bar{1}$·8719611
53 30	$\bar{1}$·5963782	68 30	$\bar{1}$·7341287	73 30	$\bar{1}$·8748406
63 36	$\bar{1}$·5991329	68 36	$\bar{1}$·7368975	73 36	$\bar{1}$·8777237
63 42	$\bar{1}$·6018871	68 42	$\bar{1}$·7396675	73 42	$\bar{1}$·8806106
63 48	$\bar{1}$·6046408	68 48	$\bar{1}$·7424387	73 48	$\bar{1}$·8835013
63 54	$\bar{1}$·6073942	68 54	$\bar{1}$·7452111	73 54	$\bar{1}$·8863958
64 0	$\bar{1}$·6181472	69 0	$\bar{1}$·7479848	74 0	$\bar{1}$·8892943
64 6	$\bar{1}$·6128998	69 6	$\bar{1}$·7507597	74 6	$\bar{1}$·8921969
64 12	$\bar{1}$·6156522	69 12	$\bar{1}$·7535361	74 12	$\bar{1}$·8951036
64 18	$\bar{1}$·6184042	69 18	$\bar{1}$·7563138	74 18	$\bar{1}$·8980144
64 24	$\bar{1}$·6211560	69 24	$\bar{1}$·7590929	74 24	$\bar{1}$·9009295
64 30	$\bar{1}$·6239076	69 30	$\bar{1}$·7618735	74 30	$\bar{1}$·9038489
64 36	$\bar{1}$·6266589	69 36	$\bar{1}$·7646556	74 36	$\bar{1}$·9067728
64 42	$\bar{1}$·6294101	69 42	$\bar{1}$·7674392	74 42	$\bar{1}$·9097012
64 48	$\bar{1}$·6321612	69 48	$\bar{1}$·7702195	74 48	$\bar{1}$·9126341
64 54	$\bar{1}$·6340121	69 54	$\bar{1}$·7730114	74 54	$\bar{1}$·9155717

Table of the values of $\log.\ \dfrac{M}{4\pi\sqrt{aa'}}$.

γ	$\text{Log}\ \dfrac{M}{4\pi\sqrt{aa'}}$.	γ	$\text{Log}\ \dfrac{M}{4\pi\sqrt{aa'}}$.	γ	$\text{Log}.\ \dfrac{M}{4\pi\sqrt{aa'}}$.
° ′		° ′		° ′	
75 0	$\bar{1}$·9185141	80 0	0·0741816	85 0	0·2654152
75 6	$\bar{1}$·9214613	80 6	0·0775316	85 6	0·2700156
75 12	$\bar{1}$·9244135	80 12	0·0808944	85 12	0·2746655
75 18	$\bar{1}$·9273707	80 18	0·0842702	85 18	0·2793670
75 24	$\bar{1}$·9303330	80 24	0·0876592	85 24	0·2841221
75 30	$\bar{1}$·9333005	80 30	0·0910619	85 30	0·2889329
75 36	$\bar{1}$·9362733	80 36	0·0944784	85 36	0·2938018
75 42	$\bar{1}$·9392515	80 42	0·0979091	85 42	0·2987312
75 48	$\bar{1}$·9422352	80 48	0·1013542	85 48	0·3037238
75 54	$\bar{1}$·9452246	80 54	0·1048142	85 54	0·3087823
76 0	$\bar{1}$·9482196	81 0	0·1082893	86 0	0·3139097
76 6	$\bar{1}$·9512205	81 6	0·1117799	86 6	0·3191092
76 12	$\bar{1}$·9542272	81 12	0·1152863	86 12	0·3243843
76 18	$\bar{1}$·9572400	81 18	0·1188089	86 18	0·3297387
76 24	$\bar{1}$·9602590	81 24	0·1223481	86 24	0·3351762
76 30	$\bar{1}$·9632841	81 30	0·1259043	86 30	0·3407012
76 36	$\bar{1}$·9663157	81 36	0·1294778	86 36	0·3463184
76 42	$\bar{1}$·9693537	81 42	0·1330691	86 42	0·3520327
76 48	$\bar{1}$·9723983	81 48	0·1366786	86 48	0·3578495
76 54	$\bar{1}$·9754497	81 54	0·1403067	86 54	0·3637749
77 0	$\bar{1}$·9785079	82 0	0·1439539	87 0	0·3698153
77 6	$\bar{1}$·9815731	82 6	0·1476207	87 6	0·3759777
77 12	$\bar{1}$·9846454	82 12	0·1513075	87 12	0·3822700
77 18	$\bar{1}$·9877249	82 18	0·1550149	87 18	0·3887006
77 24	$\bar{1}$·9908118	82 24	0·1587434	87 24	0·3952792
77 30	$\bar{1}$·9939062	82 30	0·1624935	87 30	0·4020162
77 36	$\bar{1}$·9970082	82 36	0·1662658	87 36	0·4089234
77 42	0·0001181	82 42	0·1700609	87 42	0·4160138
77 48	0·0032359	82 48	0·1738794	87 48	0·4233022
77 54	0·0063618	82 54	0·1777219	87 54	0·4308053
78 0	0·0094959	83 0	0·1815890	88 0	0·4385420
78 6	0·0126385	83 6	0·1854815	88 6	0·4465341
78 12	0·0157896	83 12	0·1894001	88 12	0·4548064
78 18	0·0189494	83 18	0·1933455	88 18	0·4633880
78 24	9·0221181	83 24	0·1973184	88 24	0·4723127
78 30	0·0252959	83 30	0·2013197	88 30	0·4816206
78 36	0·0284830	83 36	0·2053502	88 36	0·4913595
78 42	0·0316794	83 42	0·2094108	88 42	0·5015870
78 48	0·0348855	83 48	0·2135026	88 48	0·5123738
78 54	0·0381014	83 54	0·2176259	88 54	0·5238079
79 0	0·0413273	84 0	0·2217823	89 0	0·5360007
79 6	0·0445633	84 6	0·2259728	89 6	0·5490969
79 12	0·0478098	84 12	0·2301983	89 12	0·5632886
79 18	0·0510668	84 18	0·2344600	89 18	0·5788406
79 24	0·0543347	84 24	0·2387591	89 24	0·5961320
79 30	0·0576136	84 30	0·2430970	89 30	0·6157370
79 36	0·0609037	84 36	0·2474748	89 36	0·6385907
79 42	0·0642054	84 42	0·2518940	89 42	0·6663883
79 48	0·0675187	84 48	0·2563561	89 48	0·7027765
79 54	0·0708441	84 54	0·2608626	89 54	0·7586941

764. We shall pass from the case of two circles to that of a coil and a circle, or of two coils with the same axis, by the method of (743).

If a and a' are the mean radii of the two coils, and x the distance of their median planes, $2c$ and $2b$, $2c'$ and $2b'$, the dimensions of the two channels; then if M_0 is the coefficient relative to the two mean circles,

$$M = M_0 + \frac{1}{6}\left\{\frac{\partial^2 M_0}{\partial y^2}(c^2 + c'^2) + \frac{\partial^2 M_0}{\partial x^2}(b^2 + b'^2)\right\} + \ldots$$

If M_0 is expressed by formula (19), we find, taking into account known ratios,

$$\frac{\partial F}{\partial k} = \frac{E}{k(1-k^2)} - \frac{F}{k}, \quad \text{and} \quad \frac{\partial E}{\partial k} = \frac{1}{k}(E - F),$$

relative to the complete integrals,

$$\frac{\partial^2 M_0}{\partial y^2} = -\frac{\pi}{\sqrt{aa'}}\left\{\frac{E}{2\sqrt{aa'}(1-k^2)}\left[2\sqrt{aa'}k + \frac{4k^2x^2}{8\sqrt{aa'}(1-k^2)}(1 - 3k^2 + 2k^3)\right] - F\left(k + \frac{k^3x^2}{4aa'}\right)\right\},$$

$$\frac{\partial^2 M_0}{\partial x^2} = \frac{\pi k}{\sqrt{aa'}(1-k^2)}\left\{F\left[2(1 - k^2 - \frac{k^2x^2}{4aa'}(2 - k^2)\right] - E\left[2 - k^2 - k^2x^2\frac{1 - k^2 + k^4}{2aa'(1-k^2)}\right]\right\}.$$

765. The calculation will be more easily made by means of the tables, if we use the following method due to Lord Rayleigh.*

Let n_1 and n' be the numbers of windings of the coils.
a and a' their mean radii.
x the distance of the centres.
$2b$, $2c$, $2b'$, and $2c'$ the dimensions of the channels.

* MAXWELL. *Electricity and Magnetism.* 2nd Edition, Vol. II., p. 326.

If $M_0 = f(aa'x)$ denotes the coefficient of mutual iuduction of the two mean circles, we shall have

$$M = \frac{1}{6} nn' \left\{ \begin{array}{l} f(a+c, a', x) + f(a-c, a', x) \\ + f(a, a'+c', x) + f(a, a'-c', x) \\ + f(a, a', x+b) + f(a, a', x-b) \\ + f(a, a', x+b') + f(a, a', x-b') \\ - 2f(a, a', x) \end{array} \right\}.$$

We have only then to calculate the nine values of ψ, which correspond to nine values of the function f.

766. Use of Legendre's Polynomials.—Let us suppose that a system is symmetrical about an axis, and that the potential U_0 of this system at a point P_0 on the axis at a distance x from an arbitrary origin O, has been expressed by a converging series of the form

$$U_0 = A_0 + \frac{B_0}{x} + A_1 x + \frac{B_1}{x^2} + A_2 x^2 + \frac{B_2}{x^3} + \ldots,$$

where A_0, $A_1 \ldots\ldots$, B_0, $B_1 \ldots\ldots$ are constant coefficients. If we can pass from the point P_0 to a point V outside the axis without meeting acting masses, the potential U of the system at the point V is represented by what is also the convergent series,

$$(23) \quad U = A_0 + \frac{B_0}{r} + \left(A_1 r + \frac{B_1}{r^2}\right) X_1 + \left(A_2 r^2 + \frac{B_2}{r^3}\right) X_2 + \ldots,$$

in which r denotes the distance OP, and X_1, $X_2 \ldots\ldots$, X_n the polynomials of Legendre (377) or the harmonic functions of 1, 2 ... n order of the angle θ, which the right line OP makes with the axis of symmetry.

If we put

$$\mu = \cos\theta,$$

from which

$$d\mu = -\sin\theta d\theta,$$

and consider the polynomials Xn as functions of μ, Laplace's equation gives the general condition

$$n(n+1) X_n + \frac{\delta}{\delta\mu}\left[(1-\mu^2)\frac{\delta X_n}{\delta\mu}\right] = 0;$$

we get from this

$$(24) \qquad \int_\mu^1 X_n d\mu = \frac{1-\mu^2}{n(n+1)} \frac{\partial X_n}{\partial \mu} = \frac{1-\mu^2}{n(n+1)} X'_n(\theta).$$

As the functions Xn and their differentials in respect of μ play an important part in this mode of calculation, we shall give the values of the former:

$$X_1 = \mu,$$

$$X_2 = \frac{1}{2}(3\mu^2 - 1),$$

$$X_3 = \frac{\mu}{2}(5\mu^2 - 3),$$

$$X_4 = \frac{1}{4}\left(\frac{5 \cdot 7}{2}\mu^4 - 3 \cdot 5\mu^2 + \frac{3}{2}\right),$$

$$X_5 = \frac{\mu}{4}\left(\frac{7 \cdot 9}{2}\mu^4 - 5 \cdot 7\mu^2 + \frac{3 \cdot 5}{2}\right),$$

$$X_6 = \frac{1}{8}\left(\frac{7 \cdot 9 \cdot 11}{2 \cdot 3}\mu^6 - \frac{5 \cdot 7 \cdot 9}{2}\mu^4 + \frac{3 \cdot 5 \cdot 7}{2}\mu^2 - \left(\frac{5}{2}\right)\right);$$

$$X'_1 = 1,$$

$$X'_2 = 3\mu,$$

$$X'_3 = \frac{3}{2}(5\mu^2 - 1),$$

$$X'_4 = \frac{5}{2}\mu(7\mu^2 - 3),$$

$$X'_5 = \frac{15}{8}(21\mu^4 - 14\mu^2 + 1),$$

$$X'_6 = \frac{21}{8}\mu(33\mu^4 - 30\mu^2 + 5).$$

We have further for $\theta = 0$, or $\mu = 1$,

$$X_n = 1, \qquad X'_{2n} = n(2n+1),$$
$$X'_{2n+1} = (n+1((2n+1);$$

and for $\theta=\frac{\pi}{2}$, or $\mu=0$,

$$X_{2n}=(-1)^n\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\ldots 2n},\quad X'_{2n}=0,$$

$$X_{2n+1}=0,\qquad X'_{2n+1}=(-)^n\frac{1\cdot 3\cdot 5\ldots(2n+1)}{2\cdot 4\ldots 2n}.$$

767. Uniform Circular Layer.—The potential U of a uniform circular layer (367) of density equal to unity, and of radius a at a distance r from the centre, may be expressed by one or other of the two series,

$$(25)\quad \begin{aligned} U &= 2\pi a\left[1-\frac{r}{a}X_1+\frac{1}{2}\frac{r^2}{a^2}X_2+\ldots\right.\\ &\qquad \left.+(-)^{n-1}\frac{1.1.3\ldots(2n-3)}{2.4\ldots 2n}\left(\frac{r}{a}\right)^{2n}X_{2n}\right],\\ U &= 2\pi a\left[\frac{1}{2}\frac{a}{r}-\frac{1.1}{2.4}\frac{a^3}{r^3}X_2+\ldots\right.\\ &\qquad \left.+(-1)^n\frac{1.1.3\ldots(2n-1)}{2.4\ldots(2n+2)}\left(\frac{a}{r}\right)^{2n+1}X_{2n}\right],\end{aligned}$$

according as we have $r \lesseqgtr a$.

We will consider more particularly the second of these expressions, and, calling the bracket $\phi(\theta)$, we shall write

$$(26)\qquad U=2\pi a\phi(\theta).$$

The components X and Y of the action of the layer on a point P, the co-ordinates of which in respect of the axis and a perpendicular to the axis are x and y, have the value

$$X=-2\pi a\frac{\partial\phi(\theta)}{\partial x},$$

$$Y=-2\pi a\frac{\partial\phi(\theta)}{\partial y}.$$

Since we have

$$x=r\cos\theta,\qquad \frac{\partial r}{\partial x}=\cos\theta,\qquad \frac{\partial\theta}{\partial x}=-\frac{\sin\theta}{r},$$

$$y=r\sin\theta,\qquad \frac{\partial r}{\partial y}=\sin\theta,\qquad \frac{\partial\theta}{\partial y}=\frac{\cos\theta}{r},$$

it follows that

$$\frac{\partial X_n}{\partial x} = \frac{X'_n}{r}\sin^2\theta,$$

$$\frac{\partial X_n}{\partial y} = -\frac{X'_n}{r}\sin\theta\cos\theta.$$

Utilising, lastly, the known values

$$(27)\qquad \begin{aligned} X_{n+1} &= \mu X_n - \frac{1-\mu^2}{n+1}X'_n,\\ X'_{n+1} &= (n+1)X_n + \mu X'_n, \end{aligned}$$

we get

$$(28)\qquad \begin{aligned} X &= \frac{2S}{r^2}\left[\frac{1}{2}X_1 - \frac{1.3}{2.4}\frac{a^2}{r^2}X_3 + \dots \right.\\ &\qquad \left. + (-1)^n\frac{1.3\dots(2n+1)}{2.4\dots(2n+2)}\left(\frac{a}{r}\right)^{2n}X_{2n+1}\right],\\ Y &= \frac{2S}{r^2}\sin\theta\left[\frac{1}{2}X'_1 - \frac{1.1}{2.4}\frac{a^2}{r^2}X'_3 + \dots \right.\\ &\qquad \left. + (-1)^n\frac{1.1.3\dots(2n-1)}{2.4\dots(2n+2)}\left(\frac{a}{r}\right)^{2n}X'_{2n+1}\right]. \end{aligned}$$

We may put

$$(29)\qquad \begin{aligned} X &= \frac{2S}{r^2}\chi(\theta),\\ Y &= \frac{2S}{r^2}\psi(\theta). \end{aligned}$$

In the plane of the layer in particular, where $\theta = \frac{\pi}{2}$, we have

$$(3)\qquad \begin{aligned} U &= \frac{S}{r}\left[1 + \frac{1}{2}\left(\frac{1}{2}\right)^2\frac{a^2}{r^2} + \dots + \frac{1}{n+1}\left(\frac{1.3\dots(2n-1)}{2.4\dots 2n}\right)^2\left(\frac{a}{r}\right)^{2n}\right],\\ X &= 0,\\ Y &= \frac{S}{r^2}\left[1 + \frac{3}{2}\left(\frac{1}{2}\right)^2\frac{a^2}{r^2} + \dots + \frac{2n+1}{n+1}\left(\frac{1.3\dots(2n-1)}{2.4\dots 2n}\right)^2\left(\frac{a}{r}\right)^{2n}\right]. \end{aligned}$$

768. Circular Current.—From equation (14) of (367) the potential on the axis of a circular current of radius a, and intensity equal to unity, is for $x>a$

$$V_0=2\pi\left[\frac{1}{2}\frac{a^2}{x^2}-\frac{1.3}{2.4}\frac{a^4}{x^4}+\ldots+(-1)^n\frac{1.3\ldots(2n+1)}{2.4\ldots(2n+2)}\left(\frac{a}{r}\right)^{2n+2}\right].$$

The potential at the point P beyond the axis is then

$$V=2\pi\left[\frac{1}{2}\frac{a^2}{r^2}X_1-\frac{1.3}{2.4}\frac{a^4}{r^4}X_3+\ldots+(-1)^n\frac{1.3\ldots(2n+1)}{2.4\ldots(2n+2)}\left(\frac{a}{r}\right)^{2n+2}X_{2n+1}\right]$$

The components of the action are then, taking into account equations (27),

$$(31)\quad \begin{aligned} X&=\frac{2S}{r^3}\left[X_2-\frac{3}{2}\frac{a^2}{r^2}X_4+\ldots+(-1)^n\frac{1.3\ldots(2n+1)}{2.4\ldots 2n}\left(\frac{a}{r}\right)^{2n}X_{2n+2}\right],\\ Y&=\frac{2S}{r^3}\sin\theta\left[\frac{1}{2}X_2'-\frac{1.3a^2}{2.4r^2}X_4'\ldots+(-1)^n\frac{1.3\ldots(2n+1)}{2.4\ldots(2n+2)}\left(\frac{a}{r}\right)^{2n}X_{2n+2}'\right].\end{aligned}$$

When the point P is very near the plane of the current, but at a greater distance from the centre of the circle than the radius a, we may put $\theta=\frac{\pi}{2}-\delta$, the angle δ being very small. Neglecting then terms of the order of δ^3, we get

$$(32)\quad \begin{aligned} X&=-\frac{S}{r^3}\left[1+\frac{1}{2}\left(\frac{3}{2}\right)^2\frac{a^2}{r^2}+\ldots+\frac{1}{n+1}\left(\frac{1.3\ldots(2n+1)}{2.4\ldots 2n}\right)^2\left(\frac{a}{r}\right)^{2n}\right.\\ &\qquad\left.-3\,\delta^2\left(1+\frac{3.5}{2^2}\frac{a^2}{r^2}+\frac{3.5^2.7}{(2.4)^2}\frac{a^4}{r^4}\right)\right],\\ Y&=3\frac{S}{r^3}\,\delta\left[1+\frac{3.5}{2.2^2}\frac{a^2}{r^2}+\frac{5^2.7}{(2.4)^2}\frac{a^4}{r^4}\right].\end{aligned}$$

769. External Potential of a Long Coil.—Let us consider a cylindrical coil C (Fig. 141) of radius a, of length h, and formed of a single layer of wires comprising n_1 turns for unit length. The potential at an external point P is the same as that of the two bases A and B on which are the magnetic layers of densities respectively equal to $+n_1$ and $-n_1$; that is to say, from equation (26),

$$V=2\pi a n_1\left[\phi(\theta)-\phi(\theta')\right],$$

and the components of the action are

$$X = -2\pi a n_1 \left[\frac{\partial\phi(\theta)}{\partial x} - \frac{\partial\phi(\theta')}{\partial x}\right],$$

$$Y = -2\pi a n_1 \left[\frac{\partial\phi(\theta)}{\partial y} - \frac{\partial\phi(\theta')}{\partial y}\right],$$

or from equation (29)

$$(33) \qquad X = \frac{2Sn_1}{r^2}\left[\chi(\theta) - \chi(\theta')\right], \qquad Y = \frac{2Sn_1}{r^2}\left[\psi(\theta) - \psi(\theta')\right].$$

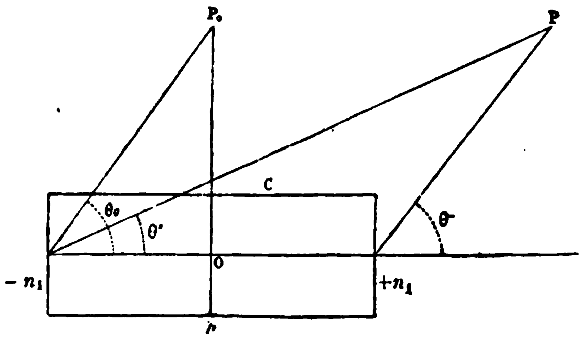


Fig. 141.

It will be sufficient then to replace the functions ψ, X, and ψ, in each particular case, by their values given above.

It is clear that generally

$$\phi(\theta) = \phi(\pi - \theta),$$
$$\chi(\theta) = -\chi(\pi - \theta),$$
$$\psi(\theta) = \psi(\pi - \theta).$$

For a point P_0, in the mean plane of the coil, we shall have then

$$(34) \qquad V = 0, \qquad X = -\frac{2Sn_1}{r^2}\, 2\chi(\theta_0), \qquad Y = 0.$$

The total number of turns being $n = n_1 h$, we may write,

$$X = -\frac{2Sn}{r^2 h} 2\chi(\theta_0), \tag{35}$$

If the angle θ_0 is very nearly a right angle—that is to say, if the ratio of the distance OP_0 to the semi length $\frac{h}{2}$ of the cylinder is very great—we have sensibly

$$2\chi(\theta_0) = \cos\theta_0 = \frac{h}{2r},$$

and therefore

$$X = -\frac{nS}{r^3},$$

which was obvious (153).

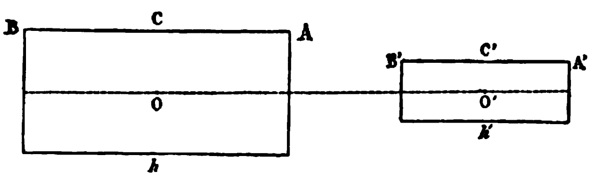


Fig. 142.

In order to obtain the action of a coil of thickness $2c$, the value of X must be expanded in increasing powers of $\frac{a}{r}$, then multiplied by $n_2 da$, and integrated between the limits $a - c$ and $a + c$.

770. Mutual Induction of Two Long Coils.—The value of the action in the interior of such a coil, if it were indefinitely long (551), would be $4\pi n_1$; its coefficient of induction on a parallel coil C′ with the same axis in the interior of section S′, and with n' turns, would be expressed by

$$M = 4\pi n_1 n' S'.$$

If the coil has a finite length, the flow of induction corresponding to the two bases must be subtracted from this value.

In order to treat the problem directly, we may first assume that the coil C′, having the same axis as the first, is on the outside (Fig. 142).

The coefficient of mutual induction is equal and of opposite sign (454) to the potential energy of the four magnetic layers A and B, A′ and B′, the densities of which are respectively $+n$ and $-n_1$, $+n_1'$ and $-n_1'$. Let x be the distance of the bases A and B′, P_m the mean potential of the layer A for a mass equal to unity on the surface B′; the mean potential of this coil on the surface is

$$n_1\left[P_m(x) - P_m(x+h)\right].$$

The potential energy of the layer B′ in respect of the coil C is then,

$$-n_1 n_1' S'\left[P_m(x) - P_m(x+h)\right].$$

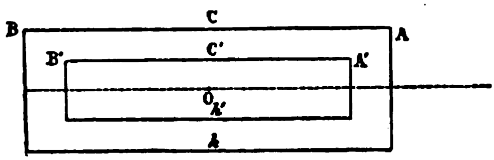


Fig. 143.

The potential energy of the layer A, where h' is the length of the coil C′, is in like manner

$$n_1 n_1' S'\left[P_m(x+h') - P_m(x+h'+h)\right].$$

From this follows for the coefficient of mutual induction of the two coils,

$$(36) \qquad M = n_1 n_1' S'\left[P_m(x) - P_m(x+h - P_m(x+h') + P_m(x+h+h')\right].$$

The values P_n might be calculated in various ways, according to particular cases.

771. Let us suppose the two coils concentric, the second having a smaller radius and being shorter (Fig. 143), the distance of the bases being x,

The mean potential of the coil C on the base B′, may be expressed by

$$-4\pi n_1 x - n_1 P_m(x) + n_1 P_m(h-x),$$

and the potential energy of the base B′ by

$$n_1 n_1' S' \left[4\pi x + P_m(x) - P_m(h - x) \right].$$

The potential energy of the base A′ is in like manner

$$- n_1 n_1' S' \left[4\pi(x + h') + P_m(x + h') - P_m(x) \right],$$

which gives for the coefficient of mutual induction

$$\begin{aligned} \text{(37)} \quad M &= n_1 n_1' S' \left[4\pi h' + P_m(h - x) + P_m(x + h') - 2P_m(x) \right] \\ &= 4\pi n_1 n' S' - n_1 n_1' S' \left[2P_m(x) - P_m(h - x) - P_m(h' + x) \right]. \end{aligned}$$

772. If, lastly, the coils are of equal lengths and concentric, $x = 0$ and $h' = h$; we have then

$$\begin{aligned} \text{(38)} \quad M &= 4\pi n_1 n' S' - 2n_1 n_1' S' \left[P_m(0) - P_m(h) \right] \\ &= 4\pi n_1 n' S' - 2n_1 n' S' \frac{1}{h} \left[P_m(0) - P_m(h) \right]. \end{aligned}$$

In order to calculate the potential P_m (0), we may utilise the former of these equations (25), which gives, making $\theta = \frac{\pi}{2}$,

$$V = 2\pi a \left[1 + \frac{1}{4} \frac{r^2}{a^2} - \ldots + (-1)^{n-1} \left(\frac{1.3\ldots(2n-3)}{2.4\ldots 2n} \right)^2 (2n - 1) \left(\frac{a}{r} \right)^{2n} \right].$$

From this follows by the ordinary rule, and making $r = a'$,

$$P_m(0) = 2\pi a \left[1 + \frac{1}{4} \frac{1}{2} \frac{a'^2}{a^2} - \ldots + (-1)^{n-1} \left(\frac{1.3\ldots(2n-3)}{2.4\ldots 2n} \right)^2 \frac{2n - 1}{n + 1} \left(\frac{a'}{a} \right)^{2n} \right].$$

In order to get the value of $P_m(h)$, we might develope the potential of a circle as a function of the ratio $\frac{a}{h}$, taking the expression (736)

$$\begin{aligned} P_m(x) &= 2\pi \left(f_0 + \frac{y^2}{2} f_1 + \frac{y^4}{3} f_2 + \ldots \right) \\ &= 2\pi \left[u - x - \frac{y^2}{2.2^2} \frac{d^2 u}{dx^2} + \frac{y^4}{3(2.4)^2} \frac{d^4 u}{dx^4} - \ldots \right]. \end{aligned}$$

If we expand $\mu = \sqrt{a^2+x^2}$ as a function of increasing powers of $\frac{1}{x}$, we have

$$u = x + \frac{1}{2}\frac{a^2}{x} - \frac{1.1}{2.4}\frac{a^4}{x^3} + \ldots + (-1)^{n+1}\frac{1.1.3..(2n-3)}{2.4\ldots 2n}\frac{a^{2n}}{x^{2n-1}},$$

which gives, by carrying out the calculations, and making $x = l_1 y = a'$,

$$\begin{aligned} P_m(h) = 2\pi a \frac{a}{h}\Bigg\{ & \frac{1}{2} - \frac{1.1}{2.4}\frac{a^2}{h^2} + \frac{1.1.3}{2.4.6}\cdot\frac{a^4}{h^4} - \ldots \\ & - \frac{1}{2.2^2}\frac{a'^2}{h^2}\left[\frac{2}{2} - \frac{3.4}{2.4}\frac{a^2}{h^2} + \frac{3.5.6}{2.4.6}\cdot\frac{a^4}{h^4} - \right] \\ & + \frac{1}{3(2.4)^2}\frac{a'^4}{h^4}\left[\frac{2.3.4}{2} - \frac{3.4.5.6}{2.4}\cdot\frac{a^2}{h^2} + \ldots\right] \\ & - \frac{1}{4(2.4.6)^2}\frac{a'^6}{h^6}\left[\frac{2.3.4.5.6}{2} - \frac{3.4.5.6.7.8}{2.4}\frac{a^2}{h^2} + \ldots\right] \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \Bigg\}. \end{aligned}$$

If the ratios $\frac{a}{h}$ and $\frac{a'^2}{a^2}$ are of the same order, and we only take the first term into account, we get

$$(39) \qquad M = 4\pi n_1 n' S' \left[1 - \frac{a}{h}\left(1 + \frac{1}{8}\frac{a'^2}{a^2} - \frac{a}{2h}\right)\right].$$

773. Potential of a Homogeneous Spherical Bowl.—Let us consider a homogeneous spherical layer, of unit density, bounded by a small circle HK (Fig. 144). Let C be the centre of the sphere, O the centre of the circle, A the pole of the circle.

Let

$$CH = u,$$
$$OH = a,$$
$$OCH = \alpha,$$
$$CP_0 = x,$$

and let us investigate the potential of the layer at P_0.

The value of the potential at P_0 of an element dS of the layer situate at the point M, on a radius which makes the angle θ with the axis, is

$$\frac{dS}{MP}=\frac{dS}{(x^2+u^2-2ux\cos\theta)^{\frac{1}{2}}}.$$

According as we have $x \lesseqgtr u$, we may express this potential by one or the other of the two converging series,

$$(40)\qquad \begin{aligned}\frac{dS}{MP}&=\frac{dS}{u}\left[1+\frac{x}{u}X_1+\frac{x^2}{u^2}X_2+\dots\right]\\ &=\frac{dS}{x}\left[1+\frac{u}{x}X_1+\frac{u^2}{x^2}X_2+\dots\right].\end{aligned}$$

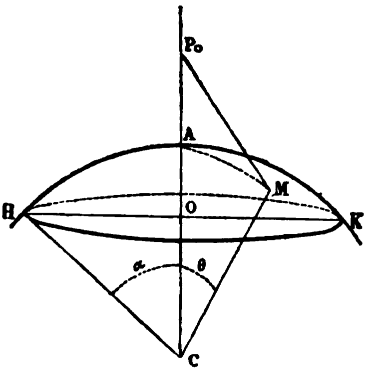


Fig. 144.

If ϕ is the angle of the two planes OCH and OCM, and putting $\mu=\cos\theta$, we may write

$$dS=u^2\sin\theta d\theta d\phi=-u^2 d\mu d\phi.$$

The potential U_0 of the spherical layer will be obtained by integrating the expression of $\frac{dS}{MP}$ in respect of ϕ from 0 to $2\pi_1$, and in respect of μ from $\mu=1$ to $\mu=\cos\mu$. We have thus the two values

$$(41)\qquad \begin{aligned}U_0&=2\pi u\left[\int_\mu^1 d\mu+\frac{x}{u}\int_\mu^1 X_1 d\mu+\frac{x^2}{u^2}\int_\mu^1 X_2 d\mu+\dots\right]\\ &=2\pi\frac{u^2}{x}\left[\int_\mu^1 d\mu+\frac{u}{x}\int_\mu^1 X_1 d\mu+\frac{u^2}{x^2}\int_\mu^1 X_2 d\mu+\dots\right].\end{aligned}$$

As equation (25) enables us to eliminate all these integrals, and seeing that the limit of μ is $\cos \alpha$,

$$(42)\quad \begin{aligned} U_0 &= 2\pi u\left(1-\mu+\frac{x}{u}\frac{1-\mu^2}{1.2}X_1'(\alpha)+\frac{x^2}{u^2}\frac{1-\mu^2}{2.3}X_2'(\alpha)+\ldots\right)\\ &= 2\pi\frac{u^2}{x}\left(1-\mu+\frac{u}{x}\frac{1-\mu^2}{1.2}X_1'(\alpha)+\frac{u^2}{x^2}\frac{1-\mu^2}{2.3}X_2'(\alpha)+\ldots\right). \end{aligned}$$

774. Potential of a Spherical Shell.—The potential V at the point P_0 of a uniform magnetic shell of unit power, limited to the same contour as the preceding layer, will be given (364) by the expression

$$V_0 = -\frac{1}{u}\frac{\partial(xU_0)}{\partial x}.$$

We obtain thus

$$(43)\quad \begin{aligned} V_0 &= -2\pi\left[1-\mu+\frac{x}{u}\frac{1-\mu^2}{1}X_1'(\alpha)+\ldots+\left(\frac{x}{u}\right)^n\frac{1-\mu^2}{n}X_n'(\alpha)\right]\\ &= 2\pi(1-\mu^2)\left[\frac{1}{2}\frac{u^2}{x^2}X_1'(\alpha)+\ldots+\frac{1}{n+1}\left(\frac{u}{x}\right)^{n+1}X_n'(\alpha)\right]. \end{aligned}$$

Lastly, the potential V of a shell at a point P at a distance r and in a direction which makes the angle θ with the axis, is from equation (23), replacing μ by $\cos \alpha$,

$$(44)\quad \begin{aligned} V &= -2\pi\left[1-\cos\alpha+\frac{r}{u}\sin^2\alpha X_1'(\alpha)X_1(\theta)+\ldots+\frac{1}{n}\left(\frac{r}{u}\right)^n\sin^2\alpha X_n'(\alpha)X_n(\theta)\right]\\ &= 2\pi\sin^2\alpha\left[\frac{1}{2}\frac{u^2}{r^2}X_1'(\alpha)X_1(\theta)+\ldots+\frac{1}{n+1}\left(\frac{u}{r}\right)^{n+1}X_n'(\alpha)X_n(\theta)\right]. \end{aligned}$$

The first series is converging for $r<u$, and the second for $u<r$. At the surface of the sphere where $r=u$, the two series have the same value if we have $\theta>\alpha$—that is to say, if the point P is outside the shell; but when we have $\theta<\alpha$—that is to say, for points of the shell itself—the difference of the two series is equal to 4π.

The potential of a circular shell, or of a circular current, is thus referred to an origin C situate at any given point of the axis.

If the axis coincides with the centre O of the shell, we have $u = a$ and $\sin \alpha = 1$; we get thus

$$V = -2\pi\left[1 + \frac{r}{a}X_1(\theta) - \ldots + (-1)^n \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\left(\frac{r}{a}\right)^{2n+1} X_{2n+1}(\theta)\right]$$

$$(45) \quad = 2\pi\left[\frac{1}{2}\frac{a^2}{r^2}X_1(\theta) - \ldots + (-1)^n \frac{1.3.5\ldots(2n+1)}{2.4.6\ldots(2n+2)}\left(\frac{a}{r}\right)^{2n+2} X_{2n+1}(\theta)\right].$$

These are the expressions which have been obtained above (768).

775. Mutual Induction of Two Circular Currents.—Let us in the first place consider two circumferences, HK and H'K' (Fig. 145), having the same axis, of radii a and a', traversed by

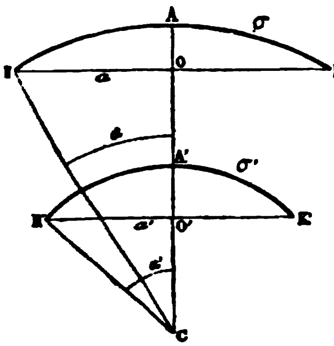


Fig. 145.

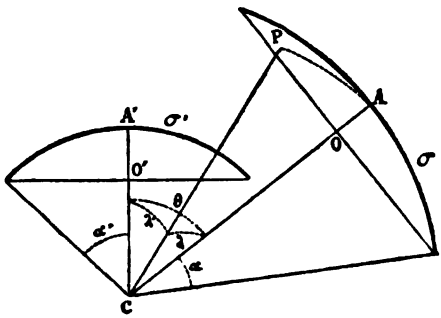


Fig. 146.

parallel currents equal to unity. Take as origin a point C of the axis, draw two spheres of radii u and u' which pass respectively through the two currents, and replace these currents by two spherical shells σ and σ'; let α and α' be the angles subtended by the radii a and a', V' the potential of the shell σ'.

The flow of force from an element σ' into an element $d\sigma$ of the first is equal to $-\frac{\partial V'}{\partial r}d\sigma$, and the coefficient of mutual induction is the integral of this expression extended to the entire surface of the shell σ.

Putting again

$$d\sigma = -u^2 d\mu\, d\phi,$$

we have

$$M = \iint \frac{\partial V'}{\partial r} u^2 d\mu d\phi = 2\pi u^2 \int_1^\mu \frac{\partial V'}{\partial r} d\mu = 2\pi u^2 \int_\mu^1 -\frac{\partial V'}{\partial r} d\mu.$$

Taking for V′ the value given by the second of the equations (44), in which we might replace u by u', a by a', and then, making $r = u$, we get

$$M = 4\pi^2 u' \sin^2 a' \left[\frac{u'}{u} X_1'(a') \int_\mu^1 X_1 d\mu + \ldots \left(\frac{u'}{u}\right)^n X_n'(a') \int_\mu^1 X_n d\mu\right].$$

The integrations should be made starting from $\mu = \cos a$. Replacing the integrals by their values (24), we get

$$(46) \qquad M = 4\pi^2 u' \sin^2 a \sin^2 a' \left[\frac{1}{2}\frac{u'}{u} X_1'(a')X_1'(a) + \ldots + \frac{1}{n(n+1)}\left(\frac{u'}{u}\right)^n X_n'(a')X_n'(a)\right].$$

If the origin is at the centre O′ of the small circle, it is sufficient to make $u' = a'$ and $\sin a' = 1$. Replacing $\sin^2 a$ by $\frac{a^2}{u^2}$, and calling S and S′ the surfaces πa^2 and $\pi a'^2$ of the two circles,

$$(47) \qquad M = \frac{2SS'}{u^3}\left[X_1'(a) + \ldots + \frac{(-1)^n}{(n+1)} \frac{1.3.5\ldots(2n-1)}{2.4\ldots 2n}\left(\frac{a'}{u}\right)^{2n} X_{2n+1}'(a)\right].$$

We thus arrive at the expansion, the first terms of which have been given in (761).

776. Suppose now that the axes (Fig. 146) of the two circular currents cut in C at the angle θ, and consider at the point P an element $d\sigma$ of the shell σ which corresponds to the surface current S.

Calling λ' the angle O′CP, λ the angle OCP, and ϕ the angle of the two planes O′CO and PCO, we may again write

$$d\sigma = u^2 \sin\lambda d\lambda d\phi = -u^2 d\mu d\phi.$$

The potential V′ of the shell σ' in the direction CP is a function of the angle λ'; the second of these equations (44) gives

$$V' = 2\pi \sin^2 \alpha' \sum \frac{1}{n+1}\left(\frac{u'}{r}\right)^{n+1} X_n'(\alpha')X_n(\lambda').$$

From this we get for the point P, where $r = u$,

$$\frac{\partial V'}{\partial r} = -\frac{2\pi u' \sin^2 \alpha'}{u^2} \sum \left(\frac{u'}{u}\right)^n X_n'(\alpha')X_n(\lambda').$$

The coefficient of mutual induction is then

$$M = \int\int -\frac{\partial V'}{\partial r}\, d\sigma = u^2 \int\int \frac{\partial V'}{\partial r}\, d\mu d\phi,$$

$$M = -2\pi u' \sin^2 \alpha' \sum \left(\frac{u'}{u}\right)^n X_n'(\alpha) \int\int X_n(\lambda') d\mu d\phi.$$

The double integration should be made from 0 to 2π for the angle ϕ, and from 0 to α for the angle λ; the angle λ' satisfies further the equation

$$\cos \lambda' = \cos \lambda \cos \theta + \sin \lambda \sin \theta \cos \phi.$$

Integrating first in respect of ϕ, we have, from a theorem of Legendre,

$$\int_0^{2\pi} X_n(\lambda') d\phi = 2\pi X_n(\lambda) X_n(\theta);$$

consequently

$$M = -4\pi^2 u' \sin^2 \alpha' \sum \left(\frac{u'}{u}\right)^n X_n'(\alpha')X_n(\theta) \int_1^\mu X_n(\lambda) d\mu.$$

Replacing finally the definite integral by its value given by equation (24), and $1 - \mu^2$ by $\sin^2 \alpha$, we get

$$(48) \quad M = 4\pi^2 u' \sin^2 \alpha \sin^2 \alpha' \sum \frac{1}{n(n+1)} \left(\frac{u'}{u}\right)^n X_n'(\alpha')X_n'(\alpha)X_n(\theta).$$

We obtain then the coefficient of induction of the two circular currents whose axes cut under an angle θ, by the value for the parallel currents, by multiplying respectively each of the terms by the polynomial $\chi_n(\theta)$ of the same order as that of which this term already contains the differentials.

777. MAXWELL'S METHOD.—We may, in certain conditions, use a method of Maxwell* which depends on the following theorem :——

Given a circle defined by its distance x from a fixed point of the axis, and by its radius r, the coefficient of induction M, *in reference to this circle of any given system of magnets, or of currents, satisfies the ratio*

$$\frac{\partial^2 M}{\partial r^2}+\frac{\partial^2 M}{\partial x^2}-\frac{1}{r}\frac{\partial M}{\partial r}=0. \tag{49}$$

For let P be a point of the circle on a radius which makes the angle θ with a radius of fixed direction, and let V be the potential of the system at the point P. The normal component of the force is equal to $-\frac{\partial V}{\partial x}$; the flow of force across an element of surface $rd\theta dr$ is

$$-rd\theta dr\frac{\partial V}{\partial x}.$$

If the radius of the circle is increased by δr, the flow of force increases by

$$-r\delta r\int_0^{2\pi}\frac{\partial V}{\partial x}d\theta;$$

as this increase is itself equal to the variation $\frac{\partial M}{\partial r}\partial r$ of the coefficient of induction, we have

$$\frac{\partial M}{\partial r}=-r\int_0^{2\pi}\frac{\partial V}{\partial x}d\theta. \tag{50}$$

Let x now have the increment ∂x, the corresponding variation $\frac{\partial M}{\partial x}\partial x$ of the coefficient of induction is equal and of opposite sign to the total flow of force cut by the lateral surface of the cylinder of radius r, and height δx. The value of this latter is

$$-r\delta x\int_0^{2\pi}\frac{\partial V}{\partial r}d\theta;$$

and therefore

$$\frac{\partial M}{\partial x}=r\int_0^{2\pi}\frac{\partial V}{\partial r}d\theta. \tag{51}$$

* MAXWELL. *Electricity and Magnetism*, Vol. II., p. 307.

Differentiating equation (50) in respect of r, and equation (51) in respect of x, we have

$$\frac{\partial^2 M}{\partial r^2} = -\int_0^{2\pi} \frac{\partial V}{\partial x} d\theta - r\int_0^{2\pi} \frac{\partial^2 V}{\partial x \partial r} d\theta,$$

$$\frac{\partial^2 M}{\partial x^2} = -r\int_0^{2\pi} \frac{\partial^2 V}{\partial x \partial r} d\theta;$$

and consequently

$$\frac{\partial^2 M}{\partial r^2} + \frac{\partial^2 M}{\partial x^2} = -\int_0^{2\pi} \frac{\partial V}{\partial x} d\theta = \frac{1}{r}\frac{\partial M}{\partial r},$$

which demonstrates the theorem (49).

778. Maxwell's method applies specially to the case of two parallel circular currents very close to each other, with the same axis, and with radii only slightly different; for instance, two adjacent windings of the same coil.

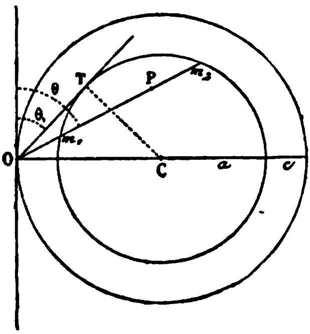


Fig. 147.

Let us, first of all, consider two concentric circumferences in the same plane; let a and $a+c$ be their radii, the difference of these radii being very small.

The action exerted by the element ds of the greatest contour at a point P of the plane, situate at a distance ρ, and in a direction which makes an angle θ with that of the element, is perpendicular to the plane, and its value is $r\int_0^{2\pi} \frac{\partial V}{\partial r}$.

The flow of force corresponding to an element of surface $\rho d\rho d\theta$ at the point P is then

$$\frac{ds \sin\theta \, d\theta d\rho}{\rho};$$

hence the total flow of force δM, or $\frac{\partial M}{\partial \sigma} ds$, due to the element ds, and which traverses the circle of radius a, has the value

$$\delta M = 2ds \int_{\theta_1}^{\frac{\pi}{2}} \int_{\rho_1}^{\rho_2} \frac{\sin\theta}{\rho} d\theta d\rho.$$

The limit θ_1 is the angle of the tangent OT with the element ds; we have evidently

$$\cos\theta_1 = \frac{a}{a+c}, \qquad \text{or} \qquad \sin^2\theta_1 = \frac{c(2a+c)}{(a+c)^2}.$$

The two limits ρ_1 and ρ_2, which correspond to the two segments om_1 and om_2 of the secant OP, are roots of the equation of the second degree,

$$\rho^2 - 2\rho(a+c)\sin\theta + c(2a+c) = 0.$$

If the ratio $\frac{c}{a}$ is very small, we may take

$$\rho_1 = \frac{c}{\sin\theta} \qquad \text{and} \qquad \rho_2 = 2a\sin\theta,$$

and a first integration with respect to ρ gives

$$\delta M = 2ds \int_{\theta_1}^{\frac{\pi}{2}} \left(l.\frac{2a}{c}\sin^2\theta\right) \sin\theta\, d\theta.$$

This expression, integrated with respect to θ, gives

$$\delta M = 2ds \left[\cos\theta \left(2 - l.\frac{2a}{c}\sin^2\theta\right) + 2l.\tan\frac{\theta}{2}\right]_{\theta_1}^{\frac{\pi}{2}};$$

as the angle θ is very small, the expression reduces to

$$\delta M = 2ds \left[l.\frac{8a}{c} - 2\right].$$

We deduce from this, for the value of the entire flux, or the coefficient of mutual induction,

$$M = 4\pi a \left[l.\frac{8a}{c} - 2\right]. \tag{52}$$

779. Let us now suppose that the planes of the two circles are at a very small distance x from each other, their radii being a and $a+y$. The flow of force from the second current, and which traverses the first of radius a, has no longer the same value as before; but if x is very small, we shall have a very close approximation if we replace the shortest distance c of the two arcs by its new value $r=\sqrt{x^2+y^2}$. We might then represent the coefficient M by an expression of the form

$$\text{(53)} \qquad M=4\pi\left[A l.\frac{8a}{r}+B\right],$$

the coefficients A and B being functions of x and y, whose approximate values are respectively a and $-2a$, and which we have now to determine.

Let us observe, in the first case, that these two functions ought not to change their value, when x is changed to $-x$; hence their expansion in series only comprise uneven powers of x. We may put then

$$A=a+A_1 y+A_2\frac{y^2}{a}+A_2'\frac{x^2}{a}+A_3\frac{y^3}{a^2}+A_3'\frac{yx^2}{a^2}+\ldots,$$

$$B=-2a+B_1 y+B_2\frac{y^2}{a}+B_2'\frac{x^2}{a}+B_3\frac{y^3}{a^2}+B_3'\frac{yx^2}{a^2}+\ldots$$

It follows, moreover, from the fundamental property of the coefficient M (341), that its value does not alter when the two circles are interchanged—that is to say, when a is replaced by $a+y$, and y by $-y$.

On the other hand, the function M should satisfy the general equation

$$\frac{\partial^2 M}{\partial y^2}+\frac{\partial^2 M}{\partial x^2}-\frac{1}{a+y}\frac{\partial M}{\partial y}=0.$$

These two conditions furnish the number of equations necessary for determining the coefficients; we find thus

$$\text{(54)} \quad \begin{aligned} A&=a\left[1+\frac{1}{2}\frac{y}{a}+\frac{y^2+3x^2}{16a^2}-\frac{y^3+3yx^2}{32a^3}+\ldots\right],\\ B&=a\left[-2-\frac{1}{2}\frac{y}{a}+\frac{3y^2-x^2}{16a^2}-\frac{y^3-6yx^2}{48a^3}+\ldots\right].\end{aligned}$$

In the case of two equal circles at the distance x, we have, if we make $y=0$,

$$(55)\quad M=4\pi a\left[1+\frac{3x^2}{16a^2}+\ldots\right]l.\frac{8a}{x}-4\pi a\left[2+\frac{x^2}{16a^2}+\ldots\right].$$

789. Self-Induction of a Coil.—The preceding formula enables us to calculate the coefficient of self-induction of a coil—that is to say, the total flow of force across a surface made up of different windings for unit current.

Let us consider the central section of the channel, and let P be the outline of this section in a spire of radius a. Neglecting the terms of correction, the flow of force from the windings which correspond to an element $dxdy$ of the section, situate at a distance r, and which traverses the circle of radius a, has the value

$$4\pi a n_1^2 dxdy\left[l.\frac{8a}{r}-2\right].$$

If this expression be integrated for the whole extent of the surface ω of the section, we shall have for the total flux relative to this circle

$$M=4\pi a n_1^2\int\int dxdy\left[l.\frac{8a}{r}-2\right].$$

Instead of a single winding at a point P, consider the number $n_1^2 dx'dy'$ of windings, which correspond to the element of surface $dx'dy'$. The value M′ of the flow of force for this element of surface is

$$M'=n_1^2 dx'dy' M.$$

The total flow for the whole coil, or the coefficient of self-induction in question, will be the integral of this expression extended over the whole surface ω. Considering first a as a constant equal to the mean radius of the coil, we may write

$$(56)\quad \begin{aligned} L&=4\pi a n_1^4\int\int\int\int dxdydx'dy'\left(l.\frac{8a}{r}-2\right)\\ &=4\pi a n_1^4\omega^2(l.8a-2)-4\pi a n_1^4\int\int\int\int l.r\,dxdydx'dy'. \end{aligned}$$

As the product $n_1^2\omega$ represents the total number of windings, if we put

$$\omega^2 l.\,R_2 = \iiiint l.\,r\,dxdydx'dy',$$

we get

$$L = 4\pi a n^2\left(l.\frac{8a}{R_2} - 2\right). \tag{57}$$

The coefficient of self-induction L of the coil is then equal to the product by n^2 of the coefficient of mutual induction of two concentric circles of radii a and $a + R_2$, or of two equal circles having as radius the mean radius of the coil, and placed at a distance R_2, which is the geometrical mean distance of the surface ω.

781. For a coil with a rectangular section of mean radius a, of breadth $2b$, and depth $2c$, the value of the quantity in brackets, which we shall represent by λ, is (758)

$$\lambda = l.\frac{8a}{\sqrt{b^2+c^2}} + \frac{1}{6}\left[\frac{c^2}{b^2}l.\sqrt{1+\frac{b^2}{c^2}} + \frac{b^2}{c^2}l.\sqrt{1+\frac{c^2}{b^2}}\right]$$
$$- \frac{2}{3}\left[\frac{c}{b}\arctan\frac{b}{c} + \frac{b}{c}\arctan\frac{c}{b}\right] + \frac{1}{12}.$$

If we put

$$\lambda = l.\frac{4a}{\sqrt{b^2+c^2}} - \lambda_1$$

we see that the quantity λ_1 only depends on the ratio $m = \frac{c}{b}$ of the two dimensions b and c, and does not alter when these quantities are interchanged; we have

$$\lambda_1 = \frac{2}{3}\left[m\arctan\frac{1}{m} + \frac{1}{m}\arctan m\right]$$
$$- \frac{1}{6}\left[m^2 l.\sqrt{1+\frac{1}{m^2}} + \frac{1}{m^2}l.\sqrt{1+m^2}\right] - \frac{1}{12}.$$

It is sufficient to know the values of λ_1 corresponding to values of m less than unity. We shall greatly abridge the calculations by means of the following table:—

m	λ_1	m	λ_1
0·00	0·50000	0·50	0·79600
0·05	0·54899	0·55	0·80815
0·10	0·59243	0·60	0·81823
0·15	0·63102	0·65	0·82648
0·20	0·66520	0·70	0·83311
0·25	0·69532	0·75	0·83831
0·30	0·72172	0·80	0·84225
0·35	0·74469	0·85	0·84509
0·40	0·76454	0·90	0·84697
0·45	0·78155	0·95	0·84801
0·50	0·79600	1·00	0·84834

This value of L only holds, however, for the case in which the dimensions of the central section ω of the coil are very small in comparison with the mean radius.

782. If instead of formula (52) we use formula (53), the coefficient of self-induction of the rectangular coil may be put in the form

$$L = 4\pi a n^2 \left[\lambda + \frac{\mu}{a^2} + \ldots\right],$$

with

$$\mu = \frac{3b^2 + c^2}{24} l . \frac{4a}{\sqrt{b^2+c^2}} - \frac{1}{120}\frac{c^4}{b^2} l . \sqrt{1 + \frac{b^2}{c^2}}$$

$$+ \frac{1}{24}\frac{b^4}{c^2} l . \sqrt{1 + \frac{c^2}{b^2}} - \frac{2}{15}\frac{b^3}{c} \operatorname{arc}\tan\frac{c}{b} + \frac{23}{160} b^2 + \frac{221}{1440} c^2 .$$

Putting as above

$$\mu = \frac{3b^2 + c^2}{24} l . \frac{4a}{\sqrt{b^2+c^2}} + \frac{b^2}{4}\mu_1 ,$$

we have

$$\mu_1 = \frac{23}{40} + \frac{221}{360} m^2 - \frac{m^4}{30} l . \sqrt{1 + \frac{1}{m^2}}$$

$$+ \frac{1}{6m^2} l . \sqrt{1 + m^2} - \frac{8}{15m} \operatorname{arc}\tan m .$$

It will be seen that a given μ_1 only depends on the ratio $\frac{c}{b}=m$. We might therefore calculate it by means of the following table:—

m	μ_1	m	μ_1
0·00	0·1250	0·50	0·3066
0·05	0·1269	0·55	0·3437
0·10	0·1325	0·60	0·3839
0·15	0·1418	0·65	0·4274
0·20	0·1548	0·70	0·4739
0·25	0·1714	0·75	0·5234
0·30	0·1916	0·80	0·5760
0·35	0·2152	0·85	0·6317
0·40	0·2423	0·90	0·6902
0·45	0·2728	0·95	0·7518
0·50	0·3066	1·00	0·8162

The value of the coefficient of induction of the coil with a rectangular channel will then be obtained by the formula*

$$(58)\quad L=4\pi a n^2\left[\left(1+\frac{3b^2+c^2}{24a^2}\right)l.\frac{4a}{\sqrt{b^2+c^2}}-\lambda_1+\frac{b^2}{4a^2\mu_1}\right].$$

783. Coil of Maximum Self-Induction.—We may solve the following problem as an application: Given a wire, as well as the section of the channel which should be similar to itself, to find the conditions which make the coefficient of self-induction a maximum. The geometrical mean distance R_2 varies in the linear dimensions of the section of the channel; the number n_1^2 of windings for unit surface being the same, the total number of windings varies as the square of these dimensions.

As, on the other hand, the total length of the wire is constant, we have the equations of condition

$$2\pi a n=\text{const.},$$

$$n=kR_2^2,$$

which give

$$\frac{dn}{n}=2\frac{dR_2}{R_2}=-\frac{da}{a}.$$

* Stefan. *Sitzungsberichte der Kaiserl. Akad. der Wissensch.* Vol. LXXXVIII., p. 201. 1883.

Taking the expression for the coefficient L in the simple form (57), we find that the condition of the maximum is

$$l.\frac{8a}{R_2}=\frac{7}{2}.$$

If the section of the channel is a circle of radius c, we have (758)

$$l.\frac{R_2}{c}=-\frac{1}{4},$$

and therefore

$$a=3{\cdot}22\,c.$$

For a square section with the side c we find

$$a=1{\cdot}85\,c.$$

784. Correction for the Insulator.—We have implicitly assumed in what precedes that the currents are distributed uniformly throughout the entire section of the channel; but if the wire is surrounded by an insulator, as is the ordinary case, the integral of the second member of the formula (56), instead of being extended to the entire section of the channel, should simply be extended to the sum of the sections of the wires. The correction which results from the separation of the wires is obviously proportional to the total length of the wire, and may be calculated for unit length.

It may in the first place be observed that the induction is greater for the bare wire than for a square wire circumscribing the insulating layer; the difference for unit length by equation (14) is equal to

$$2\left[l.\frac{y+z}{z}+0{\cdot}1380606\right].$$

In the second place, the action experienced by a wire, from those which surround it, is smaller for cylindrical wires placed in contact than for square wires closely packed occupying the same volume. It is sufficient to take into account the nearest wires.

Let us consider, for instance, the wire which occupies the centre of a square of nine wires, the action of all the others being negligable. For two circular sections the mean geometrical distance is that of

the centres; for two parallel adjacent wires the mean geometrical distance is to that of the centres as 0·99401 : 1 when the squares are placed side by side, and as 1·0011 : 1 when they are arranged diagonally. For the four wires considered above, the mean geometrical distance should then be divided by 0·99401, and for the four others by 1·0011; for the eight taken together, by the mean 0·9975 of the factors, the Napierian logarithm of which is $-0·002463$. We have then to subtract within the bracket the product of this number by 8, or 0·01971, and the correction for unit length is

$$2\left[l.\frac{y+z}{y}+0·1380606-0·01971\right]=2\left[l.\frac{y+z}{z}+0·11835\right].$$

If l is the total length of the wire comprising n windings, and M is the coefficient of mutual induction of two windings, the radius of which is equal to the mean radius and the distance equal to the geometrical mean of the distances of all points of the section taken in pairs, the coefficient of self-induction of the coil will finally be

$$(59)\qquad L=n^2M_1+2l\left[l.\frac{y+z}{y}+0·11835\right].$$

785. Reciprocal Actions.—The relative energy of two circuits A and A′ traversed by unit currents being equal to $-M$, the variation $\frac{\partial M}{\partial x}dx$, which corresponds to the displacement of one of the circuits parallel to itself and to the x axis, represents, apart from the induced currents which would produce a real displacement, the work done by the system during this displacement. The differential $\frac{\partial M}{\partial x}$ is then equal to the component ξ along the x axis of the

reciprocal action of the two circuits reckoned positive when it is repulsive, and we have

$$\xi = \frac{\partial M}{\partial x}.$$

In the same way, if one of the circuits turns about an axis through the angle $d\theta$, the moment Θ of the reciprocal actions of the two circuits in reference to this axis is expressed by

$$\Theta = \frac{\partial M}{\partial \theta}.$$

786. Circuits with the Same Axis.—When the circuits A and A′ are coils of revolution about the same axis, which we shall take as the x axis, the reciprocal action is parallel to the axis from symmetry. This action being null for a suitable position of the two circuits (the concentric position, for instance, for two cylindrical coils), and also null at a very great distance, it passes through a maximum value which corresponds to the condition

$$\frac{\partial^2 M}{\partial x^2} = 0.$$

787. Action of Two Circular Currents.—For two circular currents of radii a and a' at the distance x, the value of M may in general be expressed by means of elliptic functions (763). Taking into account the values of $\frac{dF}{dk}$ and $\frac{dE}{dk}$ (764), we find

$$\xi = -\frac{\pi k x}{\sqrt{aa'}}\left(2F - \frac{2-k^2}{1-k^2}E\right). \tag{60}$$

The use of series is often more convenient. The coefficient of induction of the circuit A (circular current or coil) on a circular current of surface S′ being equal (761) to $F_m S'$, the reciprocal action is

$$\xi = S' \frac{dF_m}{dx}.$$

When the potential of the circuit A, at a point (x, y) of the surface S′, can be advantageously expanded as a function of increasing powers of y, by one of the expressions found above, the value of ξ will be expressed by an analogous series which is also convergent.

If the circuit A is a circumference of radius a, we have, from equation (29) of (736),

$$F_m = 2\pi\left(f_0'' + \frac{y^2}{2}f_2'' + \frac{y^4}{3}f_2'' + \dots\right),$$

(61)
$$\frac{\partial F_m}{dx} = 2\pi\left(f_0''' + \frac{y^2}{2}f_1''' + \frac{y^4}{3}f_2''' + \dots\right)$$

$$= 2\pi\left(\frac{d^3u}{dx^3} - \frac{y^2}{2.2^2}\frac{d^5u}{dx^5} + \frac{y^4}{3(2.4)^2}\frac{d^7u}{dx^7} - \dots\right).$$

Putting $S = \pi a^2$, we thus find:
by equations (32) of (736), as a function of $u = \sqrt{u^2 + x^2}$,

(62)
$$\xi = -\frac{6SS'}{u^5}x\left(1 - \frac{5}{2.2^2}\frac{a'^2}{u^2}\frac{4x^2 - 3a^2}{u^2} + \frac{3.5}{2(2.4)^2}\frac{a'^4}{u^4}\frac{56x^4 - 140a^2x^2 + 35a^4}{u^4} + \dots\right);$$

by equation (37) of (740), which is especially suited for small values of x,

(63)
$$\xi = -\frac{6SS'}{a^5}x\left\{1 - \frac{5}{2}\frac{x^2}{a^2} + \frac{5.7}{2.4}\frac{x^4}{a^4} - \frac{5.7.9}{2.4.6}\frac{x^6}{a^6} + \dots \right.$$
$$+ \frac{1}{2.2^2}\frac{a'^2}{a^2}\left(3.5 - \frac{5^2.7}{2}\frac{x^2}{a^2} + \frac{5.7^2.9}{2.4}\frac{x^4}{a^4} - \dots\right),$$
$$+ \frac{1}{3(2.4)^2}\frac{a'^4}{a^4}\left(3.5^2.7 - \frac{5^2.7^2.9}{2}\frac{x^2}{a^2} + \dots\right)$$
$$\left. + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \right\};$$

and lastly, by equation (40) of (741), which holds for a great distance,

$$
\begin{aligned}
\xi = -\frac{6SS'}{x^4}\Big\{ & 1 - \frac{5}{2}\frac{a^2}{x^2} + \frac{5.7}{2.4}\frac{a^4}{x^4} - \ldots \\
& - \frac{1}{2.2^2}\cdot\frac{a'^2}{x^2}\left(4.5 - \frac{7}{2.4}4.5.6\frac{a^2}{x^2} + \ldots\right), \\
(64) \qquad & + \frac{1}{3(2.4)^2}\frac{a'^4}{x^4}\left(4.5.6.7 - \frac{9}{2.4}4.5.6.7.8\frac{a^2}{x^2} + \ldots\right), \\
& - \frac{1}{4(2.4.6)^2}\frac{a'^6}{x^6}\left(4.5.6.7.8.9 - \ldots\right), \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \Big\}.
\end{aligned}
$$

The action is at first proportional to the distance x of the two circles; it then becomes inversely as the fourth power, which might be seen *à priori* from the resemblance of currents and magnetic shells.

If the circuit A is a coil of rectangular channels of dimension $2b$, $2c$, and mean section a, the mean action on the circle A′, and the reciprocal action on the two systems, can in like manner be deduced from expressions (39), (41), and (43) of the last chapter; but in some cases it might be useful to push further the expansion in respect of increasing powers of x.

788. These expressions are no longer sufficiently convergent in the case of two very near circular currents and of the same order of magnitude; we may use then Maxwell's formula. With the values (54) we get from it

$$
\begin{aligned}
\xi = -4\pi a x \Bigg[& \frac{1}{r^2}\left(1 + \frac{y}{2a} + \frac{y^2+3x^2}{16a^2} - \frac{y^3+3yx^2}{32a^2} + \ldots\right) \\
(65) \qquad & + \frac{3}{8a^2}\left(1 - \frac{y}{2a} + \ldots\right) l.\frac{r}{8a} + \frac{1}{8a^2}\left(1 - \frac{2y}{a} + \ldots\right)\Bigg].
\end{aligned}
$$

When the circles are equal

$$
(66) \quad \xi = -\frac{4\pi a}{x}\left[\left(1 + \frac{3x^2}{16a^2} + \ldots\right) + \left(\frac{3x^2}{8a^2} + \ldots\right) l.\frac{x}{8a} + \left(\frac{x^2}{8a^2} + \ldots\right)\right].
$$

The principal term is then $-\frac{4\pi a}{x}$, so that the reciprocal action is inversely as the distance of the two circles. This action arises principally from the parallel portions of the two currents (480).

789. In a more general manner, if the coefficient of mutual induction of two parallel circular currents, having the same axis, is expressed by the aid of the polynomials X_n by equation (47), we have, making use of equation

$$(1-\mu^2)X_n'' - 2\mu X_n' + n(n+1)X_n = 0,$$

and the second of the equations (27)

$$(67)\quad \xi = \frac{\partial M}{\partial x} = -\frac{2SS'}{u^4}\left[X_2' - \frac{1}{2}\cdot\frac{3}{2}\frac{a'^2}{u^2}X_4' + \dots + \frac{(-1)^n}{n+1}\frac{1.3\dots(2n+1)}{2.4\dots 2n}\left(\frac{a'}{u}\right)^{2n} X_{2n+2}'\right].$$

790. Action of Two Coils.—For two coils with rectangular channels, we might use the value of M, expressed in elliptic functions either by Maxwell's formula (764) or by Lord Rayleigh's expression (765); but the calculations are generally very complicated, and it is better to have recourse to expansion in series.

In order to replace the circuit A′ of the preceding paragraph by a coil with a rectangular channel, of dimensions a', $2b'$, $2c'$, n_1', and n', one of the expressions found is multiplied by $n_1'^2 dxdy$, and the double integration made between the ordinary limits $a'-c'$ and $a'+c'$, $x-b'$ and $x+b'$.

When the ratios $\frac{b'}{a}$ and $\frac{c'}{a'}$ are very small, we may calculate the value $F_m(a_1')$ of the mean action on the mean circle of radius a_1' (727), and we have sensibly

$$(68)\quad \xi = n'\pi a_1'^2 \frac{\partial F_m(a_1')}{\partial x}.$$

791. Action on a Long Coil.—In order to calculate the action of a system of currents on a long coil, it will often be simpler to estimate directly the action exerted on equivalent magnetic surfaces. We shall consider a particular case as an example.

A and B are two equal coils with rectangular channels of dimensions a, $2b$, $2c$, and n, the mean planes being at a distance $2x$. A long coil A′ with n' windings, a height h', a mean diameter a', and a thickness $2c'$ is placed, so that its base is in the plane of symmetry of the two coils A and B.

The mean circle of this base is

$$S' = \pi a_1'^2 = \pi a'^2 \left(1 + \frac{1}{3}\frac{c'^2}{a'^2}\right),$$

and the total mass of the equivalent magnetic layer

$$S'\frac{n'}{h'} = \pi \frac{n'}{h'} a'^2 \left(1 + \frac{1}{3}\frac{c'^2}{a'^2}\right).$$

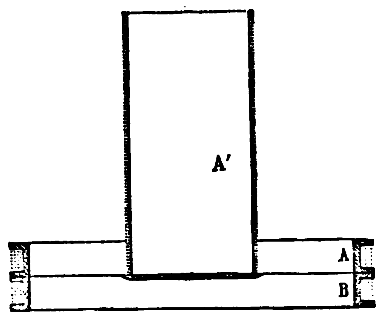


Fig. 148.

The mean action of each of the coils A and B on this base S is from equation (39) of (740)

$$F_m = \frac{2n\pi}{a}\phi,$$

putting

$$\begin{aligned}\phi = 1 &+ \frac{1}{3}\frac{c^2}{a^2} - \frac{1}{2}\frac{b^2+3x^2}{a^2} + \frac{3.5}{2.4}\frac{x^4}{a^4} + \dots, \\ &+ \frac{3}{2.2^2}\frac{y^2}{a^2}\left[1 + \frac{3.4}{2.3}\frac{c^2}{a^2} - \frac{3.5}{2.3}\frac{b^2+3x^2}{a^2} + \dots\right] \\ &+ \frac{3^2.5}{2.(2.4)^2}\frac{y^4}{a^4}\left[1 + \frac{5.6}{2.3}\frac{c^2}{a^2} - \frac{5.7}{2.3}\frac{b^2+3x^2}{a^2} + \dots\right] + \dots,\end{aligned}$$

an expression in which we shall replace y^2 by the square of the mean radius $a_1'^2$, or simply by a'^2.

If l is the total length of the wire of the coils A and B, and l' that of the coil A', the action of the two coils A and B on the lower end of the coil A' is

$$\frac{4nn'\pi^2 a'^2}{ah'}\left(1+\frac{1}{3}\frac{c'^2}{a'^2}\right)\phi=\pi\frac{n^2 l'^2}{n'h'l}\left(1+\frac{1}{3}\frac{c'^2}{a'^2}\right)\phi.$$

In order to calculate the mean action F'_m of the coils A and B on the upper end of the coil A', we shall suppose these two coils situate in their mean plane, and make use of the expression (41) of (741), which gives

$$F'_m=\frac{2n\pi}{a}\phi',$$

neglecting very small terms,

$$\phi'=\frac{a^3}{h'^3}\left[1-\frac{3}{2}\frac{a^2+a'^2}{h'^2}+\frac{3.5}{2.4}\frac{a^4+3a^2a'^2+a'^4}{h'^4}\right].$$

The total action is then

$$\xi=\pi\frac{n^2 l'^2}{n'h'l}\left(1+\frac{1}{3}\frac{c'^2}{a'^2}\right)(\phi-\phi'). \tag{69}$$

792. System of Three Symmetrical Coils.—Let us again consider as above two equal coils A and B, the mean planes of which are at the distance $2x$; and suppose that the currents circulate in opposite directions in the two coils, and that a third coil with the same axis is symmetrically placed between them. In this case the actions of the coils A and B are in the same direction.

Let ξ_0 denote the action of one of them on A' at the total distance x, ξ the total action, when the intermediate coil is displaced through δx, in the plane of symmetry, is

$$\xi=2\left[\xi_0+\frac{(\delta x)^2}{1.2}\frac{\partial^2\xi_0}{\partial x^2}+\frac{(\delta x)^4}{1.2.3.4}\frac{\partial^4\xi_0}{\partial x^4}+\dots\right].$$

For a suitable value of x, the value of ξ is a maximum which corresponds to the condition

$$\frac{\partial^3 M}{\partial x^3}=0, \qquad \text{or} \qquad \frac{\partial^3 F_m}{\partial x^3}=0.$$

The reciprocal action varies then only by a term of the fourth order, and may be regarded as constant.

Let us suppose, first of all, that the coils A, B, and A′ are reduced to their mean circumferences of radii a and a', the condition of the maximum for ξ is

$$\frac{d^5u}{dx^5}-\frac{a'^2}{2.2^2}\frac{d^7u}{dx^7}+\frac{a'^4}{2.(2.4)^2}\frac{d^9u}{dx^9}-\ldots=0.$$

When the ratio $\frac{a'}{a}$ of the two radii is very small, we get from equations (32) of (736)

$$x=\frac{a\sqrt{3}}{2}.$$

If the powers of the ratio $\frac{a'}{a}$ higher than the square may be neglected, we have in the same manner

$$4x^2=3a^2\left[1+\frac{1}{8}\frac{a'^2}{a^2}\frac{35a^4-140a^2x^2+56x^4}{(a^2+x^2)^2}\right],$$

or, replacing x in the second member by its approximate value,

$$x=\frac{a\sqrt{3}}{2}\left(1-\frac{a'^2}{a^2}\right).$$

Lastly, if the radii a and a' are but little different, we may take the expression (65).

Considering the first term only, which is the most important, the condition $\frac{d^2\xi}{dx^2}=0$ gives then

$$3y^2-x^2=0,$$

or

$$x=(a-a')\sqrt{3}.$$

When the circumferences are replaced by coils with rectangular channels of small dimensions, these different conditions relative to the mean radii give still a force which is sensibly constant.

793. Couple of Rotation of Two Circular Currents.— If the axes of two circular currents cut each other at an angle θ (Fig. 146), the moment of the couple due to the mutual actions, which tend to vary the angle θ—that is to say, to make one of the currents turn about a right line passing through the point C and the perpendicular to the plane of the axes—is

$$\Theta=\frac{\partial M}{\partial\theta}=-4\pi^2u'\sin^2 a\sin^2 a'\sin\theta\left[\frac{1}{2}\frac{u'}{u}X_1'(a')X_1'(a)X_1'(\theta)+\ldots\right.$$
(70)
$$\left.+\frac{1}{n(n+1)}\left(\frac{u'}{u}\right)^n X_n'(a')X_n'(a)X_n'(\theta)\right].$$

If the origin C is at the centre O′ of the circle of radius a', we have $\mu'=a'$, $\mu'=\cos a'=0$; we get then

$$\Theta=-\frac{2SS'}{u^3}\sin\theta\left[X_1'(a)X_1'(\theta)+\ldots\right.$$
(71)
$$\left.+\frac{(-1)^n}{n+1}\left(\frac{a'}{u}\right)^{2n}\frac{1.3.5\ldots(2n-1)}{2.4\ldots 2n}X'_{2n+1}(a)\,X'_{2n+1}(\theta)\right].$$

When the planes of the circles are almost rectangular, we may put $\theta=\frac{\pi}{2}-\delta$, the angle δ being very small, and we have

(72) $$\Theta=-\frac{2SS'}{u^3}(P-Q\delta^2)\cos\delta.$$

The value of P is

$$P=1+\ldots+\frac{2n+1}{n+1}\left(\frac{a'}{u}\right)^{2n}\left(\frac{1.3.5\ldots(2n-1)}{2.4\ldots 2n}\right)^2 X'_{2n+1}(a),$$

and the first terms of Q

$$Q=\frac{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{a'}{u}\right)^2 5X_3'(a)+\frac{5}{3}\left(\frac{1.3}{2.4}\right)^2\left(\frac{a'}{u}\right)^4 14X_5'(a)+\ldots$$

Lastly, if the circles are concentric, we have also $\mu = \alpha$, $\mu = \cos\alpha = 0$, which gives

$$P = 1 - \frac{3^2}{2}\left(\frac{1}{2}\right)^3\left(\frac{a'}{a}\right)^2 + \ldots$$

$$+ (-1)^n \frac{(2n+1)^2}{n+1}\left(\frac{1.3.5\ldots(2n-1)}{2.4\ldots 2n}\right)^3\left(\frac{a'}{a}\right)^{2n}.$$

$$Q = -5\frac{3^2}{2}\left(\frac{1}{2}\right)^3\left(\frac{a'}{a}\right)^2 + 14\frac{5^2}{3}\left(\frac{1.3}{2.4}\right)^3\left(\frac{a'}{a}\right).$$

In order to pass from this to the case of two coils with rectangular channels, we shall multiply the value of θ given by equation (70) expanded once, by $n_1^2 n_1' dxdydx'dy'$, and extend the two integrations to the sections of the channels.

If the sections of the channels are small, we shall have a first approximation by multiplying the value of the couple calculated for the mean radii m and a', by the product nn' of the numbers of windings of the two coils.

794. The preceding no longer holds if the revolution is about a line oblique to the plane of the axes, or which does not pass through their point of intersection.

Let us consider, for instance, two circular currents S and S′ with radii a and a', having their axes on the same plane and at a right angle, and such that the plane of the former passes through the centre of the second; and suppose that the second can turn about the intersection of the two planes.

Let X be the component parallel to the axis of the action of the current S_1, given by equation (31), and dS' an element of the surface S′, the expression of the couple will be

$$\Theta = \int X dS',$$

the integral being extended to the entire surface S′.

Let r be the distance of the two centres; if the ratios $\frac{a'}{r}$ and $\frac{a^2}{r^2}$ are of the same order α, and terms of a higher order than α^3 are

neglected, we get, making use of the value X given by equation (32),

$$\Theta = -\frac{SS'}{r^3}\left[1+\frac{1}{2}\left(\frac{3}{2}\right)^2\frac{a^2}{r^2}+\frac{1}{3}\left(\frac{3.5}{2.4}\right)^2\frac{a^4}{r^4}+\frac{1}{4}\left(\frac{3.5.7}{2.4.6}\right)^2\frac{a^6}{r^6}\right.$$
(73)
$$\left.+\frac{3}{2}\frac{a'^2}{r^2}-\frac{5}{4}\left(\frac{3}{2}\right)^3\frac{a^2a'^2}{r^4}\right].$$

The value of the principal term is half that which corresponds to the case in which the axis of the current passes through the centre of the current S′, which we already know.

795. If the current S′ be replaced by a magnet of length $2l$, making an angle δ with the plane of the current, we find by reasoning analogous to that of (746) that the couple produced by the action of the current S on the magnet, provided we neglect expressions of a higher order than $\frac{l^2}{r^2}$, is expressed by

$$D=\frac{\pi a^2}{r^3}\,2l\cos\delta\left[1+\frac{1}{2}\left(\frac{3}{2}\right)^2\frac{a^2}{r^2}+\frac{1}{3}\left(\frac{3.5}{2.4}\right)^2\frac{a^4}{r^4}+\dots\right.$$
$$\left.+15\frac{l^2}{r^2}\left(1-\frac{1}{2}\sin^2\delta\right)\right].$$

When the current S is replaced by a coil of n windings with a rectangular channel whose dimensions are $2b$ and $2c$, we have in the same way, by integrating the value of X given by equation (31), in which a value very near $\frac{\pi}{2}$ is assigned to θ, and neglecting terms of a higher order than the square of the dimensions of the channel,

$$D=n\frac{\pi a^2}{r^2}\,2l\cos\delta\left[1+\frac{1}{2}\left(\frac{3}{2}\right)^2\frac{a^2}{r^2}+\frac{1}{3}\left(\frac{3.5}{2.4}\right)^2\frac{a^4}{r^4}\right.$$
(74)
$$+\frac{1}{3}\frac{c^2}{a^2}+\left(\frac{3}{2}\right)^2\frac{c^2}{r^2}-\frac{3}{2}\frac{b^2}{r^2}$$
$$\left.+15\frac{l^2}{r^2}\left(1-\frac{1}{2}\sin^2\delta\right)\right].$$

PART II.

ELECTRICAL MEASUREMENTS.

CHAPTER I.

ELECTROMETRY.

796. General Characteristics of Electrometers. — An *electrometer* is an instrument intended to measure either a quantity of electricity, or the difference of potential which exists between two conductors in equilibrium, or between two points of a conductor not in equilibrium.

The simplest electrometer, which is also the oldest and the most in use, consists of an insulated conductor to which are suspended two light bodies—for instance, a double pendulum or two gold leaves. It has usually in addition two fixed conductors, called the *terminals*, placed symmetrically in respect of the gold leaves and communicating with the earth.

797. Gold Leaf Electroscope.—The gold leaves are ordinarily surrounded by a glass bell jar, which serves both as insulating support and as a protection against currents of air.

This bell jar contains drying substances, ordinarily lumps of burnt lime or of calcium chloride. It is sometimes surrounded by a second shade provided with a hole through which the rod passes, and which itself contains hygroscopic substances to protect the outside of the bell jar against moisture. The object of all these precautions is to insulate the gold leaves.

The use of a glass shade may present inconveniences in consequence of traces of electricity met with on the sides. This electrification is produced with the greatest facility on very dry glass.

It is better to use a conducting shade, such as one of wire gauze, connected with the terminals and with the ground, the stem passing through an aperture, and being supported, for instance, by a glass rod

in the interior of the shade. This forms then, in reference to the leaves, a conducting surface at constant potential, which protects the gold leaves against the action of any external electrified body (57).

Let us now suppose the leaves electrified; let V be their potential, V_0 that of the shade and terminals. The charge of the leaves is proportional to the product of their capacity, which varies generally with the angle of deflection, into the difference of potential $V - V_0$, and is of the same sign as this difference.

The divergence of the leaves, which is due to their reciprocal repulsion, and to the attraction of each of them by the adjacent terminal, only depends on the charge or on the difference of potential; it might then serve to measure either of these magnitudes. This would not be the same if the sides of the shade

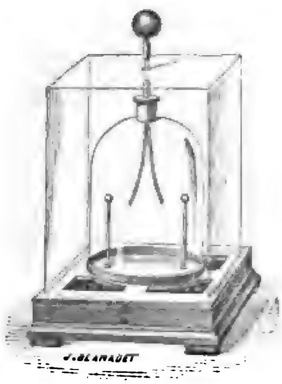


Fig. 149.

electrified partially were not at a constant potential; the terminals, being insulated, would have a charge of their own.

The advantage of a conducting cage is then manifest. Nevertheless, the projecting portion of the insulating rod might undergo an extraneous influence. It would be sometimes advantageous to get rid even of this source of error, by adopting Faraday's plan, and completely surround the apparatus and the body which is to act, by a metal cage in contact with the earth.

798. Graduation of Electrometers.—It may be proposed to determine the law of the deflection as a function of the charge, or of the potential and the constant of the instrument—that is to say, the charge or the potential which corresponds to a given deflection.

In certain instruments this double determination may be made *à priori* from the form and dimensions of the conductors in presence of each other. These are what are called *absolute* instruments. Coulomb's balance and Sir William Thomson's absolute electrometer, which we shall afterwards describe, satisfy this condition; but most frequently the law of the deflection would be determined experimentally, and the value of the constant by comparison with an absolute electrometer.

Let us consider, for instance, the gold-leaf electrometer with a metal shade, and suppose that the shade is connected to earth—that is to say, kept at zero potential. If the capacity of the system formed by the rod and the leaves is sensibly independent of the angle of deflection, the potential V is simply proportional to the charge, and the instrument might be graduated by varying the charge in a known ratio.

De Saussure* used two identical electrometers. One of them being originally electrified, the angular deflection is observed, and it is then connected with a second one, originally in the neutral state. The new angle of deflection corresponds to half the original charge, and, on the supposition of a constant capacity, to a potential one-half as great. In all cases the apparatus is thus graduated as a *measurer of the charges.*

We should obtain the same result more simply by connecting the gold leaves with a Faraday's cylinder (15), in which a succession of equal charges are introduced.

The method used by Volta† gave a graduation as a function of the potentials, the electrometer being connected with the knob of a Leyden jar charged by a variable number of sparks from the plate of an electrophorus. The sparks of an electrophorus rapidly decrease from the outset, but they soon become sensibly equal. On the other hand, the Leyden jar, in consequence of its great capacity, may be considered as always taking from the plate the same fraction of its charge, and consequently as acquiring a potential proportional to the number of sparks. Volta observed in this way that up to 20 or 25 degrees the divergence of the leaves of his electrometer was proportional to the number of sparks.

In order to determine the constant of the instrument, Volta suspended to the beam of a balance a metal disc, which he kept at a

* DE SAUSSURE. *Voyage dans les Alpes*, Vol. II., p. 165–175. Neufchatel, 1804.

† VOLTA. *Della Meteorologia Elettrica.* Second Letter.

constant distance from a horizontal conducting plane in connection with the earth. He estimated in weight the attraction exerted at this fixed distance between the electrified disc and the plane, and, on the other hand, observed the deflection of the electrometer put in connection with the disc and with the knob of a Leyden jar.

These various methods are only approximate; they might be made more exact, but they involve numerous sources of error, arising in particular from the loss by the air and by the supports. The graduation may now be made in a far simpler and more accurate method, by means of voltaic elements—Daniell's, for instance—well insulated and arranged in series. The difference of potential of the two poles is proportional to the number of elements, and we may determine the absolute value of this difference by means of an absolute electrometer.

We may, however, add that such instruments as we have described are most frequently used as simple *electroscopes*, to ascertain if a body is electrified, and what is the sign of its electrification.

799. CLASSIFICATION OF ELECTROMETERS.—A great number of different forms of electrometers have been devised.* We shall restrict ourselves to a few only among them—those which lend themselves best to accurate measurements, and of which a natural selection has brought into almost exclusive use.

Sir W. Thomson† divides the various electrometers in three principal groups.

I. This group comprises, for instance, the electrometer of Cavallo and of Saussure, the straw electrometer of Volta, Bennet's gold-leaf electroscope, Coulomb's balance, Peltier's electrometer, Riess' sine electrometer, Dellmann's electrometer, etc.

II. *Symmetrical Electrometers.*—In these a movable part, such as a needle or a gold-leaf, is arranged between two systems of conductors arranged symmetrically, but insulated from each other, and at different potentials. The deflection of the mobile part depends on its own potential, and on the difference of potential of two fixed conductors. Such are the electrometers of Bohnenberger, of Hankel, and the quadrant electrometer of Thomson.

III. *Balance Electrometers.*—In this category are included all the instruments in which the attraction between two conductors is counterbalanced by a weight. The first idea of instruments of this

* MASCART. *Traité d'Electricité Statique*, Vol. I., pp. 344 and 499. 1876.

† SIR W. THOMSON. *Report on Electrometers and Electrostatic Measurements: British Association Report for 1867. Reprint of Papers*, p. 260.

kind is due to Volta, as has been mentioned above (798), and the absolute electrometer of Thomson is now the most perfect form.

In order to complete this list, we must include the oscillation apparatus, discharge electrometers, electrical thermometers, polarisation electrometers, capillary electrometers, etc., which do not come within the preceding categories.

Sir W. Thomson proposes a classification corresponding to another point of view. *Idiostatic* electroscopes are those in which one of the parts only is electrified, and *heterostatic* those in which charges due to extraneous sources come into play. Symmetrical electrometers are necessarily heterostatic; the two other kinds may be simply idiostatic.

In idiostatic electrometers the reciprocal action of the conductors is manifestly a function of the square of the difference of potentials, and the indication does not depend on the sign of the charge.

In heterostatic instruments the direction of the deflection changes in general with the sign of the electricity, and the deflection is proportional to the potential—at least, for feeble charges. The sensitiveness for weak charges may be made far greater than with idiostatic instruments.

800. COULOMB'S BALANCE.—The form of balance employed by Coulomb,* to determine by the torsion of a wire the action exerted between two small electrified spheres, is so well known that it is sufficient to mention the principal parts. The wire supports a horizontal thread of shellac, terminating at one end in a small conducting sphere; another knob, of the same diameter, placed in the circumference which the first describes, is supported by an insulating stem. In the experiments of Coulomb the diameter of these knobs did not exceed five or six millimetres, and the radius of the circumference which the movable knob described was a little more than ten centimetres. The wire, which was of silver, was 75 cm. in length, and was so fine that a force of 0·15 mgr., applied at the end of the needle, was sufficient to twist it through 360°.

The wire is fixed at the top to a graduated drum-head, by which the torsion can be estimated. The deflection of the needle may be measured either by means of a scale fixed in the shade, or more exactly by the mirror method.

Let us suppose the position of the drum-head to be such that, for the position of equilibrium of the needle, the centre of the movable sphere coincides with that of the fixed knob. When the fixed knob

* COULOMB. *Mem. de l'Acad. des Sciences*, 1785, p. 569, etc.

is in its place, the two knobs are kept in contact by the slight torsion which results from the displacement of the movable one. If the system is electrified, the two balls divide the charge of electricity; they repel each other, and by a suitable torsion A of the micrometer they are brought back to an angular distance α, such that the torsion is $A+\alpha$, and the moment of the couple which tends to bring the needle to its position of equilibrium is $C(A+\alpha)$ (700).

If l is the distance from the axis to the centre of the movable ball, and f the repulsion between the two balls, the moment of this force in respect of the axis is equal to $fl\cos\frac{\alpha}{2}$. The equation of equilibrium is therefore

$$(1) \qquad fl\cos\frac{\alpha}{2}=C(A+\alpha),$$

which gives

$$f=\frac{C}{l}\frac{A+\alpha}{\cos\frac{\alpha}{2}}.$$

In the most general case, in which the two balls are at different distances l and l' from the axis, if d be the distance of the centres of the balls, h the perpendicular let fall from the axis on the direction of d, we have

$$hd=ll'\sin\alpha.$$

The moment of the force f being fh, the equation of equilibrium becomes

$$(2) \qquad f\frac{ll'\sin\alpha}{d}=C(A+\alpha).$$

The distance, moreover, is given by the equation

$$d^2=l^2+l'^2-2ll'\cos\alpha.$$

We must thus consider the force f to be determined as a function of the dimensions of the apparatus, of the observed angles, and of the coefficient of torsion, which will be measured by the ordinary methods (707).

801. Measurements of Masses or of Potentials.—Let m and m' be the charges of the two balls, expressed in electrostatic units. If the electricity were distributed uniformly on each of them,

and there were no extraneous action, the reciprocal action would be the same as if the masses m and m' were respectively concentrated in their centres, and we should have

$$\frac{mm'}{d^2}=f.$$

If the balls, being of the same radius, were in contact at the moment of electrification, the charges m and m' are equal, which gives

$$(3) \qquad m^2=fd^2.$$

For equal distances l and l' this expression becomes

$$m^2=4Cl(A+a)\sin\frac{a}{2}\tan\frac{a}{2}.$$

When the radii are unequal, we may obtain the ratio of the charges m and m' by means of tables calculated by M. Plana.*

If the balls were at a great distance, the potential V of each of them would simply depend on the charge it possesses; but this hypothesis is generally inadequate.

A first approximation is obtained by supposing that the external action of each mass is the same as if it were concentrated at the centre.

We have thus (177) in the case of two balls of equal radius r, and the distance of which d is represented by cr,

$$(4) \qquad \begin{aligned} m&=rV\frac{d}{d+r}=rV\frac{c}{c+1},\\ f&=V^2\left(\frac{r}{d+r}\right)^2=\frac{V^2}{(c+1)^2},\\ f&=\frac{m^2}{d^2}=\frac{m^2}{r^2}\frac{1}{c^2}. \end{aligned}$$

These formulæ are insufficient when the minimum distance of the balls does not greatly exceed their diameter, and the tables given by Sir W. Thomson† stop at the value $c=4$, which corresponds

* PLANA. *Mem. de l'Acad. de Turin* (2), Vol. VII., p. 71. 1845. See MASCART. *Traité de l'Electricité Statique*, Vol. I., p. 281.

† SIR W. THOMSON. *Reprint of Papers*, pp. 96 and 97.

to this distance; hence it is very useful to arrive at more accurate expressions.

802. Applying Murphy's method to the case of two equal spheres A and B, whose centres are O and O′, we shall assume that the electrical images (176) produced by successive influence, instead of being at different points, are all on the conjugate points P and P′ of the centres O and O′ with respect to the two surfaces—that is to say, on the principal electrical images due to the influence on each of the spheres of a homogeneous layer spread over the other. If x denote the distance OP or O′P′, and d' the distances OP′ and O′P, we have

$$x=\frac{r^2}{d}, \qquad \text{and} \qquad d'=d-x.$$

As the charge for unit potential is equal to the radius r of the spheres, we easily find, for the values of the capacities C_a and C'_a (86),

$$C_a=r+\frac{r^2}{d}\frac{r}{d'}\frac{1}{1-\left(\frac{r}{d'}\right)^2},$$

$$C'_a=\frac{r^2}{d}\frac{1}{1-\left(\frac{r}{d'}\right)^2}.$$

If m and m' are the two masses, V and V′ the corresponding potentials, the expression for the mass m is

$$m=rV-\frac{r(c^2-1)^2}{c\left[(c^2-1)^2-c^2\right]}\left[V'-\frac{c}{c^2-1}V\right],$$

and by interchanging the values V and V′ we obtain the value of the mass m'.

Consider the total mass m as made up of two parts, one $m_1=rV$, of which the distribution is uniform, and which we may suppose concentrated at the centre of the sphere; the other $-m_2=m-m_1$, which arises from the induced layers, and which we shall assume concentrated in the point P.

The reciprocal action of the two spheres is the sum of the actions exerted by the masses m_1 and $-m_2$ of one of the spheres

on the two similar masses m'_1 and $-m'_2$ of the other sphere, which gives

$$f=\frac{m_1 m'_1}{d^2}-\frac{m_2 m'_1+m'_2 m_1}{(d-x)^2}+\frac{m_2 m'_2}{(d-2x)^2}.$$

We shall thus obtain the values of m, m_1, and f as a function of the potentials of the two spheres.

When the masses m and m_1 are equal, and therefore the potentials are equal, the expressions become simpler, and give

$$(5)\quad \begin{cases} m=rV\left[1-\dfrac{c^2-1}{c(c^2+c-1)}\right] \\ f=V^2\left[\dfrac{1}{c^2}-\dfrac{2c}{(c^2-1)(c^2+c-1)}+\dfrac{(c^2-1)^2}{(c^2-2)^2(c^2+c-1)^2}\right], \end{cases}$$

from which we can deduce the ratio $\dfrac{fr^2}{m^2}$.

In order to control these formulæ, we will make $c=4$. We get then

$$m=rV\times 0{\cdot}80262,$$

$$f=V^2\times 0{\cdot}03761,$$

$$f=\frac{m^2}{r^2}\times 0{\cdot}05838.$$

If these results are compared respectively with the corresponding values 0·80258, −0·03766, 0·05846 of Sir W. Thomson's table (177), we see that the approximation given by the formulas (5) is then about 0·001; while the simple formulas leave still a relative error greater than 0·06.

When the distance d is considerably greater, we may expand the expressions (5) as a function of increasing powers of $\dfrac{1}{c}$; we thus obtain

$$(6)\quad \begin{cases} m=rV\dfrac{c}{c+1}\left[1+\dfrac{1}{c^4}\left(1-\dfrac{1}{c}+\dfrac{2}{c^2}-\dfrac{3}{c^3}+\dfrac{5}{c^4}-\dots\right)\right], \\ f=\dfrac{V^2}{(c+1)^2}\left[1-\dfrac{2}{c^3}\left(2-\dfrac{1}{c}+\dfrac{4}{c^2}-\dfrac{7}{c^3}+\dfrac{11}{c^4}-\dots\right)\right], \\ f=\dfrac{m^2}{d^2}\left[1-\dfrac{2}{c^3}\left(2+\dfrac{3}{c^2}-\dfrac{5}{c^3}+\dfrac{4}{c^4}-\dots\right)\right]. \end{cases}$$

The principal term in each of these expressions represents the first approximation given by the equations (4). It may be observed in particular that, for the expression of the force as a function of the electrical masses, the simple formula used by Coulomb does not involve a relative error of 0·02 when $c=6$—that is, when the distance of the centres is equal to three times the diameter of the spheres.

803. Influence of the Case.—When the case is of glass, no correction is possible, for we do not know the electrical condition which it acquires for a given condition of the balls. If, moreover, some parts of this surface are accidentally electrified in a few points, very serious errors may arise. Hence the internal surface must be a conductor, and in connection with the earth. The charge of the case is then equal, and opposite to the algebraic sum of the charges of the two spheres, and its potential is zero.

The presence of electricity induced on the conducting case may materially affect the reciprocal action of the spheres, and therefore the calculation of the charges as a function of the repulsive force; but it more particularly changes the value of the potentials, and this influence is greater the smaller is the case, for the quantity of induced electricity is the same in all cases.

The calculation of this correction presents in general the greatest analytical difficulties. In order to give an idea of its importance, we will consider the case of a spherical shade. We shall first assume that, the axis of suspension of the needle being eccentric, the two balls are near the centre. The induced electricity forms then an almost uniform layer at a constant internal potential; the reciprocal action of the two spheres will be expressed then as a function of the charges, just as if the case were not there.

This is no longer the case with the potentials. If, as before, V is the potential due to the charges m and m', U the true potential, and R is the radius of the cage, we have, for a point near the centre,

$$U = V - \frac{m+m'}{R}.$$

When the charges are equal, and we take the first approximation given by equation (4), we have, for the potential of each ball,

$$(7) \qquad U = V - 2\frac{r}{R}V\frac{d}{d+r} = V\left[1 - 2\frac{r}{R}\frac{d}{d+r}\right],$$

or

$$V = \frac{m}{r}\left[1 + \frac{r}{d} - \frac{2r}{R}\right]. \tag{8}$$

If we make $r = 1^c$, $R = 20c$, $d = 10r$, which represents almost the conditions of an ordinary balance, the factor of correction for the influence of the cage is $0 \cdot 1$. It may be observed that from formula (8) the potential of each ball would be simply $\frac{m}{r}$, if $R = 2d$.

804. The influence of the case is much greater when the balls are near the sides. The case being still spherical, and of radius R, let us suppose that the centre of rotation of the needle coincides with the centre of the sphere; let l and l' be the distances from the axis of the fixed and of the movable ball, whose masses are m and m'. Neglecting inequalities of distribution, the electrical image of the mass m' in respect of the case is in the prolongation of the lever l', and does not modify the deflection; the image of the mass m is at a distance D from the axis (168),

$$D = \frac{R^2}{l},$$

and it is equal to

$$m\frac{D}{R} = m\frac{R}{l}.$$

The distance x of this image to the ball m' is defined by the equation

$$x^2 = D^2 + l'^2 - 2Dl' \cos\alpha = \frac{R^4}{l^2} + l'^2 - 2\frac{R^2 l'}{l}\cos\alpha.$$

The action of this image on the ball is $\frac{mm'R}{lx^2}$, and the arm of its lever $\frac{Dl' \sin\alpha}{x}$. If f denotes the direct action of the two balls, the equation of equilibrium is then

$$f\frac{ll' \sin\alpha}{d} - \frac{mm'DRl' \sin\alpha}{lx^3} = C(A + \alpha). \tag{9}$$

If the product mm' is replaced by its approximate value fd^2, and D by $\frac{R^2}{l}$, we get

$$(10)\qquad f\frac{ll'\sin\alpha}{d}\left[1-\left(\frac{dR}{lx}\right)^3\right]=C(A+\alpha).$$

When the radius R of the sphere is great enough in reference to the distances l and l', we may replace x by D or $\frac{R^2}{l}$, which gives

$$(11)\qquad f\frac{ll'\sin\alpha}{d}\left[1-\left(\frac{d}{R}\right)^3\right]=C(A+\alpha).$$

Comparing with equation (2), we see that the factor of correction is equal to $\left(\frac{d}{R}\right)^3$.

Let us still suppose the masses equal, and the balls of the same radius, and let V be the potential of each ball produced by the mass m. The true potential will be

$$(12)\qquad U=V-\frac{1}{D-l}m\frac{R}{l}-\frac{1}{x}m\frac{R}{l}=V-m\frac{R}{l}\left[\frac{l}{R^2-l^2}+\frac{1}{x}\right].$$

Replacing m by the approximate value given by the first of the equations (4), we have

$$(13)\qquad U=V\left[1-\frac{Rr}{l}\frac{d}{d+r}\left(\frac{l}{R^2-l^2}+\frac{1}{x}\right)\right].$$

If x further be replaced by $\frac{R^2}{l}$, to have an idea of the importance of the correction, we get

$$(14)\qquad U=V\left[1-\frac{r}{R}\frac{d}{d+r}\frac{2R^2-l^2}{R^2-l^2}\right].$$

Making $l=\frac{3}{4}R$, and adopting the same numerical values as above, the correction is about 0·15.

805. Allowance should in strictness be made for the rods which support the balls. In the case in which the charges are measured, the needle and the support of the ball are insulating substances. Electricity can scarcely be prevented from gradually escaping along these surfaces, which introduces sources of error difficult to estimate. When the balance is used to measure potentials, the wire and the needle themselves act as conductors, as well as the rod which supports the fixed ball. Their influence can only be neglected when the distances are great and the diameter of the conducting parts is very small, so that their capacity is very small.

806. THOMSON'S ABSOLUTE ELECTROMETER.—This instrument weighs the attraction which is exerted between a plane and a plate parallel to it when they are raised to different potentials (**81**).

The force P, measured in absolute units, which balances electrical attraction, is given by the formula

$$P=\frac{a}{8\pi}\left(\frac{V_1-V_2}{e}\right)^2, \tag{15}$$

in which a represents the surface of the plate, e the distance of the two planes, V_1 and V_2 their respective potentials. This expression only applies, however, to the case in which the surface of the plate is covered with a layer of uniform density, and is therefore in the same condition as if it formed part of an infinite plane.

We have seen how Sir W. Thomson realises these conditions by the use of a *guard-plate.* The movable plate is surrounded by a ring B (Fig. 150) in the same plane and in connection with it, the interval 0·04 cm. to 0·06 cm. being just sufficient to allow freedom of motion without the risk of touching. The system is completed by a closed box D in connection with the guard-ring. The object of this box is to prevent electricity from being produced on the top of the movable plate, and to keep it from any electrical action from adjacent bodies. The distribution is sensibly uniform on the lower surface of the plate. The density is however a little greater at the edges, and a factor of correction must be introduced in the calculation of the surface a. It is clear that the corrected surface must be greater than the real surface and smaller than the opening of the guard-ring. We shall have then a closer value

by taking the mean $\frac{a+a'}{2}$ of the two surfaces. As they are very little different from each other, this correction may be considered sufficient.

The plate and the guard-ring in connection with each other are only at the same potential provided the metals of which they are made are the same (9). In most instruments the plate is of

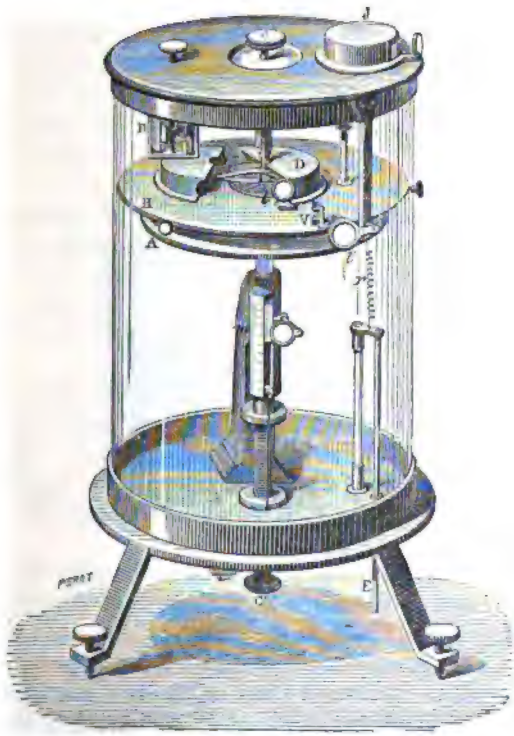


Fig. 150.

aluminum and the ring of brass; hence there is a difference of potential between them. This gives rise to an action which would be appreciable in very delicate instruments.

807. The exact measurement of the distance e of the movable plate from the plate A would present great difficulties. Sir W. Thomson avoids them by modifying the method so as only to have to measure the difference of the two distances. One of the

potentials—that of the ring, for instance—having a fixed value V_1, the plate is successively raised to the potentials V and V′, and its distance is increased until the attraction on the plate is the same in both cases. If e and e' are the corresponding distances, we shall have

$$V - V_1 = e\sqrt{\frac{8\pi F}{a}},$$

$$V' - V_1 = e'\sqrt{\frac{8\pi F}{a}};$$

and therefore

$$(16) \qquad V - V' = (e - e')\sqrt{\frac{8\pi F}{a}}.$$

In this way we measure, not the difference of potentials between two opposite faces, but that of the two potentials to which the plate A has been successively raised. It is to be observed that with this method the electrical densities on the two opposite surfaces, which are necessarily of opposite signs, have always the same absolute value.

808. In the apparatus of Sir W. Thomson the plate is supported by a flexible system consisting of three springs placed symmetrically, and shaped like carriage springs. These springs are contained in the box D, and are themselves supported by an insulating piece, which may be raised or lowered by means of a micrometric screw C.

For each observation the plate must be brought back to a fixed position, as exactly as possible in the centre of the ring. For this purpose the piece attached to the plate and the springs is provided with a cross wire stretched horizontally. A lens l gives a real image of this wire, which is formed between the two very fine points V, which are so regulated that they include the image of the wire when the plate is in the plane of the ring. The points and the image are observed with the lens l'. The position of the plate is thus fixed without error of parallax.

In order to adjust the instrument, the plate is brought back to the mark by means of the micrometer screw. It is then charged with marked weights, distributed symmetrically (0·6 grains in Sir W. Thomson's apparatus), and the screw is turned until the plate is again brought to its mark. These weights obviously represent the electrical attraction which would keep the plate in

the plane of the ring for the same position of the screw. A scale on the side indicates the number of turns, and a divided head fractions of turns. We may thus ascertain if the displacement due to a given weight varies with the temperature, or even with the time.

809. In the absolute electrometer the system of the plate of the ring and of the box is kept at constant potential. It is permanently connected with the inner coating of a Leyden jar, the dielectric of which is the glass shade of the apparatus, on the two faces of which sheets of tinfoil are fixed. Pieces of pumice, soaked with strong sulphuric acid, are placed in lead dishes, and keep the interior of the bell dry. Two accessory apparatus, the replenisher R and the gauge J, enable us to bring the potential each time a constant value.

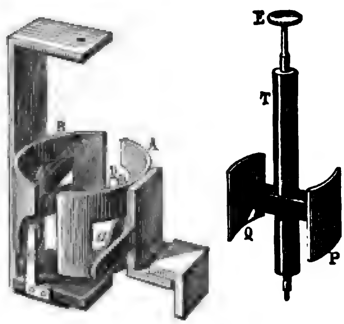


Fig. 151.

The replenisher (Fig. 151) is one of the induction apparatus, the general theory of which we have already discussed (195). The transmitter T, when turned in one direction, increases the charge of the jar; and diminishes it when turned in the opposite direction.

The gauge enables us to tell when the potential has acquired a constant value. This apparatus, which depends on the same principle as the electrometer itself, consists of a movable plate p (Fig. 152), with its guard-ring G and a plate which attracts it, the whole contained in a box J (Fig. 150). The plate and the ring are connected with the external armature of the electrometer. The plate is at a fixed distance from the disc, and is in connection with the internal armature. The potential acquires the same value whenever the plate is brought into the plane of the ring. The

gauge should be sensitive enough to indicate differences of potential without appreciable effect on the electrometer itself.

The movable plate is square, and turns about an axis formed of a fine stretched platinum wire. In the position of equilibrium of the wire, the plane of the plate is not that of the ring. When the two planes coincide, electrical attraction counterbalances the torsion of the wire.

The method of sighting, which enables us to determine the adjustment of the mark, differs slightly from that described above. The small plate of aluminum extends as a tongue *h* of the same metal a little larger than the radius of the box J, and the end of which is forked. Across this is stretched the wire which serves as sight. This end being outside the box, is under the action of external electrical masses; and from the length of the lever it

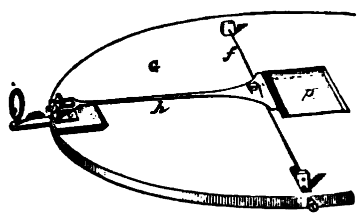


Fig. 152.

forms, serious errors might arise. Sir W. Thomson avoids them by protecting the end by metal wires connected with the box, and so arranged as not to intercept the sight. The cross wire should be brought exactly between two black marks on white paper, and at a less distance than the thickness of the wire. The wire and the two points are viewed by means of a lens. In order to avoid errors of parallax as much as possible, Sir W. Thomson uses a small plano-convex lens, the plane surface being turned towards the eye. The wire is just within the focal distance, and the eye should be at a considerable distance on the other side—at least 20 centimetres. In these conditions the field is reduced to a minimum. When the line of sight coincides with the optical axis, the wire appears as a right line thinner at the extremities. If it is a little out, the image of the wire is curved in one direction or the other.

Variations of temperature have a marked influence on the indications of the gauge, from the changes which they produce in the elasticity of the wire.

810. Quadrant Electrometer.—The quadrant electrometer does not lend itself so well as that just described to absolute measurements, but its manipulation is more convenient, and it may be rendered far more sensitive; it is especially used for comparing potentials.

The calculation made for the case of three concentric cylinders (98) represents with sufficient exactitude the theory of the quadrant electrometer.

The two fixed cylinders A and B of the theoretical figure are replaced by two pairs of quadrants AA′ and BB′ (Fig. 153), which

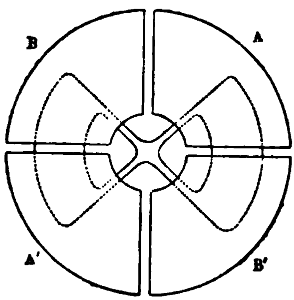


Fig. 153.

represents a flat cylindrical box, divided into four equal parts by two sections at right angles, passing through the axis. Each of these quadrants has at the top a circular groove. Two opposite quadrants are in electrical connection.

The cylinder C is replaced by a very light needle placed in the centre of the box, and movable about a vertical axis. This needle, which is cut out of thin sheet aluminum, may have any given symmetrical shape; but in order to agree as closely as possible with theory, and obtain the most uniform results, it reduces to two circular plates of 90°, attached at the centre by two narrow bands, which are represented by the extreme radii.

This needle has a bifilar suspension,* arranged so that, all the parts being neutral, the system is in equilibrium when the needle is

* A metal wire may also be used, or a single silk fibre may be used, with a small magnet.

symmetrical with the plane of separation of the quadrants. The radii are then in the mean part of the quadrants—that is to say, in a position in which the action they experience is sensibly zero.

811. Let us suppose that each pair of quadrants and the needle are kept at constant potentials, which are respectively equal to V_1, V_2, and V. If the needle turns through a very small angle θ, it is in the conditions of the movable cylinder in the case of three cylinders (**98**); if α is the capacity of the needle for unit angle, the change in energy of the system for the displacement θ, or what amounts to the same thing, the work done by electrical force during this displacement, will be expressed by

$$(16) \qquad W - W_0 = \alpha\theta(V_1 - V_2)\left[V - \frac{V_1 + V_2}{2}\right].$$

The value of the moment of the couple acting on the needle is

$$(17) \qquad \alpha(V_1 - V_2)\left[V - \frac{V_1 + V_2}{2}\right].$$

Equilibrium is obtained when this couple is equal to C sin θ, due to the rotation of the bifilar, which gives the equation

$$(18) \qquad \alpha(V_1 - V_2)\left[V - \frac{V_1 + V_2}{2}\right] = C \sin\theta.$$

The sine of the deflection may be replaced by the angle if the deflections are very small or the suspension is unifilar.

The sensitiveness of the apparatus is proportional to the ratio $\frac{\alpha}{C}$—that is to say, to the capacity of the needle and to the length of the wire, and is inversely as the weight of the needle and of the square of the displacement of the wire.

812. The constant α may be calculated. If S_1 is the surface of the circular band which corresponds to unit angle, that portion of the needle comprised within the pair of quadrants at potential V_1 has on its two faces electrical layers, the densities of which are respectively (**82**)

$$\sigma = \frac{V - V_1}{4\pi d}, \qquad \sigma' = \frac{V - V_1}{4\pi d'},$$

d and d' being the distances from these faces to the opposite sides of the box. The charge of the surface S_1 is therefore

$$S_1(\sigma+\sigma')=S_1\frac{V-V_1}{4\pi}\left(\frac{1}{d}+\frac{1}{d'}\right),$$

which gives for the constant

$$(19)\qquad \alpha=\frac{S_1(\sigma+\sigma')}{V-V_1}=\frac{S_1}{4\pi}\left(\frac{1}{d}+\frac{1}{d'}\right).$$

This expression is a minimum for $d=d'$—that is to say, when the needle is exactly at the centre of the box; the sensitiveness is then a minimum.

The value of α is in general somewhat difficult to calculate directly by measurement of S_1, d, and d'; it will in general be better to determine this constant directly by comparison with an absolute electrometer.

813. The capacity of the electrometer itself varies with the deflection. Let a be the capacity of each half of the needle on either side the long axis of symmetry, m_0 the charge of the needle when it occupies a symmetrical position in reference to the quadrants, and m the charge when it is deflected through the angle θ; we have evidently

$$m_0=a(V-V_2)+a(V-V_1)=2a\left(V-\frac{V_1+V_2}{2}\right).$$

We see already that in the position of equilibrium the charge of the needle depends not only on the potential of the needle, but on those of the quadrants. It is only independent of the potential of the quadrants when $V_1+V_2=0$; it is then equal to $2aV$.

When the deflection is θ, we have, supposing $V-V_2>V-V_1$,

$$\begin{aligned} m &= (a+\alpha\theta)(V-V_2)+(a-\alpha\theta)(V-V_1) \\ &= 2aV-a(V_1+V_2)+\alpha\theta(V_1-V_2) \\ &= 2a\left(V-\frac{V_1+V_2}{2}\right)+\alpha\theta(V_1-V_2). \end{aligned}$$

If c_0 and c are the capacities for the position of equilibrium and for the deflection θ,

$$(20) \qquad c = c_0 + a\theta \frac{V_1 - V_2}{V}.$$

The capacity changes then in general with the deflection. Nevertheless, in case $V_1 + V_2 = 0$, and considering equation (18), we have

$$(21) \qquad c = 2a + \frac{a^2(V_1 - V_2)^2}{c} \frac{\theta}{\sin\theta}.$$

The capacity is then virtually constant—at any rate, for small deflections.

814. From formula (18) the deflection θ of the needle is defined by the product of three factors. The first depends only on the construction of the instrument, the second on the difference of potential of the quadrants, the third on the potential of the needle and on the potentials of the quadrants.

The second factor is zero when the two pairs of quadrants are at the same potential; the needle is then stationary, whatever be the charge communicated to it. This would also be the case if the needle were reduced to a circular band; but from the radii it is only stationary when there is perfect symmetry in respect of the quadrants. This fact is made use of in adjusting the instrument.

The third factor becomes zero, and the needle is at rest, whenever the potential of the needle is the mean algebraical potential of the potentials of the quadrants.

If for a given state of the quadrants the needle is successively raised to equal potentials and opposite signs, the deflections

$$\theta_1 = \frac{a}{C}(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right),$$

$$\theta_2 = -\frac{a}{C}(V_1 - V_2)\left(V + \frac{V_1 + V_2}{2}\right),$$

are not necessarily of opposite signs, as follows from the preceding observation; the value of their algebraic sum is

$$\theta_1 + \theta_2 = -\frac{a}{C}(V_1 - V_2)(V_1 + V_2) = -\frac{a}{C}(V_1^2 - V_2^2),$$

and their difference

$$\theta_1 - \theta_2 = 2\frac{a}{C}(V_1 - V_2)V.$$

We have thus a means of at once determining the difference of potential of the two quadrants and the potential of the needle.

One remark must be made. If, in order to give the needle two equal and opposite potentials, it is connected with one pole of an element, the other pole of which is to earth, the preceding formulas no longer hold strictly. The connection introduces in general an electromotive force of contact, the sign of which does not change with the pole of the battery, so that the needle does not take two equal potentials of opposite signs.

815. If the potential of the needle is so high that the ratio $\frac{V_1 + V_2}{2V}$ may be neglected, formula (18) reduces to

$$\theta = \frac{a}{C}(V_1 - V_2)V. \tag{22}$$

When the potential V of the needle is constant, the deflection is simply proportional to the difference of potentials of the two quadrants, and the capacity of the needle is then independent of the deflection. The apparatus of Sir W. Thomson works under these conditions. The gauge is usually adjusted upon the potential of the Leyden jar, and hence that of the needle in connection with it attains a value of 500 to 1000 volts, according to the degree of sensitiveness which it is desired to obtain.

The deflection is still expressed by the same formula (22) when the quadrants are at constant equal potentials of opposite signs, and the variable potential to be measured is that of the instrument. The instrument is then perfectly symmetrical, and the needle, with the exception indicated, deflects equally on either side when raised to equal potentials of opposite signs.

816. When the needle is connected with one pair of quadrants—that, for instance, whose potential was denoted by V_2—the formula becomes

$$\theta = \frac{a}{2C}(V - V_1)^2.$$

In this case the deflection is proportional to the square of the difference of the two potentials, and the needle sets towards the side of that pair of quadrants with which it is not connected.

Accordingly the sign of the deflection does not alter when the sign of this difference changes. It must still be remarked that, if we put the two quadrants alternately in connection with two points at a constant difference of potential, the deflection will only be constant provided the metal of the quadrants is the same as that of the needle, for the electromotive force of contact does not change its sign at the same time as the difference of potential of the quadrants.

If the free pair of quadrants is then connected with the earth, $V_1 = 0$, and we have

$$\theta = \frac{a}{2C} V^2.$$

In this case the apparatus could be readily used for absolute measures if the factors $\frac{a}{2C}$ could be determined. For this purpose we might revert to the simple form of three cylinders, which we have used to establish the theory of the apparatus. Let us suppose the cylinders vertical, and that the movable cylinder is suspended to the plate of a balance. One of the outer cylinders being insulated, and in electrical connection with the movable cylinder, the other is in connection with the earth. The force P, necessary to keep the beam horizontal, is given by the formula

$$P = \frac{V^2}{4l \frac{R_1}{R}};$$

but in these conditions, as is readily seen, the sensitiveness of the apparatus would be very small.

817. In Sir W. Thomson's arrangement the needle is charged to a constant and very high potential (**815**); the two pairs of quadrants are connected with two points whose difference of potential is to be measured. The entire apparatus is in a conducting envelope in connection with the earth, and therefore at zero potential.

A reversed bell jar is closed by a metal lid, to which all the parts are attached. The needle has, in the direction of its axis, a platinum wire, which dips in concentrated sulphuric acid. This acid forms

one coating of a Leyden jar, the dielectric of which is the bell jar; the other coating, the tinfoil which covers part of the outside, and which is connected with the earth. The uncovered portion of the bell is usually so moist that we may suppose it also is at zero potential. We may thus keep the needle at a very high potential, and the insides of the bell jar are so well insulated that the loss of electricity is very small.

The two pairs of quadrants are supported by glass rods, which the atmosphere of the jar keeps perfectly dry, and are connected to the rods on the outside, which serve as electrodes. These rods are insulated from the lid by means of ebonite.

In order to charge the jar, and keep the potential of the inside constant, a *replenisher* and a *gauge* of appropriate sensitiveness are used. These two parts, which are also fixed to the cover of the instrument, may be detached from the internal coating, so as to avoid useless losses.

The needle itself weighs 0·07 grammes, and with the mirror does not exceed 0·12 grammes; it is supported by two independent silk fibres, the distance of which may be varied at pleasure. In ordinary conditions of sensitiveness a difference of potential of three to four volts between the quadrants sends the image beyond the scale.

The sensitiveness can be easily lessened without altering the charge of the bottle or the distance of the fibres. When one of the pairs of quadrants is insulated, one of the bodies whose potential is to be determined being then connected with the case, the deflections are ten to fifteen times less, while still proportional. We may go further by means of a metal plate called the *inductor*, placed above one of the quadrants. When the four quadrants are connected in pairs, the inductor has no action, whether it be connected to the case or to the adjacent quadrant; but if the quadrants are entirely or partially insulated, and one of them is replaced by an inductor, a series of different degrees of sensitiveness may be obtained.

818. The construction of this electrometer seems to present great difficulties. The quality of the glass is very important, for the jar ought to retain its charge, without appreciable loss, at any rate for a day. On the other hand, the fittings of ebonite alter rapidly, and it may happen that the quadrants are not sufficiently insulated for the investigation of bodies which are not in connection with sources of electricity. Lastly, the deflections of the needle are not exactly symmetrical for equal differences of potential of opposite signs, which may in some cases be inconvenient, as in observations on atmospheric electricity.

Suppose, for instance, that we have $V = 800$ volts, $V_1 = 0$, and and $V_2 = \pm 40$ volts. The ratio of the deflections θ and $-\theta'$, corresponding to the two values of the potential, is then

$$\frac{\theta}{-\theta'} = \frac{V - \frac{V_2}{2}}{V + \frac{V_2}{2}} = \frac{780}{820} = \frac{39}{41}.$$

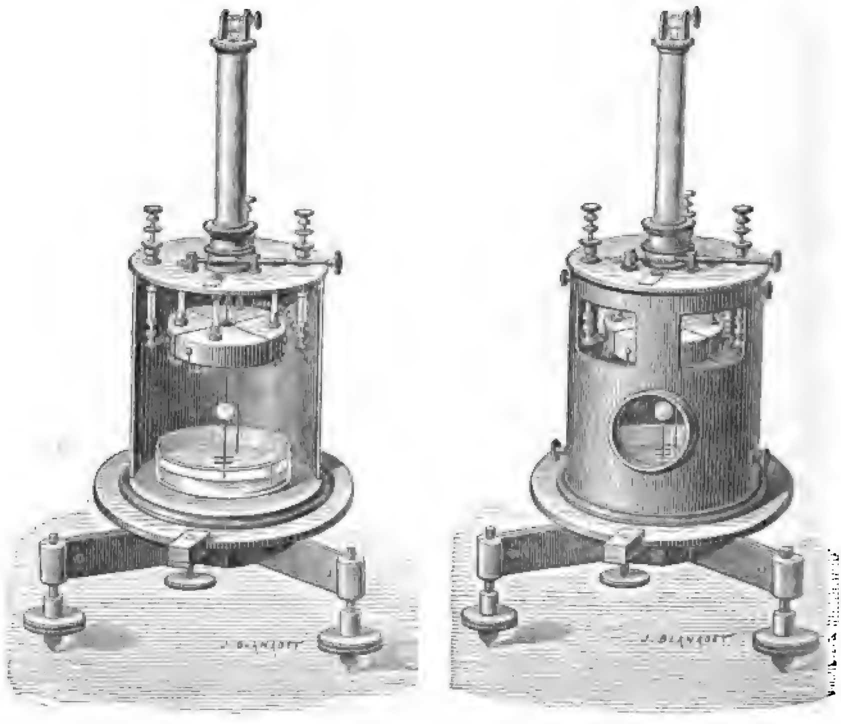


Fig. 154.

The difference of the deflection is 0·05, and the potential of air is often much higher than 40 volts. It may even happen that the two deflections are on the same side; in that case $V_2 > 2V$.

819. We may materially simplify the construction of this electrometer, without detracting from its sensitiveness and its accuracy. The Leyden jar and its accessories—replenisher, gauge, etc.—are dispensed with. The apparatus, reduced to quadrants, and the needle are contained in a metal case (Fig. 154), on the cover of

which is the tube which supports the bifilar suspension, and it allows three electrodes to pass, which are respectively in connection with the quadrants and the needle. These electrodes themselves rest on glass rods, an arrangement which was used by Sir W. Thomson in his portable electrometer.

Two of the electrodes are connected with the pairs of quadrants. The third is connected to the needle by means of the vessel containing sulphuric acid, in which are immersed, on the one hand, a platinum wire attached to the electrode, and, on the other hand, a prolongation in platinum of the axis of the needle; this prolongation has one or two small cross needles, which rapidly deaden the oscillations.

The ordinary arrangement is to make the potentials equal and of opposite sign, by connecting the electrodes respectively with the two poles of a battery formed of three small Volta's or Daniell's elements, the middle of which is to earth. Other things being equal, the sensitiveness is then proportional to the number of cells.

A thread of unspun silk, made up of single fibres, is attached to the needle by a hook, and is coiled at the top on a pulley, and passes between the teeth of a screw having threads in contrary directions, or in two V-shaped notches, the distance of which can be varied; in this way the distance of the wires at the top, and the height of the needle, can be easily regulated. The column itself can rotate by a restrained motion, or by means of a tangent screw, so that the adjustment can be made.

If the three electrodes are free, it is easy to connect them in any given way, and thus utilise, according to circumstances, the different combinations already described.

In all experiments with electrometers it is clear that the bodies placed in connection with the electrodes should be as carefully insulated as the electrodes themselves. The losses by even very moist air may be considered as *nil*, and the loss of electricity must be regarded as taking place entirely by the film of moisture which covers the supports and makes them conductors. This difficulty may be removed by keeping the glass supports, for part at any rate of their length, in an atmosphere dried by sulphuric acid. Figs. 155, 156, which explain themselves, show how this may be effected.*

820. METHOD OF OSCILLATIONS.—In order to verify the law of electrical actions, Coulomb† investigated also the oscillations

* MASCART. *Journal de Physique*, Vol. VII., p. 217. 1878.

† COULOMB. *Mem. de l'Acad. des Sciences*, p. 581. 1785.

produced when an electrified sphere of large diameter is made to act on a small disc charged with the opposite electricity, and attached to a needle suspended by a silk fibre.

As the distribution of electricity on the sphere is not appreciably altered by that of the disc, the action of the sphere is inversely as the inverse square of its distance from the disc; the time of oscillation, being inversely as the square root of the force, is then proportional to the distance.

Fig. 155.

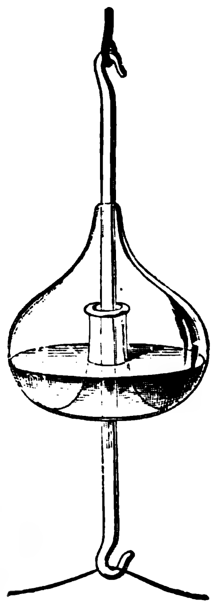


Fig. 156.

This method would also furnish absolute measurements if the dimensions of the apparatus were so chosen that the action could be calculated as a function of the electrical masses or of the potentials—for instance, in the case in which balls of the same diameter are used (802).

If the experiment itself does not involve great accuracy, as when sparks are produced, very approximate results are obtained by an arrangement resembling that of Coulomb.* The insulated needle is replaced by a conducting needle terminating in a small ball *a*, and connected with the earth by the wire of suspension; in the direction

* MASCART. *Traité d'Electricité Statique*, Vol. I., p. 52.

of equilibrium of the needle an electrified sphere A is placed. If suitable precautions are taken that the action be confined to that of the two spheres, and that the distribution on the sphere A is not appreciably modified by the electricity induced on the sphere a, then, if M and $-m$ are their electrical charges, R and r their radii, and D the distance of the centres, the values of the potentials are very approximately

$$V = \frac{M}{R} - \frac{m}{D},$$

$$0 = \frac{-m}{r} + \frac{M}{D}.$$

The electrical action is then

$$F = \frac{Mm}{D^2} = \frac{V^2R^2r}{D^3\left(1 - \frac{Rr}{D^2}\right)^2}.$$

On the other hand, if n be the number of oscillations of the needle in unit time, when there is no electrification; N, the number when the sphere A is raised to potential V; K, the moment of inertia of the needle; l, the distance from the axis to the centre of the small sphere; we have

$$Fl = (N^2 - n^2)\pi^2K = \frac{V^2R^2rl}{D^3\left(1 - \frac{Rr}{D^2}\right)^2}.$$

Determining the moment of inertia K by the ordinary methods (700), this equation will give the potential as a function of the dimensions of the apparatus. The approximation should be carried further if this method were to be used in more accurate experiments.

821. Methods Based on the Properties of the Spark.—The ordinary experiments in statical electricity bring into play charges and differences of potential far greater than can be measured by the preceding methods. In this case the methods which can be used, and which moreover are very old, are based on the properties of the electrical spark.

When conductors at different potentials are brought near each other, equilibrium is effected before contact by a sudden discharge of electricity across the dielectric. This passage, which is known as the *disruptive discharge*, is a very complicated phenomenon, the mechanism of which is unknown, and which depends on the nature, the shape, and difference of potentials of the conductors; it also

depends on the nature of the dielectric which separates them, and, if this is a gas, on the pressure and temperature of the gas, etc.

In a gas under the ordinary pressure, the disruptive discharge is produced under the form of a *spark* or *brush*, the latter taking place in conductors which present points. We may pass from one form to the other by merely modifying the capacity of the conductor. This is readily confirmed by Holtz's machine, which gives brushes in ordinary conditions, and sparks when the capacity of the conductors is increased by connecting them with Leyden jars. Experiment shows that the distance at which the discharge is produced, or the *striking distance*, corresponds in both cases to the same difference of potential. At each spark this difference suddenly sinks almost to zero, as if the spark set up a momentary connection between the conductors. With the brush the difference of potential remains nearly constant. The brush, representing discharges of small quantities of electricity which succeed each other with great rapidity, forms thus a kind of *valve*.

In rarefied gases the discharge is produced in the form of luminosities, which are very different according to the conditions. For very low pressures in particular it forms bands which are alternately bright and dark. Even with continuous sources, the luminosities are due to a very rapid succession of discharges. When the vacuum is made as complete as possible, no discharge takes place.

822. STRIKING DISTANCE.—Numerous researches have been made to determine the relation which exists between the striking distance and the difference of potential. From the experiments of Harris* and of Riess,† the striking distance is nearly proportional to this difference. If the law was general, we should conclude for the case of two parallel plates that the production of electricity almost represents the same value of the electrical density, and therefore of the electrical force and the electrostatic pressure, or, as in Maxwell's views (107), to the same condition or the same specific energy of the interposed medium. But the more recent experiments have not confirmed the generality of this law.

Sir W. Thomson measured with an absolute electrometer the difference of potential corresponding to the production of a spark between two parallel plates, one in connection with the source and the other in connection with the earth. In order to prevent the spark from striking on the edges, the latter is very slightly

* HARRIS. *Phil. Trans. Roy. Soc.*, p. 225. 1834.

† RIESS. *Reibungselectricität*, Vol. I., p. 377.

spherical; but its radius of curvature is very great, and the distribution of potentials is sensibly the same as between the planes. The following table sums up the results of the experiments.* The numbers in the third column were obtained by multiplying those in the preceding column by $\frac{3 \cdot 10^{10}}{10^8} = 300$ (610).

Distance between the Surfaces d	Difference of Potential in Electrostatic Units V	Difference of Potential in Volts V'	Electrical Force $\frac{V}{d}$
Cm.			
0·0086	2·30	690	267·1
0·0127	3·26	978	257·0
0·0190	4·26	1278	224·2
0·0281	5·64	1692	200·6
0·0408	6·18	1854	151·5
0·0563	8·11	2433	144·1
0·0584	8·15	2445	139·6
0·0688	9·69	2907	140·8
0·0904	12·20	3660	134·9
0·1056	13·95	4185	132·1
0·1325	17·36	5208	131·0

This table demonstrates the remarkable fact that the electrical force is greater for small distances than for larger ones. It results from this that the electrical density and the electrostatic pressure go on diminishing as the striking distance increases.

In the experiments of Sir W. Thomson the striking distance did not exceed 0·15 cm. It is difficult to measure directly the difference of potential for greater differences. The electrometer is connected with an insulated conductor B under the influence of a conductor A in connection with one of the electrodes between which the spark strikes, the other being in connection with the earth. So long as the system B does not take its own charge, its potential is in a constant ratio with that of the system A. This ratio, which can be varied with the distance of the two conductors, is determined in each case by a special experiment.

This method was used by Gaugain† to investigate the law of the striking distance between two balls or between two plates, and thus demonstrate the law of the variation of the electrical density

* THOMSON. *Reprint of Papers*, p. 258.

† GAUGAIN. *Ann. de Chim. et de Phys.* [4], Vol. VIII., p. 75. 1866.

between two concentric cylinders (80), the inner cylinder being in connection with the source and the outer one with the earth. Using a gold leaf electrometer as a measuring instrument, Gaugain established that the striking density is constant if the diameter of the outer cylinder is varied; but that its value changes with the diameter of the inner cylinder, and increases as this diameter diminishes.

Mr. Macfarlane* used the same method for striking distances between 0·1 cm. and 1·0 cm. in diameter. If V is the difference of potential and x the striking distance between the two plates, he finds that the results are exactly expressed by the formula

$$V = 66{\cdot}94\sqrt{x^2 + 0.205x},$$

that is to say, by the branch of a hyperbola the real axis of which

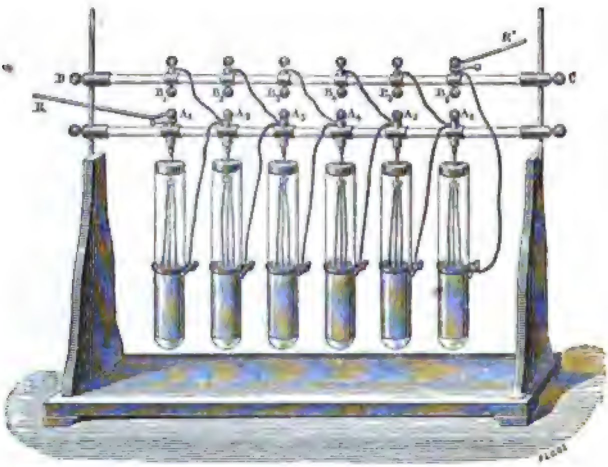


Fig. 135.

corresponds to the distances and the imaginary axis to differences of potential. In the case of a ball and a disc, or of two balls, the curve closely approaches a parabola.

In order to investigate the law of the striking distances between two balls, M. Mascart† used with advantage an electrical valve. The two knobs of a spark discharger, put separately in connection with a conductor B and a point in front of it, are connected with the two poles of a Holtz's machine, and for each striking distance the

* MACFARLANE. *Trans. Roy. Soc. Edin.*, Vol. XXVIII., p. 633. 1878.

† MASCART. *Traité d'Electricité Statique*, Vol. II., pp. 87, 93. 1876.

equivalent distance from the point to the conductor is determined, so that the discharges take place either by the brush of the safety-valve or by a spark between the balls. The knobs of the discharger are then pulled apart, the conductor B is put to earth, and the point is connected with a sphere A· which acts inductively on an oscillation apparatus (820), and the machine is turned so as to obtain a continuous brush. The time of oscillations gives then the potential in absolute value for each distance of the points to the conductor B, and therefore for the corresponding distances of the spark discharger. This arrangement is still insufficient in the case of great striking distances. A long spark may in such cases be balanced by a sum of other and shorter sparks. A series of Leyden jars are connected in cascade (Fig. 157), and connections are so made that the two coatings of each jar are connected to the two knobs A_1 and B_1, A_2 and B_2, placed opposite each other and at the same distance *d*. The outer coatings A_1 and B_6 being connected with the two knobs of a spark discharger, if the discharge takes place indifferently by all the jars in the battery or by the discharger, the difference of potential between the knobs of the discharger is equal to the sum of the known differences relative to all the jars in the battery, or to the product of one of them by the number of jars.

The following table contains the results of experiments for balls 22 mm. in diameter:—

Striking Distance d	Difference of Potential in Electrostatic Units V	Difference of Potential in Volts V'	$\frac{V}{d}$
0·1	18·3	5490	183
0·5	89·1	26730	178·2
1·0	162	48600	162
1·5	190	57000	127
2	216	64800	108
3	256	76800	85·3
4	291	77300	72·7
5	316	94800	63·2
6	338	101400	56·3
7	359	107700	51·3
8	375	112500	46·9
9	386	115800	42·9
10	397	119100	39·7
12	412	124200	34·3
15	426	127800	28·4

Dr. Warren de la Rue and H. Müller* resumed these experiments, replacing the electrical machine by a battery of high potential, the number of couples in which gave directly the difference of potential. The couples are formed of silver, silver chloride, sal-ammoniac, and zinc; their electromotive force is about 1·03 volt. The following table gives the results obtained between two discs :—

Striking Distance d	Difference of Potential		Electric Force
	in Volts V'	in Electrostatic Units V	$\frac{V}{d}$
Cm.			
0·0205	1000	3·33	163
0·0430	2000	6·67	155
0·0660	3000	10·00	152
0·0914	4000	13·33	146
0·1176	5000	16·67	142
0·1473	6000	20·00	136
0·1800	7000	23·33	130
0·2146	8000	26·67	124
0·2495	9000	30·00	120
0·2863	10000	33·33	116
0·3235	11000	36·67	113
0·3378	11330	37·77	112

The differences of potential are here a little greater than in the analogous experiments of Sir W. Thomson.

According to the same authors, the striking distance between a point and a disc increases almost as the square of the difference of potential, for distances between a fraction of a millimetre and one centimetre; for the distance of one centimetre it is 9,200 volts.

The results obtained with air are so closely concordant that, with a certain degree of approximation, differences of potential might be measured by the striking distance—at any rate, if these distances do not exceed two centimetres. Beyond that the difference of potential increases very slowly, if even it does not tend towards a limit a little higher than 400 electrostatic units, or 120,000 volts. A serious error would be made if we wished to estimate the difference of potential of a machine by the length of its spark; in like manner, we cannot conclude from the length of lightning discharges, which is frequently

* WARREN DE LA RUE and H. MÜLLER. *Phil. Trans. Roy. Soc.*, Vol. CLXIX., pp. 55, 155; Vol. CLXXI., p. 65; Vol. CLXXIV., p. 447.

enormous, that the potential of an electrified cloud is out of proportion with those obtained by ordinary frictional machines.

823. INFLUENCE OF PRESSURE.—When the pressure of a gas is diminished, the electromotive force corresponding to a given striking distance decreases rapidly. Experiments agree in showing that the law of variation is very closely represented by the branch of a hyperbola, the real axis corresponding to the pressures, and the imaginary axis to differences of potential.

With highly rarefied gases the electromotive force only diminishes to a certain limit of pressure, beyond which it again increases with extreme rapidity. There is then a pressure for which the resistance to the production of the discharge passes through a minimum. This limit varies in different gases, and for one and the same gas it is

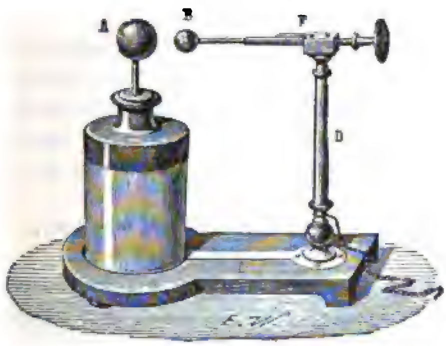


Fig. 158.

lower the narrower is the tube. De la Rue and Müller found that for air it varied from three millimetres in wide tubes to 0·38 mm. in narrow ones. As the pressure diminishes further, the spark does not pass, whatever be the electromotive force. All experiments tend thus to show that matter is necessary for the transport of electricity, and that the molecules of the dielectric serve as the vehicle. The effect of a gradual diminution of the pressure is to give more freedom to the motion of the gaseous molecules, but beyond a certain limit the diminution of the number is no longer compensated by the greater freedom of their movements.

824. MEASUREMENT OF CHARGES BY THE SPARK.—The spark is especially used in measuring charges. The simplest arrangement is that of Lane's jar (Fig. 158). It consists of a Leyden jar, the outer coating of which is connected with a knob B, which can be

brought more or less close to the knob A, connected to the inner coating. If the knob B is connected with the earth, and the knob A with the source of electricity, a spark passes between the two knobs when the potential of the inner coating is that corresponding to the striking distance, and all the sparks allow the same quantity of electricity to pass.

Lane's jar is particularly used in measuring the charge of a battery; the experiment may be arranged in two ways.

The battery being insulated, the inner coating is connected with the source of electricity, the outer coating with the knob A of Lane's jar, and the knob B with the earth. Let C be the capacity of the battery, c that of Lane's jar, V the potential of the inner coating of the battery, v that which remains on the outer coating at the end of the experiment, and n the number of sparks observed. The charge M of the battery is $M = C(V - v)$. As this battery forms a closed condenser, the outer coating has a charge equal to $-M + m$, the charge m being that which produces the potential v. The total quantity of electricity which has passed through the Leyden jar being $M - m$, if q is that which corresponds to each spark, and V_1 is the difference of potential at the striking distance, we have

$$M - m = nq, \qquad \text{or} \qquad M = m + nq.$$

The product nq, or ncV, only represents the charge of the battery provided the quantity m may be neglected, as is usually the case.

Most frequently Lane's jar is insulated; the knob A is connected with the source, and the knob B with the inner coating of the battery, the outer being to earth. As the production of the spark between two conductors only depends on the difference of potentials, and not on their absolute values, the same quantity of electricity passes at each spark; in this case the charge of the battery is exactly proportional to the number of sparks.

When the charges to be measured are feeble, Lane's jar may be replaced by Gaugain's *discharge electrometer* (Fig. 159), which plays exactly the same part. This is a gold-leaf electroscope, provided with a knob in connection with the earth, which is placed in the plane of divergence of the leaves, and within reach of one of them; when the charge is sufficient, contact takes place, and the instrument is discharged.

If electricity flows out continuously, a succession of perfectly identical discharges is produced, provided the gold-leaf falls down again immediately. When the charge of an electrified body is to be

measured, it is advantageous to connect this body to the electrometer by a bad conductor, such as a cotton thread, in order to retard the discharges. As a general rule, a complete discharge is not effected, and there remains on the system of conductors a residual charge incapable of giving contact to the gold-leaf in the electrometers; this residue may be measured by various methods, on which we need not enter.*

If the cotton thread connects the electrometer, not with an electrified body, but with a source at constant potential, or with a conductor of large capacity, such as a battery, the number of discharges in unit time measures the *yield* of electricity, and therefore the strength of the current which traverses the wire.

On the other hand, if the potential of the source is very high compared with the variable potential of the gold-leaf during the

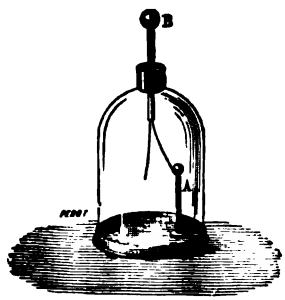


Fig. 159.

alternating insulation and discharge, the yield is proportional to the potential of the source. This experiment furnishes thus a very simple means of comparing potentials of the magnitude of those we meet with in the phenomena of statical electricity.

If the discharge electrometer is connected with a closed conducting cylinder (58), we might also compare electrical charges—the division, for instance, of electricity between two conductors of any given shape; it would be only necessary to count the number of discharges which each of them gives, after they have been in contact, when it is introduced into the cylinder.

825. Capillary Electrometer.—This very delicate instrument, invented by M. Lippmann,† enables us to determine the difference of

* Mascart. *Traité d'Electricité Statique*, Vol. I., p. 417. 1876.

† Lippmann. *Ann. de Chim. et de Phys.* [5], Vol. v., p. 494. 1873.

potential of two points, provided this difference does not exceed 0·9 volt. It consists of a tube A (Fig. 160), drawn out at the bottom in a very fine point. This tube, which is open at both ends, contains a column of mercury kept up by capillary action; two platinum wires α and β, one connected with the mercury of the tube, and the other with the mercury of the vessel, form the electrodes. When these electrodes are connected by a conductor which contains any

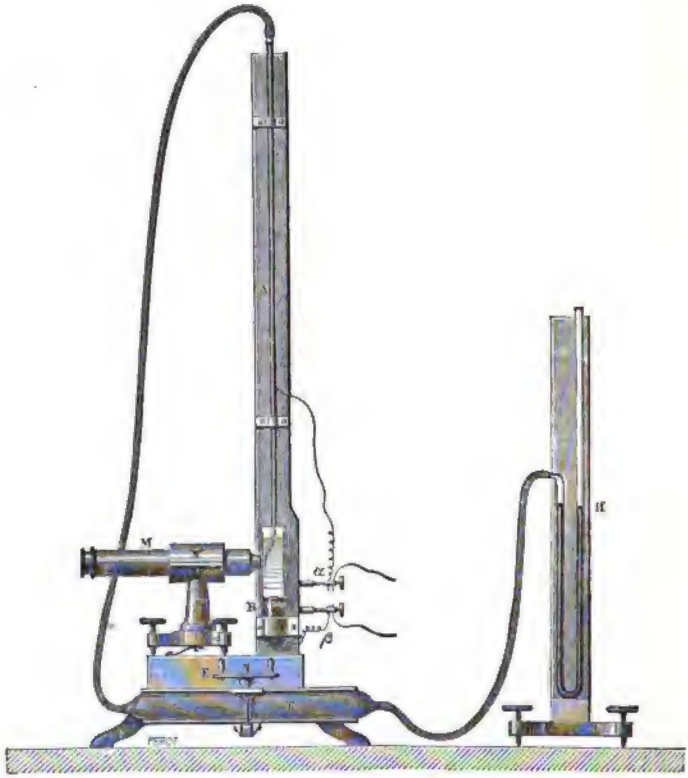


Fig. 160.

convenient electromotive force, the polarisation increases the value of the capillary constant of the mercury (270), and the bottom of the column rises; it is brought back to its original level, which is defined by a mark on the microscope M, by exerting a pressure at the top of the tube by means of a caoutchouc cylinder T.

The electromotive force is deduced from the difference of pressure measured by the manometer H. For as the total pressure is inversely

as the diameter of the tube at the bottom and proportional to the capillary constant, the difference of pressure relative to the variation of the capillary constant produced by a given electromotive force is proportional to the original pressure.

The following table was calculated for a column of mercury of 75 centimetres. The electromotive force is expressed in fractions of that of a Daniells' element.

The increase of pressure is at first proportional to the electromotive force; it then passes through a maximum for an electromotive force of 0·9; the capillary constant attains then 1·47 times its normal value.

Electromotive Force.	Increase Pressure. Cm.	Electromotive Force.	Increase Pressure. Cm.
0·016	1·5	0·500	28·8
0·024	2·15	0·588	31·4
0·040	4·0	0·833	35·65
0·109	8·9	0·900	35·85
0·140	11·1	0·909	35·85
0·170	13·1	1·000	35·3
0·197	14·8	1·261	30·1
0·269	18·85	1·444	23·9
0·364	23·5	1·833	11·0
0·450	27·05	2·000	9·4

It is important that the electrode *a* which corresponds to the capillary tube should be always negative, in order to avoid oxidation of the mercury. It is also necessary to wait until the capillary tube is well moistened by acid, which is effected by causing the column of mercury to oscillate slightly.

The capacity of the electrometer is proportional to the surface of the mercury in the drawn-out tube. Although this surface is very small, its capacity is far greater than that of the instruments described above; for a polarized surface acts always as a condenser with plates very near each other. The capacity of polarization of the platinum is at least 0·1 microf~~ormed~~arad for a square millimetre, and therefore equal to that of a sphere of about 900 metres radius. The capacity of mercury must be a quantity of the same order.

CHAPTER II.

MEASUREMENT OF CURRENTS.

826. Different Methods.—The intensity of currents may be estimated either by a direct measurement of the yield (**824**) in the form of successive sparks, or by any of the physical effects which accompany it, such as the heating of a conductor in conformity with Joule's law, the local heating of a soldering, chemical action, etc.; but use is more especially made of the electromagnetic properties of currents, which give methods of observation which are more rapid and capable of greater accuracy.

The intensity at a point in the magnetic field of a coil traversed by a current is proportional to the strength of the current. The measurement of the current is thus reduced to a measurement of the strength of the field.

Galvanometers are instruments in which the field is measured by its action on a magnetised needle. Both the idea and the name of galvanometer are due to Ampère.* Schweigger† devised the use of the *multiplier*—that is to say, a coil which multiplies the strength of the field or the action of the current on the needle.

The magnetic moment of a coil is equal to the product of the strength of the current by the total surface—more exactly, by the sum of the projections on the mean plane of the coil of the surfaces comprised by the different windings. We might thus also deduce the intensity of a current from the action exerted by a magnetic field on a movable coil.

We may finally cause a fixed current, traversed by a current I, to act on a movable current I′ which is or is not different from the first. The action is proportional to the product of the intensities II′, and may serve to measure either of them. Instruments constructed on this principle are called *electrodynamometers.*

* Ampère. *Ann. de Chim. et de Phys.* [2], Vol. xv., p. 59. 1820.

† Schweigger. *Allgemeine Litteraturzeitung*, No. 296. Nov., 1820.

827. When two intensities are to be merely compared, it is unnecessary to know the dimension of the coils; the factors which depend on it disappear in the ratio being common to the two terms. We thus attempt more particularly to obtain instruments the sensitiveness of which is suited to the currents to be estimated, and the coils are constructed of those dimensions which are most advantageous from this point of view (733).

To get the absolute measure of the current, we must calculate the electromagnetic action (728), or the surface of the coil as a function of its dimensions. Circular frames with small rectangular sections, in which the wire is regularly coiled, are the only ones the calculation of which does not present too great difficulties, and which it is desirable to adopt in what are called *absolute* instruments.

828. Absolute and Relative Sensitiveness.—Whatever be the method employed, if the intensity of the current is determined as a function of a quantity (x) resulting from observation

$$I = f(x),$$

the *absolute sensitiveness* S_a of the instrument is the ratio of the increase dx of the variable to the corresponding increase dI of the current

$$S_a = \frac{dx}{dI} = \frac{1}{f'(x)};$$

the *relative sensitiveness* is the ratio of the increase dx to the relative variation $\frac{dI}{I}$ of the current,

$$S_r = I\frac{dx}{dI} = \frac{f(x)}{f'(x)}.$$

829. Galvanometers.—Suppose that we cause a coil, traversed by a current I, to act on a magnetised needle about an axis. When the needle is infinitely small, if M is the projection of its magnetic moment on a plane perpendicular to the axis of rotation, GI the projection on the same plane of the action of the current at the point where the needle is, β the angle of the two directions GI and M, the moment of the couple which acts on the needle is equal to IMG $\sin\beta$. The factor G_1, which represents the action of unit current, is the *galvanometric constant* of the coil.

If the length of the needle is considerable in respect of the dimensions of the coil and the value G holds for the centre of the needle, the moment may be put in the form

$$(1) \qquad IMG(1+\gamma)\sin\beta = IMG_1 \sin\beta,$$

the factor G_1 being a mean value of the action of the coil in the space occupied by the needle. The term of correction γ which enables us to express G_1 as a function of G depends on the dimensions of the needle and on the angle which it makes with the mean plane of the coil. The factor G_1 may also be regarded as constant when the needle in the various experiments occupies the same position in respect of the coil, or is very little displaced from its mean position.

If the coil is one of revolution, the force G is parallel to the axis of the frame for all points of the axis or of the mean plane. We shall say that the needle is in a *principal position* when its centre occupies one of these points. If δ is the angle which the direction of the needle makes with the plane of the coil, we have, apart from the term of correction γ,

$$(2) \qquad IMG \sin\beta = IMG \cos\delta.$$

This couple is a maximum when the needle is parallel to the coil. We know further that if the coil is symmetrical with respect of its mean plane, the factor G is a maximum at the centre relative to points on the axis.

The various methods differ in respect of the way in which the couple IMG $\sin\beta$ is measured to obtain from it the intensity of the current in relative or absolute values.

We shall see that in galvanometers properly so called, we estimate the moment of the current as a function of the couple MH, which an external field of intensity H exerts on the needle. If the field is uniform, or the needle so small that the term of correction may be neglected, the magnetic moment M of the needle drops out as a factor common to the two actions. The indications of the galvanometer are then independent of the magnetisation and of the shape of the needle.

830. Method of Torsion.—We may counterbalance the action of the current by that of a torsion couple. Suppose the needle suspended to a metal wire and in equilibrium in the magnetic meridian, the torsion is then zero. If θ is the angle through which the wire must be turned to keep the needle in this position notwithstanding

the action of the current I, then if C is the coefficient of the wire, we have

$$\mathrm{IMG} \sin \beta = \mathrm{C}\theta.$$

If the needle occupies a principal position, and its original direction is parallel to the plane of the coil,

$$(3) \qquad \mathrm{I} = \frac{\mathrm{C}}{\mathrm{MG}} \theta.$$

Suppose that this wire having an initial torsion, the direction of the needle at the fiducial position makes an angle α with the meridian; if θ_0 is the angle through which the wire must be turned to bring the needle back to the meridian, we have

$$\mathrm{MH} \sin \alpha = \mathrm{C}(\theta_0 - \alpha).$$

This defect of adjustment may be eliminated by making a second observation with the current reversed. The observed torsions θ and θ' satisfy the ratios

$$\mathrm{IMG} \sin \beta + \mathrm{MH} \sin \alpha = \mathrm{C}(\theta + \theta_0 - \alpha),$$
$$\mathrm{IMG} \sin \beta - \mathrm{MH} \sin \alpha = \mathrm{C}(\theta' - \theta_0 + \alpha),$$

which give, when added

$$(4) \qquad \mathrm{IMG} \sin \beta = \mathrm{C} \frac{\theta + \theta'}{2}.$$

The mean of the torsions observed on the right and the left is equal to that which would be obtained with a wire originally without torsion.

This method, which was devised by Ohm,* has more particularly the disadvantage of requiring a manipulation which disturbs the needle whenever the wire is touched, to bring it back to the fiducial position; we must wait then until the needle has come to rest, which necessitates great loss of time. It can only be applied to constant currents. If we want to use it in comparing variable currents, a series of alternate tests would enable us to eliminate the variations, provided they are not too rapid.

* OHM. *Pogg. Ann.*, Vol. IV., p. 79. 1825.

the maximum of the two degrees of sensitiveness is when the rotation of the coil is near 90°.

833. TANGENT GALVANOMETER.—For galvanometers, properly so called, the coil is fixed, and the needle takes a direction of equilibrium under the combined influence of the current and the external field.

If the plane of the coil makes the angle α with the external field, and the current I deflects the needle through the angle δ, we have

$$\mathrm{IMG}\cos(\delta+\alpha)=\mathrm{HM}\sin\delta,$$

or

$$\mathrm{I}=\frac{\mathrm{H}}{\mathrm{G}}\frac{\sin\delta}{\cos(\alpha+\delta)}. \tag{7}$$

The angle α is generally zero, and therefore

$$\mathrm{I}=\frac{\mathrm{H}}{\mathrm{G}}\tan\delta; \tag{8}$$

the intensity is proportional to the *tangent* of the deflection. This method is also due to Pouillet.

For the law of tangents to be correct, the term of correction ϕ, which should be introduced into the value G (**822**), must be very small; or, in other words, the length of the needle must be small compared with the mean radius of the coil.

We have, for the tangent galvanometer,

$$\mathrm{S}_a=\frac{\mathrm{G}}{\mathrm{H}}\cos^2\delta,$$

$$\mathrm{S}_r=\frac{1}{2}\sin 2\delta;$$

the absolute sensitiveness is a maximum for very weak deflections, and the relative sensitiveness is so for the deflection 45°.

834. If the direction of the field is not parallel to the mean plane of the coil, we may correct the defect of adjustment by observing the fresh deflection δ' obtained after reversing the current. Assuming that the value of G is the same for the two positions of the needle, the two equations of equilibrium are

$$\begin{aligned}\mathrm{IG}\cos(\delta+\alpha)&=\mathrm{H}\sin\delta,\\ \mathrm{IG}\cos(\delta'-\alpha)&=\mathrm{H}\sin\delta';\end{aligned} \tag{9}$$

they give by addition

$$IG \cos\left(\frac{\delta - \delta'}{2} + \alpha\right) = H \tan\frac{\delta + \delta'}{2} \cos\frac{\delta - \delta'}{2}.$$

If the difference $\delta - \delta'$ is so small that its square may be neglected, the angle α is of the same order of magnitude, and we have sensibly

$$(10) \qquad I = \frac{H}{G} \tan\frac{\delta + \delta'}{2}.$$

When this simplification is inadmissible, equations (9) give

$$I^2 = \frac{H^2}{G^2} \frac{(\tan\delta' - \tan\delta)^2 + 4\tan^2\delta \tan^2\delta'}{(\tan\delta' + \tan\delta)^2}.$$

In both cases the mean of the deflections to the right and to the left is taken as the value of G.

835. If the needle is on a pivot, the friction, however small it is, prevents it, in strict accuracy, from attaining its position of equilibrium, and by no method can this cause of error be altogether corrected. It is lessened by taking a very light needle, and making it rest, by means of a polished agate cap, on a very fine steel point. When the needle, through a very small angle, is deflected from its position of equilibrium, it should revert to it within the limits of the errors of reading.

It is generally preferable to suspend the needle by a thread, although the centring is less accurate. If the torsion of the wire is not a negligeable quantity, it may be readily eliminated. When the current is suppressed, the needle only sets exactly in the magnetic meridian provided the wire is without torsion. Let us assume that it deflects through an angle β_0; if θ_0 is the angle through which the wire must be twisted to bring the needle into the meridian, and C is the coefficient of the wire, we have

$$MH \sin\beta_0 = C(\theta_0 - \beta_0).$$

If we turn the needle at the top through an angle θ, the needle turns through an angle β, and makes then the angle $\beta + \beta_0$ with the

meridian; repeating this operation with a torsion θ', which gives a deflection β', the conditions of equilibrium are

$$\mathrm{MH}\sin(\beta+\beta_0)=\mathrm{C}(\theta-\beta+\theta_0-\beta_0),$$
$$\mathrm{MH}\sin(\beta'+\beta_0)=\mathrm{C}(\theta'-\beta+\theta_0-\beta_0).$$

These three equations determine the angles θ_0 and β_0, which, if necessary, enable us to rectify the original position, and the ratio $\epsilon=\dfrac{\mathrm{C}}{\mathrm{MH}}$ of the coefficient of the wire to the couple produced by the field upon the needle.

As the deflection is generally very small, we get from it

$$\epsilon=\frac{\sin\beta\,(1+\cos\beta')-\sin\beta'(1+\cos\beta)}{(\theta-\beta)(1+\cos\beta')-(\theta'-\beta')(1+\cos\beta)}.$$

If the initial torsion is zero, we have also $\beta_0=0$, and

$$\epsilon=\frac{\sin\beta}{\theta-\beta}.$$

Lastly, if the deflections β are themselves very small, as when cocoon threads are used, this expression simply becomes

$$\epsilon=\frac{\beta}{\theta}.$$

A very simple means of producing torsion in a wire, when it is not suspended to a torsion circle, consists in making the needle turn through a whole circumference by means of an external magnet; in that case we have

$$\theta=2\pi.$$

Allowing for this angle, and for the angle α, which the direction of the field makes with the mean plane of the coil, the equations of equilibrium for the direct and inverse directions of the current are

$$\mathrm{IMG}\cos(\delta+\alpha-\beta_0)=\mathrm{HM}\sin(\delta-\beta_0)+\mathrm{C}(\theta_0-\beta_0+\delta),$$
$$\mathrm{IMG}\cos(\delta'-\alpha+\beta_0)=\mathrm{HM}\sin(\delta'+\beta_0)-\mathrm{C}(\theta_0-\beta_0-\delta');$$

from this follows

$$\text{IMG} \cos\frac{\delta+\delta'}{2}\cos\left(\frac{\delta-\delta'}{2}+\alpha-\beta_0\right)$$
$$=\text{HM}\sin\frac{\delta+\delta'}{2}\cos\left(\frac{\delta-\delta'}{2}-\beta_0\right)+\text{C}\frac{(\delta+\delta')}{2}.$$

When the angles α and β_0 are very small, this is also the case with the difference $\delta-\delta'$, and we have sensibly

(11) $$\text{I}=\frac{\text{H}}{\text{G}}\tan\frac{\delta+\delta'}{2}+\epsilon\frac{\frac{\delta+\delta'}{2}}{\cos\frac{\delta+\delta'}{2}}$$

If the deflections themselves, δ and δ', are very small, we may again write

$$\text{I}=\frac{\text{H}}{\text{G}}\tan\frac{\delta+\delta'}{2}+\epsilon\frac{\delta+\delta'}{2}.$$

836. From what we have seen above (746), the factor of correction γ, relative to a needle of magnetic length $2l$, the middle of which is on the axis at a distance x from the centre of the coil, has the following value, if we confine ourselves to quantities of the second order,

$$\gamma=\frac{3}{4}\frac{a^2-4x^2}{u^2}(1-5\sin^2\delta)\frac{l^2}{u^2},$$

an expression in which a is the mean radius of the coil, and u the distance $\sqrt{a^2+x^2}$. Equation (8) gives then, to the same degree of approximation,

$$\text{I}=\frac{\text{H}}{\text{G}}\left[1-\frac{3}{4}\frac{a^2-4x^2}{u^2}(1-5\sin^2\delta)\frac{l^2}{u^2}\right]\tan\frac{\delta+\delta'}{2},$$

or, for a needle in the centre of the coil,

$$\text{I}=\frac{\text{H}}{\text{G}}\left[1-\frac{3}{4}(1-5\sin^2\delta)\frac{l^2}{a^2}\right]\tan\frac{\delta+\delta'}{2}.$$

The length $2l$ is rather difficult to determine for a magnet of any given form, but for a short cylindrical bar it may be assumed, in conformity with the experiments of Coulomb, that the pole is at a third of the same length from each end, and that consequently the value of l is equal to $\frac{2}{3}$ of half the length of the magnet.

When the angle δ varies from zero to 45°, the factor $(1 - 5 \sin^2 \delta)$ remains less than 1·5. If, then, the ratio $\frac{l^2}{a^2}$ is of the order of experimental errors—or, at any rate, of the approximation which we aim at—the law of tangents will be applicable within the same interval. Beyond this limit the sensitiveness rapidly diminishes.

In order to extend these limits, M. Bertin* has proposed to place the coil at an angle of 45° with the magnetic meridian, and to pass the current so that the needle is brought towards the coil. Equation (7) becomes then

$$I = \frac{H}{G} \frac{\sin \delta}{\cos (\alpha - \delta)},$$

or, if θ is the angle $\alpha - \delta$ of the needle with the coil, and $\alpha = 45^\circ$,

$$I = \frac{H}{G\sqrt{2}} (1 - \tan \theta).$$

The angle θ may thus vary from $+45^\circ$ to -45°, so that the total deflection may amount to 90°. The strength of the current which can be measured by the galvanometer without introducing a correction is thus doubled. It is not, however, possible by this method to reverse the current and get rid of the defects of adjustment.

837. In the original instrument of Pouillet the coil consisted of a single circle 25 to 30 cm. in diameter, with a needle 7 to 8 cm. in length playing on a pivot and provided with an index movable on a graduated circle. The ratio $\frac{l^2}{a^2}$ was then about $\frac{1}{16}$, and the term of correction of the same order.

* BERTIN. *Ann. de Chim. et de Phys.* [4], Vol. XVI., p. 25. 1869.

The inexactitude of the law of tangents in this apparatus was first made clear by the experiments of Despretz,* and Blanchet calculated the correction (746) which should be introduced into the readings.

Dr. Joule† appears to have been the first to use a tangent galvanometer in conditions more in agreement with theory. The needle, consisting of a small bar 5 mm. to 6 mm. in length, is suspended by a silk fibre in the centre of a coil 15 cm. in diameter. We have then $\frac{l}{a}=\frac{1}{30}$, and the correction does not exceed $\frac{1}{800}$ even with a deflection of 45°. To read the deflections, the needle is provided with a glass thread, the end of which moves over a graduated circle. The friction of this thread in air produces a powerful damping; and the mean of the readings, after reversing the current, gives a measure of the deflection with an error which, according to Dr. Joule, does not exceed 2′.

In accurate experiments it is generally preferable to give to the galvanometers a smaller degree of sensitiveness by diminishing the number of windings, or increasing the diameter of the coil, and measuring the deflections by the mirror. The term of correction γ is then sensibly constant.

Gaugain's coil (748), conically wound, abolishes the term of correction to the second order, when the needle is at the apex of the cone; but this mode of construction presents practical difficulties, and a dissymmetry which soon led to its use being given up.

The two equal coils of Von Helmholtz (749) get rid of the term of the second order when the distance of the coils is equal to the mean radius; and even the term of the fourth order, which depends on the coil when the dimensions of the channels are in a convenient ratio. Lastly, coils with four sets of windings (750), and such with three (751), give other means of obtaining a sensibly uniform field; but these arrangements seem to have been but rarely utilised.

Another arrangement, used by Weber,‡ consists in placing the magnetised needle outside of the coil and in a principal position on the axis or in the mean plane, the axis being always perpendicular to the meridian.

* DESPRETZ. *Comptes rendus*, Vol. XXXV., p. 449. 1852.

† JOULE. *Brit. Assoc. Report.* Cork, 1843. *Scientific Papers*, Vol. I., p. 404.

‡ WEBER. *Electrodyn. Maasbestimmungen*, Vol. I., p. 16. 1846.

In the former case, if the distance x is sufficiently great, the term of correction may be put in the form

$$\gamma = -3(1-5\sin^2\delta)\frac{l^2}{u^2}.$$

When the needle is in the mean plane, at a distance r from the centre, we have (795)

$$\gamma = \frac{15\frac{l^2}{r^2}\left(1-\frac{1}{2}\sin^2\delta\right)}{1+\frac{1}{2}\left(\frac{3}{2}\right)^2\frac{a^2}{r^2}+\ldots} = 15\frac{l^2}{r^2}\frac{\frac{1+\cos^2\delta}{2}}{1+\frac{9}{8}\frac{a^2}{r^2}}.$$

The tangent galvanometer is one of the instruments best suited for measuring currents in absolute value. In that case, independently of the constant G, which should be calculated as a function of the dimensions of the coil, it is necessary to determine the intensity H of the field in absolute value. We shall examine subsequently the methods used in this latter determination.

838. Fig. 161 represents a tangent galvanometer, with movable coils, constructed entirely of wood. Each of the two coils is wound on a frame which, sliding on two wooden rails, can be fixed by a screw, and can be adjusted by a two-way screw. The distance of the frames may be determined by means of two verniers.

The needle, provided with a mirror and suspended by a fibre of silk, is placed on the axis of the coils. The reading of the angle is effected by means of a telescope and a graduated scale. As the whole of the apparatus turns about a vertical axis, it may be adjusted so that for the normal position the needle is parallel to the plane of the coils.

The condition of Von Helmholtz's galvanometer is satisfied by a suitable displacement of the coils. In any case, the length of the needle is only a tenth of the diameter of the coil, and the small correction for the law of tangents may be obtained with sufficient approximation. The sensitiveness is a maximum when the two coils are in contact with the tube containing the suspension wire.

839. Ordinary Galvanometers.—In an instrument which is not intended for absolute measurements, the object aimed at is to get great sensitiveness as well as rapid and easy readings.

The sensitiveness is proportional to G, and is inversely as H. In order to increase the factor G, we must bring the windings as closely

as possible to the needle, and coil the wire so that the curve which bounds the section of the channel satisfies equation $u^2 = c^2 \sin\theta$. It would be still better to distribute the wire in layers which satisfy this condition, and in each of which the diameter of the wire increases as the parameter c.

Lastly, very small needles are used, so as to reduce to a minimum the central cavity, which removes just those windings whose special action is greatest.

We may add that in the construction of the coil it is particularly important to use very pure copper, and especially free from iron.

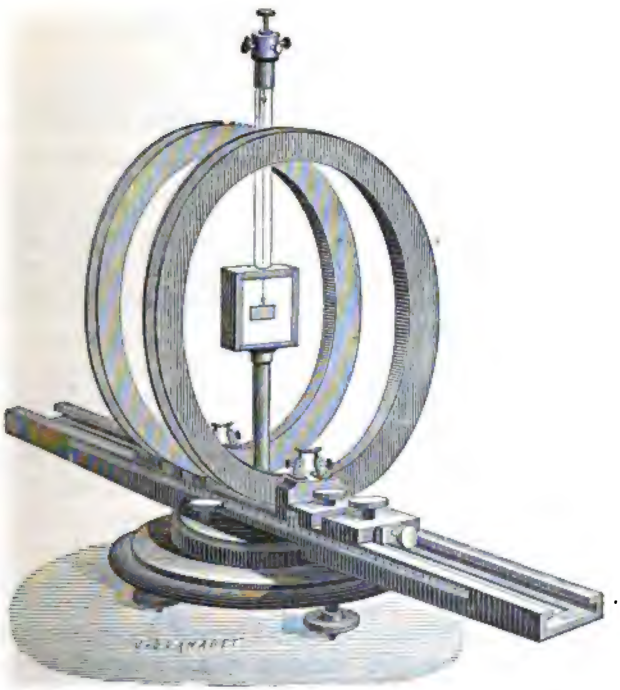


Fig. 161.

The presence of other elements rapidly diminishes the conductivity of copper. Traces of iron give rise to local action, which become considerable when the mass of the wire comes close to the needle. The errors due to this cause were for a long time a source of serious difficulties in using galvanometers of great sensitiveness.

840. ARTIFICIAL FIELD.—Several arrangements may be used to diminish the action of the external field. If the bar is placed symmetrically in repect of the axis of rotation of the needle, and all the points are at a great distance from the needle, its magnetic field is sensibly uniform and horizontal in the space it occupies.

If H′ and H are the horizontal components of the field of the bar and of the terrestrial field, and θ the angle of their respective directions, the intensity R of the resultant field and the angle α which it makes with the meridian are given by the equation

$$R^2 = H^2 + H'^2 + 2HH' \cos\theta,$$

$$(H + H' \cos\theta) \tan\alpha = H' \sin\theta.$$

The minimum of R, which corresponds to $\theta = \pi$ is $\pm(H - H')$. If the ratio of the fields H and H′ is near unity, the angle α is near $\frac{\theta}{2}$. It tends towards $\frac{\pi}{2}$ when θ approaches π—that is to say, when the position of equilibrium of the needle tends to make a right angle with the meridian, as for a system of quasi-astatic needles (299).

The apparatus is then very sensitive to variations in intensity or in direction of the component fields. Thus we have the changes dH of the terrestrial field,

$$\frac{dR}{R} = \frac{H + H' \cos\theta}{R^2} dH, \qquad \text{or sensibly} \qquad = \frac{dH}{H - H'},$$

and for changes in direction

$$\frac{d\alpha}{d\theta} = \frac{H'(H \cos\theta + H')}{R^2} \ldots\ldots = -\frac{H'}{H - H'}.$$

In the condition of maximum sensitiveness, the difference H − H′ being very small, the relative variation of the resultant field is very great, and the position of equilibrium changes considerably with variations in the declination.

The compensating magnet is usually supported by a vertical rod placed on the galvanometer in the prolongation of the axis of rotation of the needle. The magnet can be fixed at any given height by means of a screw, and by a tangent screw the rod itself may be turned about its axis. The magnet has the form of the arc of a circle, so as to have a more uniform field in the

space comprised within the two poles and to be able at need to place the two poles in the prolongation of the needle.

841. Astatic Needles.—A second method, due to Nobili, consists in using a system of astatic needles (**299**). The action of the earth on the system may be reduced at pleasure. By placing one of the needles within the coil and leaving the other outside, the action of the coil is slightly increased, the action on the external needle, which is moreover very slight, being always in the same direction as the principal action.

The apparatus is more symmetrical when two superposed coils are used, with a needle in the middle of each, and passing the current in contrary directions in the two coils. As a system almost astatic tends to set at right angles to the meridian, this property, as well as the time of oscillation may be used to investigate the degree of compensation of the system.

842. Bifilar Suspension.—A third method, pointed out by Gauss,* consists in supporting the bar by a bifilar suspension, so that its position of equilibrium is still in the meridian, but opposite that which it takes under the action of the earth. The resultant directing couple is equal to the excess of the bifilar couple over the terrestrial couple. This difference may be made very small; but the experimental arrangement is less convenient than the preceding, and is only suitable for very heavy magnets.

843. Damping.—With instruments which are not damped the observations are always long and tedious. Although it is unnecessary to wait until the magnet is quite at rest, yet we must wait until the amplitudes are small enough to deduce the position of equilibrium of three successive elongations (**695**). When the galvanometers have a very slight damping of their own, the needle may be stopped by means of a small magnet held in the hand, which is so manipulated as to act in opposition to the oscillations. It is convenient in that case to use a magnet jointed like a compass, the action of which is inappreciable when the two branches touch. It is often advisable to adjust near the galvanometer a coil through which the current of an auxiliary battery can be sent in either direction by means of a key in the hands of the observer. Yet for ordinary galvanometers it is better that the instrument should have an appreciable damping of its own.

The natural damping of the oscillations is due to the resistance of the air and to the induction currents developed by the motion

* Gauss. *Œuvres*, Vol. v., p. 367. *Resultate des Magn. Vereins*, Vol. I. 1837.

of the magnet either in the wires of the coil or in masses of metal in the vicinity.

When the oscillations decrease in geometrical progression, the logarithmic decrement λ may be taken as a measure of the damping. The directing couple which is due to the field is equal to RM. If, as above (678), C_1 is the coefficient of the resistance of the medium, K the moment of inertia of the system, and, disregarding inductive actions (681),

$$\lambda^2 = \epsilon^2\tau^2 = \pi^2\frac{\epsilon^2}{n^2-\epsilon^2} = \frac{C_1^2}{4KMR - C_1^2}.$$

844. The damping by the air may be very great as in Joule's galvanometer (837). The needle is sometimes provided with vanes of aluminum or mica, which greatly increase the friction of the air. This means is still more efficacious if the vanes are enclosed within a box, and are very near the sides. In reflecting galvanometers the mirror itself acts as a damper. When it is placed in a very narrow space, we may arrive at an aperiodic motion (Sir W. Thomson's *dead beat* galvanometer).

845. The damping by the coil is a much more complicated effect. Let α be the angle of the mean plane of the coil with the direction of the external field H, x_0 the deflection corresponding to equilibrium of the needle for the constant current I_0. When the needle in its oscillations makes an angle x with its position of equilibrium, the moment of the action of the coil for unit current is $MG\cos(\alpha + x_0 + x)$. For a displacement dx, the work of this action on the needle, or the variation of the flow of force from the needle across the circuit, is

$$dQ = MG\cos(\alpha + x_0 + x)\,dx.$$

If R is the resistance and L the coefficient of self-induction of the circuit, the strength I of the current induced by the needle satisfies equation

$$(12)\qquad L\frac{dI}{dt} + RI + MG\cos(\alpha + x_0 + x)\frac{dx}{dt} = 0\,;$$

we have further

$$(13)\qquad I_0 MG\cos(\alpha + x_0) = HM\sin x_0.$$

On the other hand, the value of the moment of the couple which acts on the needle at the time in question is

$$(I_0+I)MG\cos(\alpha+x_0+x)-C_1\frac{dx}{dt}-HM\sin(x_0+x);$$

the equation of the motion of the needle is then (675)

$$(14)\quad K\frac{d^2x}{dt^2}+C_1\frac{dx}{dt}+HM\sin(x_0+x)=(I_0+I)MG\cos(\alpha+x_0+x).$$

The elimination of I between equations (12) and (14) leads to a differential equation of the third order, with transcendental coefficients, even if we suppose the value of G to be a constant.

The equation becomes somewhat simpler when we consider very small oscillations about the position of equilibrium, but the coefficients of the differential equations only become constant when the deflection α and the initial displacement are themselves very small. Taking into account equation (13), we may then write

$$(12)'\qquad L\frac{dI}{dt}+RI+MG\frac{dx}{dt}=0,$$

$$(14)'\qquad K\frac{d^2x}{dt^2}+C_1\frac{dx}{dt}+HMx=MGI.$$

In this particular case, the motion of the needle on either side of its position of equilibrium is independent of the original current I_0.

If we put

$$(15)\qquad 2\epsilon_0=\frac{C_1}{K},\qquad n_0^2=\frac{HM}{K},\qquad \delta=\frac{M^2G^2}{2KR},$$

and eliminate the intensity I of the induced current between equations (12) and (14), we get the linear differential equation of the third order

$$(16)\quad L\left(\frac{d^3x}{dt^3}+2\epsilon_0\frac{d^2x}{dt^2}+n_0^2\frac{dx}{dt}\right)+R\left(\frac{d^2x}{dt^2}+2(\epsilon_0+\delta)\frac{dx}{dt}+n_0^2\right)=0.$$

When the circuit is open we have $I=0$, and the phenomenon is defined by the equation

$$\frac{d^2x}{dt^2}+2\epsilon_0\frac{dx}{dt}+n_0^2x=0.$$

If the damping is so weak that the motion is not aperiodic, and if we take as origin of the time the moment at which the needle passes through the position of equilibrium (681), the integral is of the form

$$x=A_0e^{-\epsilon_0 t}\sin\gamma_0 t=A_0e^{-\frac{\lambda_0}{\tau_0}t}\sin\pi\frac{t}{\tau_0},$$

the coefficient A_0 depending on the initial condition. We have further the condition

$$\gamma_0^2+\epsilon_0^2=n_0^2.$$

If we put $\phi_0^2=1+\frac{\lambda_0^2}{\pi^2}$, and if T is the time of oscillations without damping, we have also

$$n_0^2=\frac{HM}{K}=\frac{\pi^2}{T^2}=\pi^2\frac{\phi_0^2}{\tau_0^2}.$$

When the circuit is closed, the general integral of the equation is of the form

$$x=A_1e^{\rho_1 t}+A_2e^{\rho_2 t}+A_3e^{\rho_3 t},$$

the values of ρ_1, ρ_2, and ρ_3 being roots of the equation

$$(17)\qquad L(\rho^3+2\epsilon_0\rho^2+n_0^2\rho)+R(\rho^2+2(\epsilon_0+\delta)\rho+n_0^2)=0.$$

As the ratio of L to R is very small, one of the roots ρ_1 is very large, and nearly $-\frac{R}{L}$. The two other roots, ρ_2 and ρ_3, differ very little from the roots of the equation

$$u^2+2(\epsilon_0+\delta)u+n_0^2=0,$$

which are imaginary.

Equation (17) being put in the form

$$Lf(\rho)+R\phi(\rho)=0,$$

if we replace ρ by $u+y$, y being a very small quantity, and expand by Taylor's theorem, taking only first terms, and observing that $\phi(u)=0$, we obtain for y the approximate value

$$y=-\frac{L}{R}\frac{f(u)}{\phi'(u)}=-\frac{L}{R}\frac{u}{2}\frac{u^2+2\epsilon_0 u+n_0^2}{u+\epsilon_0+\delta}=\frac{L}{R}\delta\frac{u^2}{u+\epsilon_0+\delta}.$$

Putting

$$\beta^2=n_0^2-(\epsilon+\delta)^2,$$

(18) $$\delta'=2\frac{L}{R}\delta(\epsilon_0+\delta),$$

$$\beta'=\frac{L}{R}\frac{\delta}{\beta}(2\beta^2-n_0^2),$$

we obtain for values of ρ_2 and ρ_3

$$\rho=-(\epsilon_0+\delta+\delta')\pm(\beta+\beta')\sqrt{-1}.$$

If we take as origin of the time the period at which the needle passes through an elongation of amplitude a, the constants A_1, A_2, and A_3 will be defined by the condition that when $t=0$, we have

$$x=a,\qquad \frac{dx}{dt}=0\qquad\text{and}\qquad I=0.$$

The coefficient A_1 is proportional to the cube of the ratio $\frac{L}{R}$, and the corresponding time may be neglected. t_0 being the time of passage through the position of equilibrium, the value of x may then be again put in the form

$$x=Ae^{-\epsilon t}\sin\gamma(t-t_0)=Ae^{-\frac{\lambda}{\tau}t}\sin\pi\frac{t-t_0}{\tau},$$

and, neglecting the square of the ratio $\frac{L}{R}$, we have

$$(19)\qquad \begin{aligned} \epsilon &= \epsilon_0 + \delta + \delta' = \epsilon_0 + \delta\left(1 + \frac{2L}{R}\epsilon\right), \\ \gamma &= \beta + \beta'. \end{aligned}$$

We deduce from the first of these equations

$$(20)\qquad \frac{1}{\delta} = \frac{2KR}{M^2G^2} = R\frac{2H}{G^2M}\frac{\tau_0^2}{\pi^2\phi_0^2} = \frac{1}{\epsilon - \epsilon_0}\left(1 + \frac{2L}{R}\epsilon\right).$$

To the same degree of approximations equations (18) and (19) give, putting $\phi^2 = 1 + \frac{\lambda^2}{\pi^2}$,

$$\epsilon^2 + \gamma^2 = \frac{\pi^2\phi^2}{\tau^2} = n_0^2\left(1 + 2\frac{L}{R}\delta\right) = \frac{\pi^2\phi_0^2}{\tau_0^2}\left(1 + 2\frac{L}{R}\delta\right),$$

$$\epsilon - \epsilon_0 = \frac{\phi_0}{\tau_0}\left(\frac{\lambda}{\phi} - \frac{\lambda_0}{\phi_0}\right)\left(1 + \frac{L}{R}\epsilon\right).$$

Unless the damping is considerable, the times of oscillation τ and τ_0 differ but little from each other.

If the ratio $\frac{L}{R}$ be neglected, we deduce from equation (20)

$$\frac{\lambda}{\tau} = \frac{\lambda_0}{\tau_0} + \frac{M^2G^2}{2RK}.$$

The damping λ increases then with the constant G of the coil, with the ratio of the square of the magnetic moment of the needle to its moment of inertia, and inversely as the resistance of the circuit. It is a maximum when the coil is closed.

If we add to the circuit a resistance R′, the approximate values of the elements λ' and τ' of the new oscillations satisfy the equation

$$R\left(\frac{\lambda}{\tau} - \frac{\lambda_0}{\tau_0}\right) = (R + R')\left(\frac{\lambda'}{\tau'} - \frac{\lambda_0}{\tau_0}\right),$$

or

$$\frac{\lambda'}{\tau'}=\frac{R}{R+R'}\frac{\lambda}{\tau}+\frac{R'}{R+R'}\frac{\lambda_0}{\tau_0}.$$

Equation (20), as we shall afterwards see, is one of those which may be used to determine the resistance R as a function of the experimental data; it gives, in fact, to the same degree of approximation,*

$$(20)' \qquad R=\frac{G^2}{2}\frac{M}{H}\frac{\pi^2\phi_0^2}{\tau_0^2(\epsilon-\epsilon_0)}+2L\epsilon.$$

If we replace $\epsilon-\epsilon_0$ by its value, we may write

$$(20)'' \qquad R=\frac{G^2M}{2H}\frac{\pi^2\phi_0}{\tau_0\left(\frac{\lambda}{\phi}-\frac{\lambda_0}{\phi_0}\right)}+L\frac{\lambda}{\tau}.$$

846. In most of the ordinary galvanometers the coil has of itself a considerable resistance, so that the damping, due to induced currents in the wire, is always very weak.

Since Gambey's discovery† on the damping of magnets by metal plates, it is usual to place above or around the needle a very thick mass of copper, which is therefore a good conductor. This arrangement may be advantageously applied to all instruments such as tangent galvanometers, where the frame produces no appreciable damping in consequence of its distance from the needle. In order that the ratio of the magnetic moment to the moment of inertia of the movable system shall be as great as possible, and in order to avoid any accessory pieces, Weber‡ used as needle a small circular mirror, magnetised along one diameter, and oscillating in the middle of a cavity arranged in the centre of a very thick copper sphere. This sphere, which forms a conducting screen between the magnet and the coil, almost nullifies the damping due to the coil, even when this is very near the needle.

* E. DORN. *Wied. Ann.*, Vol. XXII., p. 265. 1884.

† ARAGO. *Ann. de Chim. et de Phys.* [2], Vol. XXVIII., p. 325. 1824.

‡ W. WEBER. *Electrodyn. Maasbestimmungen*, Vol. I., p. 17. 1846.

With a somewhat thick copper sphere, without a break, and a powerfully magnetised needle, an aperiodic movement is easily obtained (685).

This arrangement, however, is not suited for sensitive galvanometers, for the introduction of a heavy mass of metal about the needle increases the central cavity of the coil, the effect of which is to diminish the sensitiveness.

Having stated these general principles, we will proceed to pass under review the principal types of galvanometers in ordinary use.

847. NOBILI'S GALVANOMETER.—The galvanometer which has been for a long time in most extended use is that of Nobili.* The

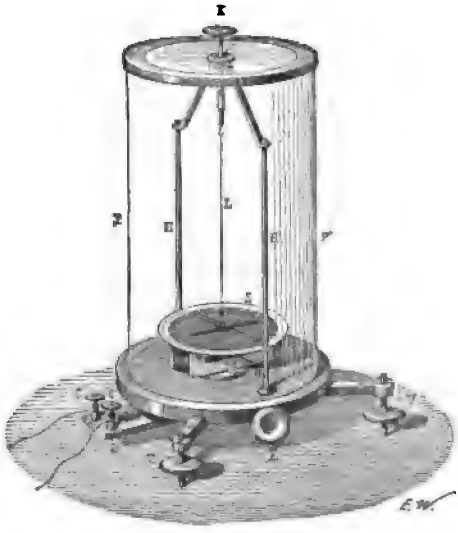


Fig. 162.

needles, which form an astatic system, are portions of sewing needles, 5 cm. to 6 cm. in length; the coil is rectangular (Fig. 162), and the frame is just large enough to allow the free play of one of the needles; its breadth is 4 cm. to 5 cm. The lower needle is in the interior of the frame; the other is outside, and is provided with a drawn-out index. The system is attached to a fibre of silk L, and may be raised or lowered by means of the screw K.

* NOBILI. *Descrizione d'un Nuovo Galvanometro.* Maggio, 1825. *Memorie ed Osservazioni*, etc., Vol. I., p. I.

A plate of copper S is interposed between the frame and the needle; it supports the divided scale, and at the same time serves to damp the oscillations. At the top the windings form two sets, and there is a space between them for the introduction of the astatic system; there should be a corresponding slit in the copper plate, which has the drawback that there is a break near where the needle has its greatest velocity, and seriously injures the damping. A glass shade protects the apparatus from air-currents.

This form of galvanometer has been the object of important researches by Nobili, Melloni, Péclet, Dubois Raymond, De la Provostage, and Desains, on radiant heat, or in physiology. The most delicate instruments are liable to the objection, which has been discussed above, of giving several positions of equilibrium, in consequence of the presence of some traces of iron in the copper wire. In order to get rid, in great part, of the difficulties relative to zero, Péclet* coiled all the wire in one bundle, and supported the needle by a bent stirrup, which passed round the coil; this arrangement got rid of the diametrical slit of the copper plate.

848. Weber's Galvanometer.—The methods used by Weber† led him to increase greatly the dimensions of the magnetised bar; he uses a cylinder 10 cm. in length and 15 mm. in diameter, usually hollow, so as to diminish the moment of inertia without materially diminishing the magnetic moment. The magnet is surrounded by a very thick elliptical copper frame, which serves both as damper and as core for the coil. The suspension is by a bundle of silk fibres, and the reading is effected by means of the mirror.

This arrangement does not allow of the use of systems of astatic needles, but the sensitiveness may be increased by a bifilar suspension which tends to give the bar a direction opposed to that of the terrestrial field (**842**).

849. Thomson's Galvanometers. — In the construction of galvanometers Sir W. Thomson has endeavoured to realise as closely as possible the theoretical conditions.

The needle consists of a thin plate of steel, of about 0·8 cm. diameter, fixed to the back of the mirror which serves for the observations. Instead of a single plate, four or five may be used, placed parallel to each other. The weight of the mirror and of the needle are made as small as possible, and in well-constructed instruments this weight does not exceed 0·05 gramme. The system is suspended

* Péclet. *Ann. de Chim. et de Phys.* [3], Vol. II., p. 103. 1841.

† Weber. *Electrodyn. Maasbestimmungen*, p. 337.

by a silk fibre, about 1 cm. in length, in a cavity of rectangular section, just sufficient to allow of very slight motions on either side of the position of equilibrium. Induction currents have only an unimportant part in the damping, from the smallness of the needle, its distance from the coil, and from the great resistance which the wire usually presents. The coil is circular, and the section of the channel is a rectangle, constructed according to the theoretical curve of best winding (733). By means of a correcting magnet, the directing force which acts on the needle can be varied at will.

Sir W. Thomson has constructed galvanometers of this form, in which the diameter of the several wires varies according to the law indicated by theory (735). Each layer communicates with an external binding screw, so that in the central part we may take that length of the wire the resistance of which is equal to that of the external circuit. We may thus work within the conditions of maximum sensitiveness for all experiments.

In the astatic galvanometer (Fig. 163) the two needles of the system are placed separately at the centre of a circular coil. Each of these coils consists of two, which fit against each other, only having sufficient space between them to permit the passage of the aluminum rod which connects the needles. This rod supports the mirror of one of the coils, and in the centre of the other there is a thin plate of aluminum or mica, for damping. The current is passed in opposite directions in the two coils, so that the actions may add themselves. A correcting magnet, which modifies unequally the action of the field on the two needles, renders it possible to vary at will the sensitiveness of the apparatus. If the system of needles is very nearly astatic, the position of equilibrium depends on the magnet only, and the zero is more stable.

The resistance of the coils, unless for instruments intended for experiments on radiant heat, is generally considerable, and amounts to 8,000 or 10,000 ohms.

These galvanometers have very rapid oscillations, which speedily damp themselves. Their sensitiveness is very great, and they are particularly suited for methods of reduction to zero, in which the existence and the direction of a very feeble current is to be determined.

850. Siemens' Galvanometer.—This galvanometer is especially noticeable from the needle, which is a kind of horse-shoe magnet in the shape of a cylindrical bell, grooved in a diametrical plane for great part of its length. This turns about its axis in a hollow cylindrical cavity in the centre of a solid sphere of copper. As the

oscillations are about the axis of revolution, the magnet is always in the same position in respect of the sphere; it follows from this that, different to what is the case in other arrangements, the induced

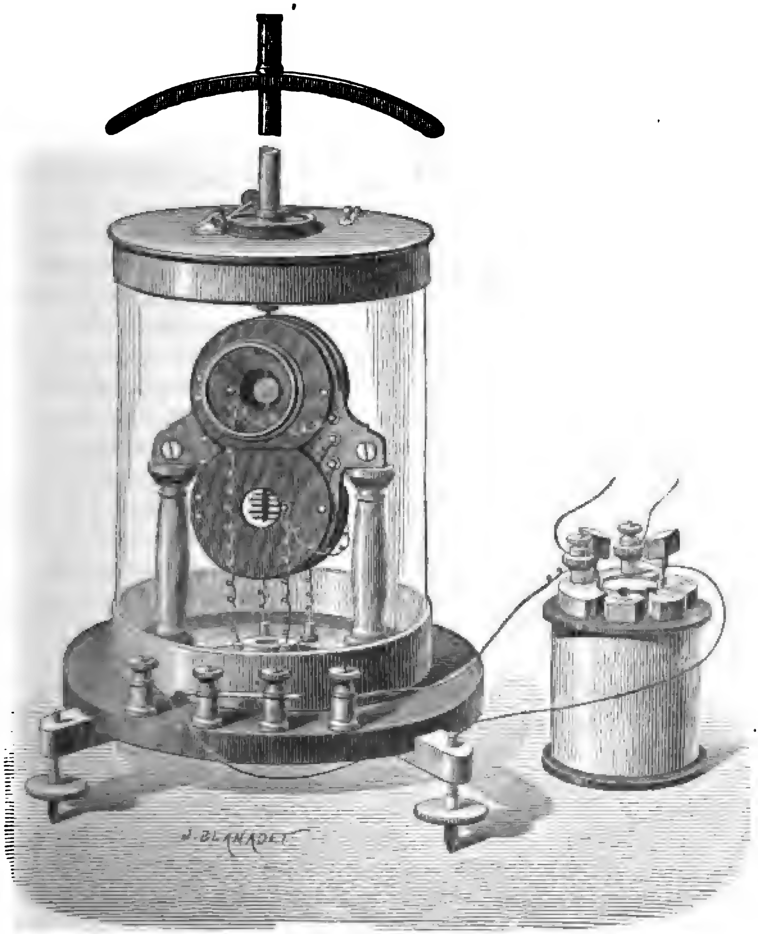


Fig. 163.

currents are strictly proportional to the velocity, whatever be the amplitude of the oscillations.

As these induced currents are very strong, and the oscillating system has a very small moment of inertia, so great a damping is easily obtained that the motion becomes aperiodic.

The coil consists of two cylindrical parts, which are almost in contact, each comprising in its core half the copper sphere. When an astatic system is used, two similar needles are thus placed, each in the centre of the coil, in which the current circulates in contrary directions.

In order to modify the external field, two equal magnets, having their centres in the axis of rotation, are placed under the base of the instrument, and by means of a suitable screw-gear may be placed in any given direction, either in reference to each other, or in respect of the instrument. The time of oscillation is much greater than in the apparatus on Thomson's plan.

851. VARIOUS INSTRUMENTS.—In connection with the instruments we have described, and which are the principal types at present used in scientific researches, we shall mention a few others, which in some of their features have a theoretical interest.

We will describe, first of all, the marine galvanometer of Sir W. Thomson. The motion of a vessel, and its change of direction, make it impossible to utilise the terrestrial field. The coil is placed between the limbs of a horse-shoe magnet, which produces a sensibly uniform magnetic field, and such that the axis of the needle, in its position of equilibrium, is almost in the line of the poles. The field of the earth is so weak compared with that of the magnet, that its action is scarcely perceptible. It is almost completely suppressed, as well as that of the masses of iron in the vessel, if we enclose the entire instrument in a thick iron cylinder, which forms a magnetic screen. A correcting magnet, parallel to the needle, and having its poles opposite those of the fixed magnet, may be displaced parallel to itself by means of a screw. The pendulum motions of the needle are got rid of by suspending it from a short silk fibre, attached to a spring, which exerts a suitable strain. The resistance of these galvanometers may be as much as 30,000 or 40,000 ohms.

852. In order to give galvanometers considerable directive force and great fixity of zero, M. Deprez* also uses the powerful field of a horse-shoe magnet, but he replaces the ordinary magnetised needle by a system of soft iron needles, mounted parallel to each other on the same axis *(fish-back needle)*. As the magnetisation of the needle is almost proportional to the intensity of the resultant field, the directing couple and the action of the current are very powerful, and the time of oscillations is very small. On the other hand, the electromotive force of induction produced by the motion of the

* DEPREZ. *Journal de Physique*, Vol. IX., p. 227. 1880.

needle is very great, and the damping may be such that the system stops without appreciable oscillation in its position of equilibrium.

The plane of the limbs of the horse-shoe magnet is horizontal, as well as that of the coil. The system of needles turns about a horizontal axis, and it is provided with a straw which serves as index against a graduated scale. By a system of levers the deflection of the index may be multiplied.

This apparatus is solidly constructed, and is thus adapted for industrial applications.

853. Differential Galvanometer.—The differential galvanometer invented by A. Becquerel* is an instrument constructed of two coils, which act in contrary directions on a magnetised needle. The currents which traverse the two coils may be either independent, or may form two branches of the same circuit. The deflection of the needle is then proportional to the difference of the actions of the currents, and the experiment consists, in general, in modifying the intensity of one of them, so as to bring the needle to zero. Let I and I′ be the intensities of the two currents, G and G′ the constants of the coils; the deflection δ of the needle will be given by the equation

$$GI - G'I' = H \tan \delta.$$

The deflection δ is zero when

$$GI = G'I';$$

if the instrument has been constructed so that the constants G and G′ are equal, it follows that

$$I = I'.$$

The two coils are usually symmetrical, and of resistance R and R_1, which are equal to each other.

In order to verify a differential galvanometer, the same current should be passed through the two coils in opposite directions; the deflection should be zero if the galvanometer constants are equal. The current is then divided between the two coils; the needle should remain at zero if the resistances are equal.

A simple method of at once realising these two conditions consists in taking a strand of two equal wires, which is then wound on the

* Becquerel. *Ann. de Chim. et de Phys.* [2], Vol. XXXII. 1826. *Traité d'Élect. et de Magn.*, Vol. III., p. 74.

frame. Whatever be the winding and the position of the needle, the actions of the two wires are equal. As the winding of the strand is not very regular, it is usual to wind the two wires simultaneously and parallel to each other on the frame. The system is then equivalent to that of two identical coils, one of which has been displaced parallel to itself by a quantity equal to the diameter of the wire; it follows from this that the actions of the two coils become unequal when the centre of the needle is not in the plane of symmetry. This inconvenience might, it is true, be remedied by taking care to invert the order of the wires at each fresh layer—that is to say, to put on the right that which was previously on the left in the preceding layer; the adjustment of the needle need not then be so exact.

It is, moreover, unnecessary that the coils shall coincide. Thomson's astatic galvanometer, consisting of two coils, with a superposed double frame (**849**), may be used as a differential galvanometer; the currents must separately traverse the two coils in such a direction that their actions on the corresponding needles are in opposite directions. If M and M′ are the magnetic moments of the needles, H and H′ the corresponding fields produced by the combined action of the earth and of a correcting magnet, we have

$$GMI - G'M'I' = (MH - M'H') \tan \delta.$$

Besides equalising the resistance, we should realise the condition $GM = G'M'$; but a more symmetrical arrangement is obtained when the two frames of each coil are connected. In order to facilitate the adjustment, one of the systems has a less resistance than the other; it is then completed by a small additional coil, the action of which coincides with that of the principal coil, and which can be approached to or removed from the corresponding needle. Equality in the resistances being thus obtained, a position is found by trial for this auxiliary coil, so that the same current, traversing the two systems in opposite directions, produces no deflection; the galvanometric constants are then equal.

854. The differential galvanometer is most frequently employed in establishing the identity of two currents; it might, in case of need, measure their difference. The arrangement adopted by Mr. Fleming Jenkin* enables us to measure the ratio of the two currents.

* Jenkin. *Report of the Committee of the British Association.* Dundee, 1867.

Suppose that two identical circular coils, like those of a tangent galvanometer, are fixed at right angles to each other, with a common vertical diameter, and suppose that a needle is suspended in the centre. If the currents I and I′ are passed separately in the two coils, a position may be found for the system such that the needle is in the meridian. If ϕ is the angle which one of the coils makes with the meridian, G and G′ the constants of the two coils, the equation of equilibrium of the needle is then

$$GI\cos\phi - G'I'\sin\phi = 0.$$

From this we get

$$\tan\phi = \frac{GI}{G'I'}.$$

If the galvanometer constants G and G′ are equal by construction, it follows that

$$\frac{I}{I'} = \tan\phi.$$

855. Electromagnetic Balance.—A. Becquerel* was also the first to conceive the idea of weighing the action of the current. Let us suppose that a magnet, suspended vertically to the beam of a balance, is kept in the axis of a coil through which a current passes. There is attraction or repulsion, according to the direction of the current; from symmetry the action would be zero if the middle of the magnet were in the centre of the coil, and it acquires a maximum value when the middle of the magnet is at a certain point P of the axis, beyond the coil. The beam is made horizontal by adding or removing weights, so as to balance the action of the current on the magnet.

The coil should have the form of an elongated cylinder, and the magnet itself should be tolerably long, for otherwise the force would be very feeble. In the case of a repulsion, the equilibrium is stable when the centre of the magnet is more distant from the centre of the coil than the point P, which corresponds to the maximum. It is also stable in the case of attraction, if the centre of the magnet was at the point P, or in the neighbourhood, for the action is then sensibly constant, and the balance has of itself a stable equilibrium. This is

* A. Becquerel. *Comptes rendus*, Vol. v., p. 35. 1837.

evidently the position which should be chosen as standard; it may also be observed that it is the only one for which the sensitiveness of the balance is not modified.

If the magnetisation were constant, the action would be proportional to the intensity of the current. This is the case with very weak currents; but in general the magnetisation produces also a temporary magnetisation, which is at first, for mean currents, proportional to the intensity, and always gives an attractive action.

The condition of equilibrium may then be expressed by an equation of the form

$$p = (A \pm A'I)\, I,$$

in which p is the weight, A and A′ two constants to be determined by experiment.

This formula was used by Lenz and Jacobi,* who made some modification in Becquerel's balance. Their apparatus comprised two coils, and two magnets suspended to the ends of the same beam. The two wires of suspension are unequal, and the two poles V being downwards, one of the magnets is above the corresponding coil, and the other below; the action is one of repulsion.

The proportionality of the temporary magnetisation to the intensity of the current would be inadmissible for very powerful currents, and the apparatus should then be graduated experimentally.

If the magnet is replaced by a mass of soft iron, the action, which is always one of attraction, commences by being proportional to the intensity of the current, and then increases less rapidly; in this case also an empirical graduation is necessary.

The second arrangement is to be preferred for strong, and the first for weak, currents.

856. Movable Coils.—When a coil of surface S, is traversed by a current I, its magnetic moment is SI. If the coil is attached to a unifilar or bifilar suspension, so that, for the position of equilibrium, its axis is perpendicular to the magnetic meridian, the passage of this current tends to bring the axis into the meridian. We may again either bring the coil to its original position by a convenient torsion of the suspension, as in the method of torsion applied to magnets (830), or, as in ordinary galvanometers, we may leave the apparatus to itself, and observe the deflection which corresponds to the fresh position of

* Lenz and Jacobi. *Pogg. Ann.*, Vol. XLVII., p. 227. 1839.

equilibrium; this second method is generally employed. We have then, for a bifilar suspension,

$$\mathrm{HSI}\cos\delta = \mathrm{C}\sin\delta,$$

or

$$(22) \qquad \mathrm{I} = \frac{\mathrm{C}}{\mathrm{HS}}\tan\delta,$$

and for a unifilar suspension,

$$(23) \qquad \mathrm{I} = \frac{\mathrm{C}}{\mathrm{HS}}\frac{\delta}{\cos\delta}.$$

This method is due to Weber.* If the suspension is bifilar, the two wires are used to convey the current to the coil; if it is unifilar, the circuit is completed by a second vertical wire above the coil, and attached to a slight spring, or by a rod which dips in a mercury cup.

In order to regulate the original position of the coil, its axis is put almost in the meridian, and the suspension is adjusted until the passage of a current in the coil produces no deflection; the suspension on the whole apparatus is then turned through 90°.

The slight defect of adjustment which might still remain is corrected by reversing the direction of the current in the coil. If α is the angle of the axis of the coil, with the perpendicular to the meridian, δ_1 and δ the deflections on each side for the two directions of the current, with a bifilar suspension, the equations of equilibrium

$$\mathrm{HSI}\cos(\delta+\alpha) = \mathrm{C}\sin\delta,$$

$$\mathrm{HSI}\cos(\delta'-\alpha) = \mathrm{C}\sin\delta',$$

would enable us, as has been seen above (834), to calculate the angle α; but if the deflections δ and δ' do not differ much, the angle α which represents the error of adjustment is itself very small, and we have sensibly

$$\mathrm{I} = \frac{\mathrm{C}}{\mathrm{HS}}\tan\frac{\delta+\delta'}{2}.$$

* W. Weber. *Electrodyn. Maasbestimmungen*, Vol. I., p. 16. 1846.

A unifilar suspension in like manner would give

$$I=\frac{C}{HS}\frac{\frac{\delta+\delta'}{2}}{\cos\frac{\delta+\delta'}{2}}.$$

The conditions of sensitiveness of the apparatus, in the case of bifilar suspension, are the same as for a tangent galvanometer.

With unifilar suspension we have

$$S_a=\frac{HS}{C}\frac{\cos\delta}{1+\delta\tan\delta},$$

$$S_r=\frac{\delta}{1+\delta\tan\delta}.$$

The absolute sensitiveness, which is proportional to the intensity of the field and to the surface of the coil, is again a maximum for zero deflection; and the relative sensitiveness is a maximum for $\delta=\cos\delta$, or about $\delta=40°$.

857. The intensity of the current in the tangent galvanometer is proportional to the horizontal component of the terrestrial field; for a movable coil, on the contrary, it is inversely proportional. A combination of the two methods would thus enable us to get rid of the action of the earth, and determine the intensity of the current as a function of the dimensions of the two instruments and of the directing couple of the suspension of the coil.

If, as Kohlrausch* has pointed out, we pass one and the same current I through a tangent galvanometer and through a movable coil with bifilar suspension, then indicating by an accent the deflection and the elements of the coil, and supposing all the corrections for adjustment made, we shall have

$$I=\frac{H}{G}\tan\delta=\frac{C'}{HS'}\tan\delta'.$$

* F. Kohlrausch, *Pogg. Ann.*, Vol. CXXXVIII., p. 1. 1869.

These two equations give separately the intensity I and the component H :—

$$(24)\qquad I^2 = \frac{C'}{GS'}\tan\delta\tan\delta',\qquad H^2 = \frac{GC'}{S'}\frac{\tan\delta'}{\tan\delta}.$$

The magnetic field is not in general identical for both instruments. The ratio of the intensities H and H' of the two fields should then be determined; for instance, by the oscillations of the same needle.

858. Sir W. Thomson has proposed* to combine the experiment so as to obtain two deflections δ and δ' by the same instrument. This double galvanometer is made of a rather large frame with a small needle at the centre, as in the tangent compass; both frame and needle, however, being movable. Both being parallel to the magnetic meridian in the original position of equilibrium, the deflections δ and δ' are observed which the needle and the coil experience in opposite directions during the passage of the current.

The equilibrium of the needle is defined by equation

$$(25)\qquad \mathrm{IMG}\cos(\delta+\delta') = \mathrm{MH}\sin\delta;$$

the action of the needle on the coil being equal to that of the coil on the needle, we have, on the other hand, for the equilibrium of the needle with a bifilar suspension

$$(26)\qquad \mathrm{HIS}\cos\delta' + \mathrm{IMG}\cos(\delta+\delta') = C\sin\delta'.$$

From these two equations it follows that

$$(27)\qquad I = \frac{C}{HS}\,\frac{\tan\delta'}{1+\dfrac{MG}{HS}\dfrac{\cos(\delta+\delta')}{\cos\delta'}} = \frac{C}{HS}\,\frac{\tan\delta'}{1+\dfrac{M}{IS}\dfrac{\sin\delta}{\cos\delta'}}.$$

To deduce from these equations the values of I and of H, it is necessary to know one of the ratios $\frac{MG}{HS}$ or $\frac{M}{IS}$; but as the needle

* MAXWELL. *Electricity and Magnetism*, Vol. II., p. 337.

is very small the factor of correction is very small, and we might determine the ratio $\frac{M}{H}$ approximately by the ordinary methods.

The apparatus would indeed allow this correction to be made directly. For if the needle is fixed in its original position independently of the coil, and we read the deflection δ' of the coil produced by the same current, the equation of equilibrium is then

$$C \tan \delta_1 = HIS \left(1 + \frac{MG}{HS} \right);$$

comparing with equation (27), we get

$$(28) \qquad \frac{MG}{HS} \frac{\cos(\delta + \delta')}{\cos \delta'} = \frac{\tan \delta_1 - \tan \delta'}{\frac{\sin \delta'}{\cos(\delta + \delta')} - \tan \delta_1}.$$

859. The idea of Sir W. Thomson does not seem to have as yet been realised in this simple form. The accuracy of the method requires that the deflections of the coil and of the needle are of the same order of magnitude. We could only attain this result with a very light coil, having only a few turns of wire and supported by a suspension with very feeble coefficient. In these conditions it wquid be difficult to make the coil sufficiently rigid to know exactly its dimensions, and to determine its moment of inertia so as to deduce from it the couple of torsion. These drawbacks may all be avoided by using one of Weber's arrangements for the tangent galvanometer.

The needle is placed outside the coil in a principal position, on the axis or on the mean plane, and at so great a distance that the action of the current is materially diminished. The coil may in that case have a great number of windings, and a suspension may be used, the directing couple of which is easily determined.

Suppose, for instance, the suspension so adjusted that the mean plane of the coil in equilibrium is in the meridian, and that the needle is placed on the axis at a distance d from the centre. If δ and δ' are the deflections of the needlé, and of the coil, produced by the passage of a current I, y and x the co-ordinates of the

middle P of the needle in respect of the axis and of a right line in the mean plane of the coil for its new position, we have first

$$y = d \sin \delta',$$
$$x = d \cos \delta'.$$

If the deflection δ is very small, the components X and Y at the point P of the action of the coil, which we suppose reduced to the winding of mean radius a for unit current, may be calculated as in **736**.

The expression for the couple produced on an infinitely small needle is

$$M\Big[X \cos(\delta + \delta') + Y \sin(\delta + \delta')\Big] = M \cos(\delta + \delta') \Big[X + Y \tan(\delta + \delta')\Big].$$

Replacing X and Y by their expansion in series as function of y, and disregarding terms of a higher order than the second, we have

$$X + Y \tan(\delta + \delta') = 2\pi \frac{a^2}{u^3}\left\{1 + \frac{3}{2}\frac{d^2}{u^2}\sin\delta'\Big[\cos\delta' \tan(\delta + \delta') - 2\left(1 - \frac{5}{4}\frac{a^2}{u^2}\right)\sin\delta'\Big]\right\}.$$

If the needle is at some distance from the coil, we may in the term of correction replace u by d, the sines and tangents by their corresponding angles, and neglect the product of the square of the deflection by the ratio $\frac{a^2}{d^2}$; we then get

$$X + Y \tan(\delta + \delta') = 2\pi \frac{a^2}{u^3}\Big[1 + \frac{3}{2}\delta'(\delta - \delta')\Big].$$

To allow for the length of the needle, this result is multiplied by the usual factor (**746**), which reduces sensibly to $1 - 3\frac{l^2}{d^2}$.

We get then the expression

$$D = X + Y \tan(\delta + \delta') = 2\pi \frac{a^2}{u^3}\Big[1 + \frac{3}{2}\delta'(\delta - \delta') - 3\frac{l^2}{d^2}\Big],$$

in which we shall replace u^2 by $x^2 + a^2 = d^2 \cos^2\delta' + a^2$.

Lastly, in order to proceed to the case of a coil, we will substitute for the principal factor $2\pi\frac{a^2}{u^3}$, which represents the action of the coil on the axis at the distance x, the value of G given by equation (12) of (729).

The value of the factor D being thus calculated from the dimensions of the coil and the distance of the needle, with the corrections for the deflections and for the length of the needle, the equations of equilibrium are then for the needle

$$(29) \qquad \mathrm{IMD}\cos(\delta+\delta') = \mathrm{MH}\sin\delta,$$

and for the coil

$$(30) \qquad \mathrm{HIS}\cos\delta' + \mathrm{IMD}\cos(\delta+\delta') = \mathrm{C}\sin\delta';$$

they only differ from equations (25) and (26) by the substitution of D for G.

In order to determine directly the term of correction relative to the magnetisation of the needle, it is sufficient to fix it in its original position, and observe the fresh deflection δ_1 of the coil.

Equation (30) gives, in that case, $\delta = 0$,

$$\mathrm{C}\tan\delta_1 = \mathrm{HIS}\left(1 + \frac{\mathrm{MD}}{\mathrm{HS}}\right);$$

we shall get from it the ratio $\frac{\mathrm{MD}}{\mathrm{HS}}$, as a function of the angles δ, δ', and δ_1 by an expression similar to equation (28).

We should calculate in a similar way by the formulas of (795) the experiments for the case in which the needle is in the mean plane of the coil.

To construct such an instrument, the coil is mounted so as to be movable about a vertical axis provided with a graduated circle. The needle is placed in a case itself movable about a vertical axis, and supported by a carriage which slides along a divided scale. The divided scale can finally also rotate about the same axis as the mounting of the coil.

It will easily be seen that these various arrangements enable us, either directly or with rotations of 90° and appropriate reversals, 1, To place the mean plane of the coil in the meridian; 2, To

place the needle on the axis of the coil or in its mean plane; 3. To increase the distance d from the needle to the central coil.

The needle and the coil support, moreover, two mirrors in which an observer can, from a fixed position, view with two telescopes the images of two divided scales, having also a key to reverse the current so as to get rid of the remaining errors of adjustment in the usual way.

860. Syphon Recorder.—Apparatus with movable coils may acquire great sensitiveness if the intensity of the external field is

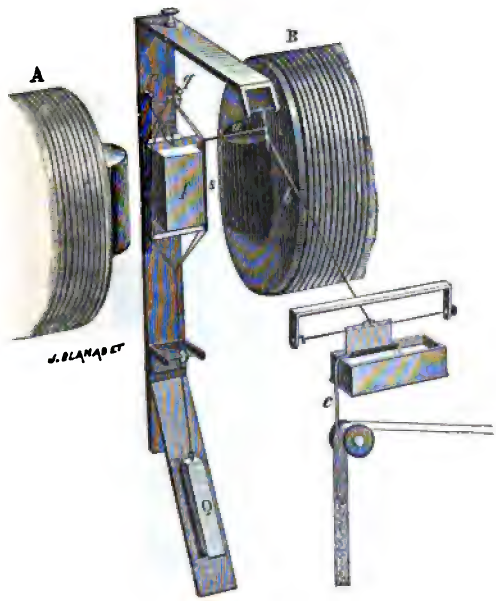


Fig. 164.

sufficient. Sir W. Thomson made use of this property in constructing his *syphon recorder*, which is used as a receiving instrument in submarine telegraphy (Fig. 164). The coil s is wound on a thin rectangular frame placed between the poles of an electro-magnet, the ends of which A and B are marked on the figure. A fixed mass of soft iron f occupies the space left in the centre of the coil, and increases the strength of the field in the region which the wire traverses. The coil is attached to a bifilar suspension. The lower part carries a weight Q which slides along a board, which is more or less inclined to the vertical and serves to regulate the

tension. The current enters by two very flexible spirals connected with the terminals p and q.

The damping is rapid, for the induction due to the displacement of the coil is very great.

The name "syphon recorder" is due to a detail of its construction. The coil acts by a wire *ab*, and amplifying levers, on a capillary glass tube *c*, bent in the shape of a syphon, and the shorter branch of which is immersed in a coloured liquid. The liquid, being electrified

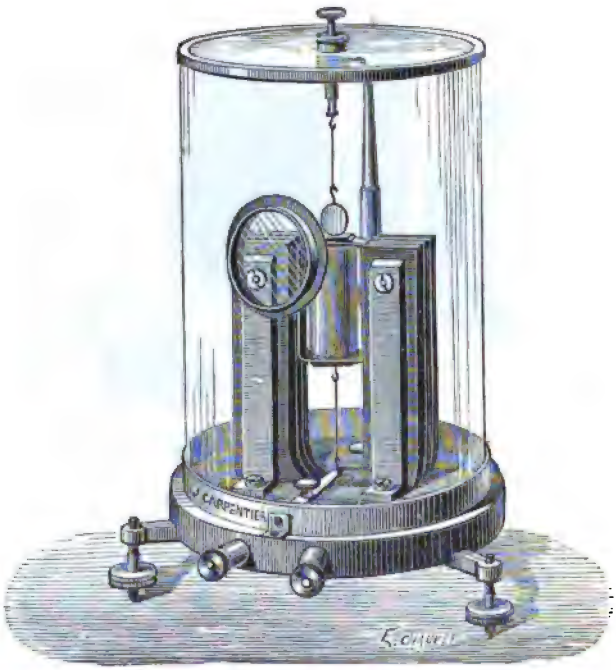


Fig. 165.

by a small rotating electrical multiplier (**195**) (*mouse mill*), spirts out of the tube against a paper strip which is being paid out in front. The motion of the coil is transmitted to the syphon, and produces a continuous line on the paper, with indentations on the right or left, corresponding to the + or − signs of the current.

861. Deprez and d'Arsonval* have constructed a galvanometer

* DEPREZ and D'ARSONVAL. *Comptes rendus*, Vol. XCIV., p. 1347. 1882.

on the same principle, which is equally remarkable for the rapidity of its damping (Fig. 165). The frame, which is rectangular, is suspended by two metal wires which convey the current, and the tension of which is regulated by a screw. The field is produced by a horse-shoe magnet, with branches very close, and a hollow cylinder of soft iron is placed inside the coil. This cylinder is magnetised almost as a conductor would be electrified in an electrical field of the same shape (**387**). The deflections are read by means of a mirror.

862. LIPPMANN'S GALVANOMETER.—With the same order of ideas we may associate the galvanometer of M. Lippmann.* Two vertical tubes containing mercury (Fig. 166) are connected at the

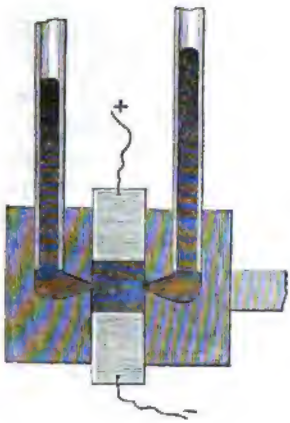


Fig. 166.

bottom with a very narrow cavity comprised between two parallel glass plates, and placed in a magnetic field, the direction of which is at right angles to the plane of the plate.

The current conveyed by two platinum plates passes vertically through the liquid contained in this cavity. If the cavity is rectangular, and of height a, the action of the field H on the current is equal (**458**) to I$\mathrm{H}a$, and tends to displace the liquid in a certain direction; a difference of level is produced between the two branches of the mercury. If p is the difference of the corresponding pressure, and e the thickness of the cavity, the condition of equilibrium of the liquid is

$$pea = \mathrm{IH}a,$$

* LIPPMANN. *Comptes rendus*, Vol. XCVIII., p. 1256. 1884.

or

$$I = \frac{e}{H} p.$$

For the same current the difference of pressure is proportional to the intensity of the field, and inversely as the thickness of the cavity.

863. ELECTRODYNAMOMETERS.—Weber's electrodynamometer* is a movable coil, with unifilar or bifilar suspension, on which the current of a fixed coil is made to act. In the original position of equilibrium the two coils are at a right angle, and in order to diminish the action of the earth as much as possible, the axis of the movable coil is perpendicular to the magnetic meridian.

If S' is the surface of the movable coil, I' the current which passes through it, I the current of the fixed coil, these currents tend to bring the axes of the coils near each other; and for a deflection δ from the original direction the couple of the reciprocal action is $II'GS' \cos \delta$, the factor G being a mean value of the galvanometric constant of the fixed coil in the space occupied by the movable coil.

This couple may be counterbalanced by twisting the suspension at the top through an angle θ, which brings the coil into the original position. We have then, for a bifilar,

$$II'GS' = C \sin \theta,$$

and for a unifilar,

$$II'GS' = C\theta.$$

The angle θ changes its sign when the direction of one of the currents is reversed, but it remains the same when the direction of both are reversed; this is a characteristic property of the electrodynamometer. When the two currents are equal we have, according to the case,

$$I^2 = \frac{C}{GS'} \sin \theta, \qquad \text{or} \qquad I^2 = \frac{C}{GS'} \theta.$$

If, on the contrary, the deflection δ is observed, which the movable coil takes up, the external field affects the condition of equilibrium, and we have then, according to the mode of suspension,

$$(31) \quad \begin{aligned} II'GS' \cos \delta &= (C \pm S'I'H) \sin \delta, \\ II'GS' \cos \delta &= C\delta \pm S'I'H \sin \delta. \end{aligned}$$

* W. WEBER. *Electrodyn. Maasbestimmungen*, Part I. 1846.

864. These equations only hold if the apparatus is well adjusted. Suppose that, in the original condition, the coils are not exactly rectangular, but that the axis of the movable coil makes an angle α, and the plane of the fixed coil an angle β, with the magnetic meridian.

The deflections δ_1 and δ_2, obtained by the two different directions of the current in the movable coil, give, with a bifilar suspension, equations

$$II'GS' \cos(\delta_1 + \alpha + \beta) = C \sin\delta_1 + I'S'H \sin(\delta_1 + \alpha),$$
$$II'GS' \cos(\delta_2 - \alpha - \beta) = C \sin\delta_2 - I'S'H \sin(\delta_2 - \alpha).$$

The deflections δ_3 and δ_4, for the change in direction of the current in the fixed coil, give similarly

$$II'GS' \cos(\delta_3 - \alpha - \beta) = C \sin\delta_3 + I'S'H \sin(\delta_3 - \alpha),$$
$$II'GS' \cos(\delta_4 + \alpha + \beta) = C \sin\delta_4 - I'S'H \sin(\delta_4 + \alpha).$$

If the conditions of adjustment are nearly satisfied, and the coefficient C is great in respect of the product I'S'H, the angles α and β are very small, and the values of the four deflections very near each other. By combining these equations as above (834) and neglecting the squares of small angles, we have

$$II' = \frac{C}{GS'}\frac{1}{2}\left(\tan\frac{\delta_1 + \delta_3}{2} + \tan\frac{\delta_2 + \delta_4}{2}\right),$$

or sensibly

(32) $$II' = \frac{C}{GS'} \tan\frac{\delta_1 + \delta_2 + \delta_3 + \delta_4}{4}.$$

In the preceding conditions, we may thus neglect the action of the earth, and take as standard deflection the mean of four readings.

With the method of torsion, we shall take as standard value the mean of the four angles necessary to bring the needle back to its original position.

The experiment being somewhat long, alternate tests must be made to get rid of variations in the current.

865. With the electrodynamometer it would be difficult to attain the same degree of sensitiveness as with the galvanometer. But as

the method gives results which are independent of the intensity of the external magnetic field, or at any rate in which terms of correction only come in, it is particularly suitable for absolute measurements. It is then necessary to calculate the coefficients G and S′ for the two coils.

In Weber's apparatus the two coils are concentric, and the movable coil, consisting of a large number of windings, occupies a considerable part of the space left by the fixed coil; with this mode of construction the calculation of G in particular presents great difficulties.

In the dynamometer constructed by Latimer Clark* for the British Association, the two coils are still concentric; but the diameter of the movable coil is small compared with that of the fixed one, and each of them is closed, as was Helmholtz's coil, by two equal coils, the distance of which is equal to their mean radius. The calculation of G is then easily deduced from the dimensions of the coils by formula (70) of (**793**).

The two coils, being always at right angles to each other in the original condition of equilibrium, which facilitates calculation, the coils can be displaced from each other, so that, for instance, the movable coil occupies a principal position—that is to say, that its centre is in the mean plane, or on the axis of the fixed coil. These arrangements were adopted in Weber's experiments on the elementary action of currents on currents. The formulas in (**793**) and (**794**) enable us to calculate the couple due to the reciprocal action of the two coils.

866. Balance Electrodynamometers.—Another mode of using the electrodynamometer consists in weighing, by means of a balance, the attraction or repulsion between circuits traversed by currents.

We have seen (**786**) that the reciprocal action exerted between two coils A and A′, having the same axis, passes through a maximum value when the distance of the centres is suitably chosen. It is evidently advantageous to take this position of maximum for that of equilibrium of the movable coil, for the action is then virtually constant—at least, within certain limits of oscillation—and that therefore the sensitiveness of the balance is not modified.

In strict accuracy, equilibrium would be unstable from the electrical point of view; but the stability which the balance itself gives is sufficient to allow of observations.

* Maxwell. *Electricity and Magnetism*, Vol. II., p. 339.

Joule* applied this method to determine variations of current in his calorimetrical researches. He used the system of three symmetrical coils whose properties have been studied above (792). The three coils were of equal dimensions, and each formed of a flat spiral. In these conditions the formulæ are difficult of application. Dr. Joule contented himself with observing that, if we only take into account the actions of portions of the adjacent wires, regarding each element as under the action of an infinite parallel current, and if l is the total length of the wire in each coil, S the surface, and p the weight necessary to keep the movable coil at an equal distance from the fixed ones, the intensity of the current is

$$I = \frac{1}{2l}\sqrt{\frac{Sgp}{\pi}}(1 + A),$$

in which A stands for a correction depending on the dimensions of the apparatus, and which is determined by comparison with a tangent compass.

Lallemand† had already used a similar arrangement, with this difference, that the movable spiral was placed at the end of a horizontal lever supported by a metal wire, the torsion of which could be varied. Here the action of the earth is not null, as in Joule's balance; but it is eliminated by placing at the other end of the lever a coil symmetrical with the first, and traversed in the same direction by the current (489).

Such, also, is the electrodynamometer used by Maxwell‡ in his investigations on the ratio of the electrostatic and electromagnetic units. A horizontal lever, suspended in the middle to a wire, supports two flat vertical coils. Each of these is placed between two other fixed coils of larger diameter, the distance of which satisfies the condition of the maximum (792).

On this subject it may be observed that the square of the intensity of a current has the same dimensions (609) as a force. The factor by which the action exerted on the movable coil must be multiplied so as to obtain the square of the intensity of the current is thus an abstract number. We see from the values of ξ in (787) and (792) that we can consider this factor as containing

* JOULE. *British Association Report, 1864. Scientific Papers*, Vol. I., p. 584.

† LALLEMAND. *Ann. de Chim et de Phys.* [3], Vol. XXXII., p. 432. 1854.

‡ MAXWELL. *Phil. Trans. Roy. Soc., 1868*, p. 643.

the ratio of the distance $2x$ of the fixed coils to their mean radius a, and the ratio of the mean radii a and a'. This latter ratio may be reduced to a comparison of the two resistances, and may be determined electrically with the greatest precision (876).

Instead of circular coils, Cazin* used two parallel rectangular coils. The calculation for this action was given in (492).

Mascart† also used, with the same object, the action (791) of two equal and parallel coils on a long coil with the same axis as the first, and the lower plane of which is in the plane of symmetry of the two fixed coils. The movable coil A′ is suspended to the plate of a balance, and the current is conveyed there by very flexible platinum wires. The reciprocal action being near the maximum, the equilibrium is perfectly stable.

As the intensity of the current is almost always subject to continual variations, it is difficult to obtain an exact balance; but small differences may be estimated by observing the displacement of the index. If we want to observe in the position of equilibrium itself, and the current is diminishing, for instance, the action being attractive, weights which are slightly too small are added, and the moment of the passage of the needle through zero is observed. The same operation, repeated from time to time, would enable us to fix points for constructing the curve of intensities from which the value of the current is deduced at a given time, or the integral in respect of the time. This integral will give the total quantity of electricity which has passed.

A similar device may be applied with all apparatus, whether of torsion or depending on the time, which require the return of a movable part to a fixed position.

We may, in conclusion, cite an arrangement of Von Helmholtz,‡ which, though not suited for absolute measurements, is very convenient. To the pans of an ordinary balance are suspended two identical coils, the diameter of which is equal to the height, and which can move freely inside two fixed coils of the same height. These are supported by a horizontal metal arm, which may be fixed on the pillar of the balance. One acts by attraction and the other by repulsion on the corresponding movable coil. The position of the fixed coils is regulated by the condition that the

* CAZIN. *Ann. de Chim et de Phys.* [4], Vol. I., p. 257. 1864.

† MASCART. *Journal de Physique* [2], Vol. I., p. 109. 1882.

‡ HELMHOLTZ. *Proceedings of the Roy. Soc., 1881.* — *Wissenschaftliche Abhandlungen*, Vol. I., p. 922.

balance shall have the same sensitiveness before and after the passage of the current. The action is then a maximum.

The current is transmitted to the movable coils by strips of thin tinsel 5 mm. to 6 mm. in breadth. The resistance of these strips is very small. Owing to their large surface they do not become appreciably heated, and they do not prevent oscillations.

867. BRANCH CIRCUITS AND SHUNTS.—When the intensity of the current to be measured is too great for the sensitiveness of the galvanometer, a given fraction of it is measured by a branch circuit. The principal current, terminating at the two terminals of a galvanometer, is divided into two parts, one of which passes through the coil and the other by a branch circuit, to which is usually given the name of *shunt.*

If g is the resistance of the galvanometer, s that of the shunt, I the principal current, i that which passes in the galvanometer with the shunt, we have

$$(33) \qquad gi = s(\mathrm{I} - i) = \frac{s}{g+s}\mathrm{I}, \qquad \text{or} \qquad \mathrm{I} = \frac{g+s}{s}i.$$

The factor $\frac{g+s}{s} = m$, by which the current observed must be multiplied to obtain the value of the principal current, is called *the multiplying power* of the shunt.

The resistance g_1 of the system, formed by the shunt and the galvanometer, is

$$g_1 = \frac{1}{\frac{1}{g} + \frac{1}{s}} = \frac{g}{\frac{g+s}{s}} = \frac{g}{m}.$$

In order to have a shunt of power m it is better, instead of measuring the resistance $s = \frac{g}{m-1}$, to adjust the wire on the two terminals of the galvanometer until the resistance of the whole is equal to $\frac{g}{m}$.

Let us suppose that the circuit contains a constant electromotive force E. Let R be the resistance external to the terminals of the galvanometer, I_0 the intensity which the current would have in the

circuit R alone, I_1 its intensity in the resistance $R+s$ of the circuit and the shunt, I_2 in the resistance R and the galvanometer, and I when a shunt of multiplying power m is added to the galvanometer. The equations

$$(34) \qquad E = I_0 R = I_1(R+s) = I_2(R+g) = I\left(R + \frac{g}{m}\right)$$

will enable us to calculate the different intensities I_0, I_1, and I_2 as a function of the total intensity I, and therefore as a function of the observed intensity i.

In order that the intensity in the external circuit shall be the same, whatever shunt is added to the galvanometer, a resistance ρ must be inserted at the same time in the undivided circuit, such that

$$R + g = R + \frac{g}{m} + \rho,$$

$$(35) \qquad \rho = g\left(1 - \frac{1}{m}\right).$$

With this object the arrangement represented in Fig. 167 is

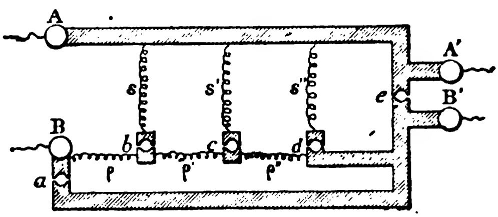


Fig. 167.

used, which explains itself: s, s', s'' are the various shunts of resistances $\frac{g}{m-1}$, $\frac{g}{m'-1}$, $\frac{g}{m''-1}$, and ρ, ρ', ρ'' resistances which satisfy equations

$$\rho = g\left(1 - \frac{1}{m}\right),$$

$$\rho + \rho' = g\left(1 - \frac{1}{m'}\right),$$

$$\rho + \rho' + \rho'' = g\left(1 - \frac{1}{m''}\right).$$

A plug placed in *a* allows the entire current to pass through the galvanometer G; placed at *b*, it introduces both the shunt s and the compensating resistance ρ; and so on for the other positions *c* and *d*. When the plug is at *e*, it closes the galvanometer, thus making it safe.

A system of shunts, provided with compensating resistances, forms the necessary complement of any accurate galvanometer (Fig. 163). The shunts have usually the resistances $\frac{g}{9}, \frac{g}{99}, \frac{g}{999}$, so that their multiplying powers are then 10, 100, 1000.

The use of shunts greatly facilitates galvanometric operations, but it involves some inconveniences which require great care whenever exact measures are to be made. One of the principal arises from changes of temperature, which, as they affect unequally the wire of the coil and that of the shunt, alter the value of the multiplier to an unknown extent.

868. Measurement of Currents by Fall of Potential.—An arrangement frequently adopted, and which is particularly advantageous in the case of powerful currents, consists in placing the ends of the galvanometer wire in connection with the two points A and B of the principal circuit, comprising between them a resistance s which plays the part of a shunt. If the resistance of the galvanometer is considerable in comparison with that of the shunt, the compound resistance

$$g_1 = \frac{g}{m} = \frac{gs}{g+s} = s\frac{g+s}{g}$$

differs little from the resistance s, and the intensity I_1 from intensity I, so that the introduction of the galvanometer as a shunt does not sensibly alter the intensity of the total current. We can then estimate directly the difference of potential $E = V_1 - V_2$ between the two points A and B by comparing the deflection δ obtained with the deflection δ_0 given by a standard electromotive force E_0, such as that of a Daniell's cell, the internal resistance of which is very small compared with that of the galvanometer. We have

$$(36) \qquad \frac{E}{\delta} = \frac{E_0}{\delta_0}, \qquad \text{from which} \qquad i = \frac{E}{s} = \frac{E_0}{s}\frac{\delta}{\delta_0}.$$

The introduction of a galvanometer of very great resistance between two points A and B is equivalent to the use of an electrometer, the electrodes of which are connected to the same points. The electrometer, once graduated for a standard electromotive force, gives the difference of potential E, from which is deduced the intensity i of the current.

As a particular case, if we use the quadrant electrometer as in (816), connecting the needle with one pair of quadrants, and those separately to the points A and B, then if k is a constant for the instrument, the deflection δ may be represented by the formula

$$\delta = \frac{k}{2}(V_1 - V_2)^2 = \frac{k}{2}E^2.$$

If δ_0 is the deflection produced by the standard electromotive force E_0, we have again

$$\delta_0 = \frac{k}{2}E_0^2,$$

and therefore

$$i = \frac{E}{s} = \frac{E_0}{s}\frac{E}{E_0} = \frac{E_0}{s}\sqrt{\frac{\delta}{\delta_0}}.$$

869. The difference of potential measured between two points A and B by a galvanometer of great resistance, or by an electrometer, may be due merely to the passage of a current in the resistance, or it may comprise an electromotive force of any nature whatever. In any case, knowing the value I of the current and the difference of potential E of the points A and B, the product EI represents the electrical work between these two points.

We may determine these factors separately by a galvanometer of great resistance or by an electrometer. The apparatus is successively connected with two points A and B, comprising the total electromotive force E, and with two other points A′ and B′, simply separated by a known resistance R.

The work EI may be directly estimated by the quadrant electrometer.† If V_1, V_2, V_1', and V_2' are the respective potentials of the

* JOUBERT. *Comptes rendus*, Vol. XCI., p. 161. 1880.—*Annales de l'École Normale* [2], Vol. X., p. 131. 1881.

† POTIER. *Journal de Physique*, Vol. IX., p. 445. 1881.

points A and B, A′ and B′, the two pairs of quadrants are connected with A′ and B′ separated by a resistance R. If the needle is connected with the point A, the deflection α observed satisfies the equation

$$\alpha = k(V'_2 - V'_1)\left[V_1 - \frac{V'_2 + V'_1}{2}\right];$$

if the needle is connected to the point B, the deflection β gives also

$$\beta = k(V'_2 - V'_1)\left[V_2 - \frac{V'_2 + V'_1}{2}\right].$$

Subtracting these two equations from each other, we get

$$\beta - \alpha = k(V'_2 - V'_1)(V_2 - V_1);$$

as $V'_2 - V'_1 = RI$, and $V_2 - V_1 = E$, it follows that

$$EI = \frac{\beta - \alpha}{kR}.$$

We may, finally, make use of an electrodynamometer the fixed coil of which is in the principal circuit, and the movable coil, which is of great resistance, is placed as a shunt at the two points A and B. The deflection observed is in that case proportional to the work of the electricity between these two points.*

870. Graduation of Galvanometers.—In mirror instruments, where the deflections are very small, the intensity of the current is proportional to the tangent of the angle of deflection, or to the angle itself, and this is also the case with all galvanometers in which the deflections are within certain limits. This proportionality ceases, however, to hold if the needles are long in comparison with the dimensions of the coil; an empirical graduation is then necessary.

If we have at command a galvanometer with a systematic graduation, such as a sine or a tangent galvanometer, the same current is passed through the standard instrument and the galvanometer under trial, using a shunt if necessary, so as to bring the

* M. Deprez. *Comptes rendus*, Vol. xc., p. 592. 1880.

deflections of the two instruments to a convenient magnitude. If, then, the strength of the current is varied, the indications of the graduated instrument will give those of the other. Yet we should avoid, as much as possible, the use of an auxiliary apparatus.

871. In experiments on radiant heat we seek to obtain deflections which represent numbers proportional to the quantities of heat which fall in unit time on a thermoelectrical pile placed in the circuit. The mode of graduation should, in that case, be relative to the quantities of heat, and it only represents an electrical graduation if the intensity of the current is proportional to the calorific radiation on the pile, or, more exactly, to the difference of radiations which fall on the two faces.

The method used by Melloni* consists in placing on each side of the pile constant sources of heat S and S′. On one of the faces the radiation from the source S is allowed to fall, and a deflection δ is observed. The action of the source S being suppressed, the source S′ is allowed to act on the other face. This gives a deflection δ' opposite that of the first. Finally, the two sources acting simultaneously give a deflection δ'', for instance, of the same sign as the first. If we have $\delta'' = \delta - \delta'$, it may be assumed that the deflections are proportional to the radiations. In most of the galvanometers constructed on Nobili's plan, this proportionality holds up to 20 or 25 degrees; beyond that $\delta'' > \delta - \delta'$, and a table of graduation must be constructed. If the deflection δ alone is outside the limits of proportionality, we have

$$\frac{S'}{\delta'} = \frac{S - S'}{\delta''} = \frac{S}{\delta' + \delta''};$$

the observed deflection δ corresponds to the graduation $\delta' + \delta''$, and so on from step to step.

We may also place the two sources on the same side of the pile, and make them act at the same angle; or even with a single source make use of the law of the square of the distances. M. P. Desaines uses a single source in front of which he places a screen perforated by an aperture consisting of four equal sectors. When oné, two, three, or four of the sectors were left open, he obtains calorific radiations, which are in the ratio 1, 2, 3, and 4, which he causes to act successively on the pile. If the experiments are repeated at different distances, we shall have all the elements

* MELLONI. *Ann. de Chim. et de Phys.* [2], Vol. LIII., p. 5. 1833.

necessary for a complete graduation of the galvanometer as far as the limits of the scale.

872. Poggendorff's method* consists in transforming the galvanometer into a sine one to make the graduation. If the instrument does not possess a horizontal circle, it is placed over a graduated circle which can turn about a vertical axis. The coil being at first parallel to the meridian, a constant current is passed, and the deflection δ_0 gives an equation of the form

$$I = H \frac{\sin \delta_0}{\phi(\delta_0)}.$$

The coil is then turned through the quantities α', α'', α''' from the initial position, and the corresponding deflections δ', δ'', δ''' of the needle with the coil are observed. We have

$$\frac{I}{H} = \frac{\sin \delta_0}{\phi(\delta_0)} = \frac{\sin(\alpha' + \delta')}{\phi(\delta')} = \frac{\sin(\alpha'' + \delta'')}{\phi(\delta'')} = \ldots\ldots$$

The values of $\phi(\delta)$ are then proportional to the sine of the angles determined by experiment, which gives the table of the values of $\frac{\sin \delta}{\phi(\delta)}$ for the normal position of the coil.

This method would enable us, for instance, to determine the factor of correction of a tangent galvanometer. If the law of the tangents was exact, the value of $\phi(\delta)$ should in fact, to within a factor, be equal to $\cos \delta$.

873. Petrina† used the method of shunt circuits by placing the galvanometer wire as a shunt between two points of a homogenous rectilinear conductor traversed by a current. If the resistance g of the galvanometer is very great compared with the resistance comprised between two points (868), the intensity I of the principal current and that i of the galvanometer satisfy the ratio

$$i = \frac{s}{g} I.$$

The intensity I being constant, the intensity i is proportional to the

* POGGENDORFF. *Pogg. Ann.*, Vol. LVI., p. 324. 1842.

† PETRINA. *Holger's Zeitschrift*, Vol. I., p. 171. 1842.

resistance. Hence, if the distance AB of the points of contact are measured, the deflections observed for different values of this distance with the same current give the table of graduation.

874. The most rapid method is to utilise Ohm's law by varying according to a known law the intensity I of the current produced by a constant electromotive force, such as a Daniell's cell, by the addition of graduated resistances. If the internal resistance of the couple is very small compared with the total resistance, the various intensities I, I′, I″..... of the current, which corresponds to the resistances R, R′, R″ of the portion of the current external to the galvanometer, give the ratios

$$\frac{\mathrm{I}}{\frac{1}{\mathrm{R}+g}}=\frac{\mathrm{I}'}{\frac{1}{\mathrm{R}'+g}}=\frac{\mathrm{I}''}{\frac{1}{\mathrm{R}''+g}}=\ldots\ldots$$

If the resistances R, R′, R″ are themselves very great compared with the galvanometer, the equations reduce to

$$\frac{\mathrm{I}}{\frac{1}{\mathrm{R}}}=\frac{\mathrm{I}'}{\frac{1}{\mathrm{R}'}}=\frac{\mathrm{I}''}{\frac{1}{\mathrm{R}''}}=\ldots\ldots$$

The experiment is very simple when the galvanometer is provided with a shunt well adjusted with a compensating resistance. The shunt being placed to the galvanometer, the external resistance is regulated so that a deflection of n divisions is obtained. The shunt is removed. The principal current is not altered, but it becomes m times as great in the galvanometer, and gives a fresh deflection n'. If we take for m a number which is not very high (2, for instance), a table is readily constructed for the instrument by varying the value of n.

Several other methods have been proposed with the same object; but we shall not dwell on this question, which has lost most of its interest.

875. COMPARISON OF GALVANOMETERS.—Let us assume that for two galvanometers G and G′ the constants of the coils are supposed constant; let H and H′ be the horizontal intensities of the external field on each of the needles, δ and δ' the deflections

corrected by the graduation which correspond to the passage of the same current in the two instruments, we have

$$(37) \qquad I = \frac{H}{G}\delta = \frac{H'}{G'}\delta'.$$

When the sensitiveness of the two instruments are of the same order, the deflections δ and δ' are given directly by the common current; if not, a shunt is interposed in the most sensitive instrument, and the corresponding deflection is equal to the product of the deflection observed, by the multiplying power of the shunt.

If the galvanometer of comparison—the first, for instance—is an absolute instrument, this experiment at once gives the factors by which the indications of the second galvanometer must be multiplied to deduce from it the intensity of the current in absolute values.

876. If the external field were the same for the two instruments, we should deduce from equation (37) the ratio of the galvanometric constants of the two coils; but local influences generally prevent this identity, even for points which are pretty near.

In order to determine directly the galvanometric constant G' of a galvanometer, it is placed in the centre of the coil of a tangent galvanometer, the mean planes of the windings being parallel, and the two coils are shunted at two points of the same circuit, so as to form a sort of differential galvanometer. If I is the intensity in the tangent galvanometer, I' that in the other, and δ the deflection, we have

$$H \tan \delta = GI - G'I'.$$

The action of the galvanometer will generally preponderate, but it can be reduced by a suitable resistance; if we bring the needle to zero, we have

$$\frac{G'}{G} = \frac{I}{I'}.$$

The total resistances of the two instruments being g and g', we may also write

$$\frac{G'}{G} = \frac{I}{I'} = \frac{g'}{g}.$$

It is important to observe that, as comparing the resistances g and g', this experiment gives the ratio of the radii of mean action, if the shapes of the channel are simple, so that the terms of correction can be easily calculated.

877. Formula (20′) of (**845**) gives the equation

$$G^2 = 2\left(R - 2L\frac{\lambda}{\tau}\right)\frac{H}{M}\frac{\tau_0^2}{\pi^2+\lambda_0^2}\left(\frac{\lambda}{\tau}-\frac{\lambda_0}{\tau_0}\right),$$

which would enable us to determine the galvanometric constant by investigating the oscillations for the coil when open and closed, provided we know the resistance and the coefficient of self-induction of this coil, as well as the ratio $\frac{M}{H}$ of the magnetic moment of the needle to the horizontal intensity of the field.

By observing the deflections produced in the galvanometer with a current of known intensity I, we might eliminate from this formula the value of H; but this method does not seem to possess any practical interest.

878. Regulation of a Galvanometer.—In galvanometers with astatic needles, or with correction, the sensitiveness may change considerably, either by changes of temperature or by displacement of magnets. It would be necessary frequently to repeat the comparison with an absolute instrument; but this method is not sufficiently expeditious for practical measurements which require no great accuracy.

The simplest is to use a standard of electromotive force E—a Daniell's element, for instance—which is connected to the galvanometer by a suitable resistance R.* If, further, a shunt of multiplying power m is used, and the deflection observed gives a number of divisions on any given scale, the intensity i of the current in the galvanometer is then

$$i = \frac{I}{m} = \frac{E}{mR+g}.$$

* 20,000 or 30,000 ohms with Thomson's astatic galvanometer (**849**) and a shunt of a thousandth.

As this intensity is sensibly proportional to the deflection, if N is the number of divisions corresponding to unit current,

$$i=\frac{x}{N}=\frac{E}{mR+g}.$$

If R_1 is the resistance which the circuit should have, so that the standard electromotive force produces a deflection of one division, we have

$$\frac{1}{N}=\frac{E}{R_1}, \qquad \text{or} \qquad R_1=NE.$$

The coefficient R_1 measures the sensitiveness; it is often called the *figure of merit* of the galvanometer.

We have further

$$N=\frac{mR+g}{E}x,$$

$$R_1=(mR+g)x.$$

With very sensitive galvanometers such as are used at the present time, the resistance R is generally very great in respect of $\frac{g}{m}$, and we may simply write

$$R_1=NE=mxR.$$

If the electromotive force E is expressed in volts and the resistance R in ohms, the intensity i is in ampères.

879. When the resistance R is not very great, the internal resistance of the couple cannot be neglected; but this source of error may be eliminated by one of the following methods, due to the late Mr. Charles Hockin.*

The arrangement of the apparatus being represented by Fig. 168, we note the deflection of the galvanometer G provided with a shunt s. If a, b, and c are the resistances of the three

* HOCKIN. *Report of the British Association Committee on Electrical Standards*, edited by F. JENKIN, p. 149.

branches indicated by these same letters; g_1 that of the galvanometer with its shunt; I the intensity of the current in the branch a; and I_1 the intensity of the current in the branch b, we have

$$I_1 = (I - I_1)\frac{c}{g_1 + b} = I\frac{1}{1 + \frac{g_1 + b}{c}},$$

$$E = I\left[\rho + a + \frac{1}{\frac{1}{c} + \frac{1}{g_1 + b}}\right] = I_1\left[(\rho + a)\left(1 + \frac{g_1 + b}{c}\right) + (g_1 + b)\right].$$

The resistance a is then replaced by another very different one a', and a corresponding resistance b' is placed in the branch b, such

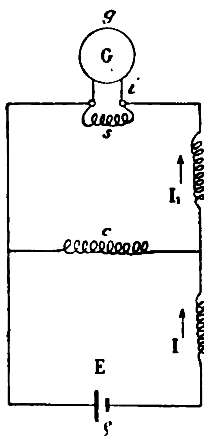


Fig. 168.

that the current in the galvanometer does not alter, the condition follows

$$\frac{E}{I_1} = (\rho + a)\left(1 + \frac{g_1 + b}{c}\right) + \frac{g_1 + b}{c} = (\rho + a')\left(1 + \frac{g_1 + b'}{c}\right) + g_1 + b',$$

which gives

$$\frac{E}{I_1} = c\left[\frac{a' - a}{b' - b}\left(1 + \frac{g_1 + b}{c}\right)\left(1 + \frac{g_1 + b'}{c}\right) - 1\right].$$

The intensity i of the current in the galvanometer being

$$i=\frac{I_1}{m}=\frac{x}{N}=\frac{Ex}{R_1},$$

we get from it the equation

$$R_1=mx\frac{E}{I_1}=mxc\left[\frac{a'-a}{b'-b}\left(1+\frac{g_1+b}{c}\right)\left(1+\frac{g_1+b'}{c}\right)-1\right],$$

which gives the coefficient R_1, by the known resistances g_1, b, b', c, and by the differences $a'-a$, $b'-b$.

A more direct method is to observe the current in the first condition, and then cut the conductor c and replace the resistance b by a resistance b_1, such that the deflection does not vary. We have then as a condition

$$(\rho+a)\left(1+\frac{g_1+b}{c}\right)+g_1+b=\rho+a+g_1+b_1=\frac{b_1-b}{g_1+b}c+g_1+b_1;$$

whence

$$R_1=mcx\left[\frac{b_1-b}{g_1+b}+\frac{g_1+b_1}{c}\right],$$

and there are only four resistances to measure.

880. Observation of Deflections.—With galvanometers the damping of which is rapid, equilibrium is established after so short a time that the deflection for the permanent current is observed without difficulty. In other cases, an auxiliary damping magnet or current (842) is used; but we may also employ the current itself by one of the following methods, which are due to Gauss,* and which are particularly suitable for observing magnetic systems the moment of inertia of which is considerable.

The passage of the current suddenly displaces the needle through an angle δ from its position of equilibrium. When the motion is that of a pendulum, the arc of the first throw relative to the permanent current is 2δ; but if this current is broken when the arc traversed is $\frac{\delta}{2}$, which corresponds to a third of the time of oscil-

* Gauss. *Resultate aus den Beobachtungen des Magn. Vereins.* 1839. *Œuvres*, Vol. v., p. 395.

lation T, the needle has acquired sufficient velocity to complete the arc δ—that is to say, the position of equilibrium with velocity zero at the period $\frac{2T}{3}$; but if the current is re-established the needle will be stationary. As there is always a little damping, and the current is not opened and closed at exactly the proper times, the needle still makes small oscillations of which the mean may be taken.

In like manner to reduce the needle to zero, with velocity zero, the current must be opened for a third of the oscillation, closed for a second third, and then definitely opened.

This rule is not sufficient if the damping is considerable. By the formulas of (681), and for different values of λ, Gauss calculated the fraction $\frac{t_1 - t_0}{T}$ of the time of an oscillation during which the current must be closed, as well as the fraction $\frac{t_2 - t_1}{T}$ during which it must be broken, in order that at the time t_2 the needle reaches its position of equilibrium with no velocity.

The following is an extract from this table:—

λ	$\frac{t_1 - t_0}{T}$	$\frac{t_2 - t^1}{T}$
0	0·333	0·333
0·1	0·376	0·291
0·2	0·418	0·251
0·3	0·459	0·214
0·4	0·498	0·180
0·5	0·535	0·149
0·6	0·569	0·123

The fraction $\frac{t_1 - t_0}{T}$ increases with the damping, and the fraction $\frac{t_2 - t_1}{T}$ varies in the contrary direction. Their sum is at first equal to $\frac{2}{3}$, and increases very slowly with the value of λ.

It is useless to carry the table further; for if the damping is more rapid, the oscillations of the needle extinguish themselves so rapidly that we need not trouble about them.

881. Changes in intensity of the external field are corrected by often repeating the control of the galvanometer; but we are more particularly concerned to eliminate the changes of direction, which, for the terrestrial field, correspond to changes of declination. Instead of determining zero at each observation, it is better to reverse the current and observe the elongations on either side, which gets rid of the position of zero. This inversion is particularly necessary in continuous observation: the current is then reversed at regular intervals—every three oscillations, for instance. In order that they may be very small, the current is broken for the time of a single oscillation before making it in the opposite direction. If x_1, x_2, x_3, x_4, ... etc., are the deflections observed on each side, starting from an arbitrary zero, we shall have for the permanent deflection, a series of values such that

$$(38) \qquad a_0 = \frac{1}{2}\left(\frac{x_1 + 2x_2 + x_3}{4} - \frac{x_4 + 2x_5 + x_6}{4}\right).$$

882. MEASUREMENT BY FIRST SWING: METHOD OF MULTIPLICATION.—We may also calculate the permanent deflection a_0 from the position of equilibrium. The needle being first of all at rest, this deflection is $2a_0$ when the damping is zero. If the damping is considerable, the angle of throw is an oscillation the first part of which is a_0 and the second $a_0 e^{-\lambda}$, so that the total swing a_1 is

$$a_1 = a_0(1 + e^{-\lambda}), \qquad \text{from which} \qquad a_0 = \frac{a_1}{1 + e^{-\lambda}}.$$

When the deflection is very small, we may obtain vibrations which are more easily measurable by appropriate reversals of the current. When the needle stops at its extreme position, and is just about to move back, it is sufficient to change the direction of the current, keeping it constant during the whole of the next vibration.

The angular deflection of the first swing is

$$a_1 = a_0(1 + q) = a_0 q + a_0,$$

and the following

$$a_2 = (a_0 + a_1)q + a_0 = a_0(1 + q)(1 + q),$$
$$a_3 = (a_0 + a_2)q + a = a_0(1 + q)(1 + q + q^2),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_n = (a_0 + a_{n-1})q + a_0 = a_0(1 + q)(1 + q + \dots + q^{n-1}).$$

The limiting deflection a for a permanent condition of reversals is

$$a = a_0 \frac{1 + q}{1 - q} = a_0 \frac{1 + e^{-\lambda}}{1 - e^{-\lambda}},$$

whence

$$a_0 = a \frac{1 - e^{-\lambda}}{1 + e^{-\lambda}}.$$

The *method of multiplication* is not very well suited for accurate measurements, but it is excellent for demonstrating a very weak current.

883. TRANSIENT CURRENTS. — BALLISTIC GALVANOMETER. — The quantity of electricity which a transient current or a discharge of any form expends, may be determined by the first throw of the needle of a galvanometer.

For this purpose the duration of the discharge must be very small compared with that of the oscillation of the needle—in other words, the current must stop before the needle is appreciably deflected from its position of equilibrium. This is a ballistic method analogous to that used in measuring the velocity of projectiles.

If K is the moment of inertia of the needle, M its magnetic moment, T the time of oscillations without damping, ω_0 the angular throw imparted to the needle by the instantaneous discharge of a quantity of electricity m is (506)

$$\omega_0 = \frac{\mathrm{MG}}{\mathrm{K}} m = \frac{\mathrm{G}}{\mathrm{H}} \frac{\pi^2}{\mathrm{T}^2} m.$$

On the other hand, if the deflections are very small, the initial velocity ω_0 is connected with the corresponding angle of throw by the expression

$$\omega_0 = a_1 \frac{\pi}{\mathrm{T}} e^{\frac{\lambda}{\pi} \operatorname{arc\,tan} \frac{\pi}{\lambda}};$$

it follows that

$$(39) \qquad m = a_1 \frac{H}{G}\frac{T}{\pi} e^{\frac{\lambda}{\pi}\operatorname{arc\,tan}\frac{\pi}{\lambda}} = a_1 \frac{H}{G}\frac{\tau}{\pi}\sqrt{1+\frac{\lambda^2}{\pi^2}}\, e^{\frac{\lambda}{\pi}\operatorname{arc\,tan}\frac{\pi}{\lambda}}.$$

When the damping is very weak, we may take the approximate value

$$m = \frac{H}{G}\frac{T}{\pi}\left(a_1 + \frac{a_3 + a_1}{4}\right).$$

The sensitiveness of the instrument is measured by the ratio of the throw to the discharge, or by the expression

$$\frac{G}{H}\frac{\pi}{T} = G\sqrt{\frac{M}{HK}}.$$

It is necessary that the time of vibration of the needle T be sufficiently great not only to eliminate the influence of the time of discharge, but also to enable the deflection to be read, and accordingly the best experimental conditions are those in which the intensity H of the field is diminished. In order further to approach the theoretical conditions, the damping by the air is reduced as much as possible, as its law is in that case somewhat uncertain, and we only retain that arising from currents induced in the coil. If, finally, we want to repeat a series of successive observations, the needle is stopped each time by an auxiliary damper, magnet, or a current.

As at the outset the needle is never absolutely rigid, its total initial amplitude $2a_0$ is first observed. The angle of deflection observed a_1 is then increased or diminished by a_0, according as the throw is in the direction of the initial motion or is opposed to it.

The measurement of the quantity m in absolute value requires a knowledge of the ratio $\frac{G}{H}$, which is determined either directly or by comparison with an absolute galvanometer (874).

884. The use of shunts in a ballistic galvanometer may give rise to serious errors, as the discharges only divide themselves between the shunt and galvanometer according to the law of branch circuits provided the needle is stationary throughout the discharge (547), a condition which is difficult to realise.

The following experiment of Mr. Latimer Clark* illustrates very clearly the influence of the motion of the needle. A condenser is discharged through two identical galvanometers shunted in respect of each other on the same circuit; both give the same deflection, which is half that observed in each of them when it alone receives the discharge. If the experiment is again made after fixing one of the needles, the deflection of the free needle is much less. In the first case the two needles experience the same swing and expend the same work; in the second case there is electromagnetic action in only one of the circuits, and only for this alone is there an apparent increase of the resistance (**535**). The current, therefore, does not divide equally between the two branches.

With differential galvanometers where the needle is stationary, shunts can always be used, even for instantaneous currents.

885. The ballistic method involves two kinds of errors† which it seems difficult to eliminate completely, and which in certain cases may have a considerable influence.

In the first case the current which traverses the galvanometer acts during a very short time, but with great energy. This action, which is perpendicular to the axis of the needle, is, it may be, of a nature to produce a temporary change in its magnetic moment so great that the swing is no longer equal to that which would correspond to the original moment.

In the second place, we apply to the angle of throw the correction for the logarithmic decrement deduced from observations of continuous oscillations. This method would be perfectly correct if the damping were due merely to currents induced by the motion of the needle; but when a considerable part of the effect is due to resistance of the air, we may inquire whether this resistance acts in the same way on a movable body oscillating regularly as it does in this same movable body originally at rest, and which moves under the influence of a shock, and particularly if it is near its position of equilibrium. The resistance of the air should be greater in the second case than in the first, hence the correction as made would be inadequate. It is true that the effect of this same cause would be to increase the time of oscillation, and therefore to introduce a new error in the contrary direction to the first, and which would partially destroy it.

* L. CLARK. *Journal of the Society of Telegraph Engineers*, Vol. II., p. 16. 1873.

† L. RAYLEIGH. *Experiments to Determine the Value of Unit of Resistance.* —*Phil. Trans.*, *1882*, Part ii., p. 619.

886. Observation of Swings.—When we can control a phenomenon under identical conditions, the experiment may be repeated a great number of times alternately in the two directions, so as to eliminate accidental errors of reading as well as displacements of zero, and the mean of observations is taken. In this case the *method of multiplication* enables us to still increase the angle of swing if we make the discharges alternately each time the needle passes zero.

The needle, being first started with a velocity ω_0 by a first positive discharge, reverts to its position of equilibrium after a time τ with the velocity $-\omega_0 q$; at this instant a negative discharge gives it the velocity $-\omega_0$, which makes $-\omega_0(1+q)$; on its return to zero it will have a fresh velocity $+\omega_0$, and so forth. The successive deflections $a_1, a_2, a_3 \ldots$ being proportional to the velocities at the various transits, we shall have

$$a_1 = a_0 q + a_0 = a_0(1+q),$$
$$a_2 = a_1 q + a_0 = a_0(1+q+q^2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_n = a_{n-1} q + a_0 = a_0(1+q+\ldots+q^n).$$

The limiting elongation a_1 corresponding to an established system of inversions, and the initial elongation a, which will be produced by a single discharge, are connected by the ratio

$$a = a_0 \frac{1}{1-q} = a_0 \frac{1}{1-e^{-\lambda}}.$$

The value of a_0 defined by this equation, is expressed in divisions of a scale, and we shall deduce the corresponding angle α_1 from the distance of the scale.

This method has, moreover, the advantage of substituting a permanent and regular regime for a single swing, which facilitates readings. But, unless the initial swing is very feeble or the damping considerable, we soon attain undue elongations; for the value of a increases without limit when λ tends towards zero.

We may by other methods effect a regular regime with deflections of the same order as those of the first swing.

887. The *method of recoil* was devised by Weber.* The needle having been swung in the positive direction, it is allowed to acquire

* W. Weber. *Abh. der Königl. Gesell. zu Göttingen*, Vol. I., p. 349.—*Resultate des Magn. Vereins, 1838*, p. 98.

its first positive elongation A (Fig. 169), then the following negative elongation B′. The instant it is at zero, having moved in the positive direction, a negative charge is sent; the needle stops suddenly, and returns in the negative direction. It is allowed to attain its first negative elongation A′, then the following positive one B;

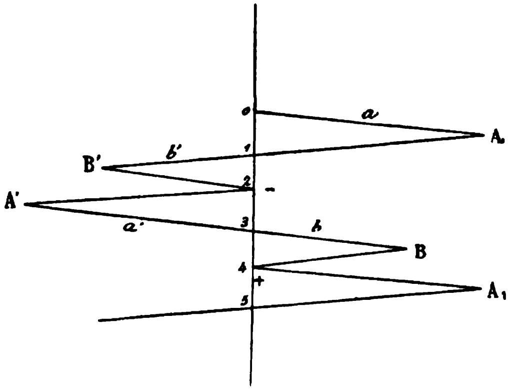


Fig. 169.

and when it again passes zero a positive discharge is sent, which stops the needle and restores it to a positive elongation A_1, etc.

If a, b', a', b, a_1.... are the successive elongations

$$\begin{aligned} b' &= aq, \\ a' &= a_0 - b'q, \\ b &= a'q, \\ a_1 &= a_0 - bq. \end{aligned}$$

From which we deduce

$$q = \frac{b'}{a} = \frac{b}{a'} = \frac{b+b'}{a+a'} = e^{-\lambda},$$

$$2a_0 = a' + a_1 + (b' + b)q = a_1 + a' + b + b' - (b + b')(1 - q).$$

The decrement λ and the elongation for a single throw are thus determined by the sum of the swings to the right and left—that is to say, by the difference of the readings of the scale without its being necessary to know the position of zero. This position, moreover, would be determined by two successive elongations A and B′, A′ and B..... of a free oscillation, taking into account the damping.

As those free oscillations are repeated in pairs, they enable us to follow variations of zero throughout the duration of the experiment.

When the damping is not very small, a permanent regime is soon established, the period of which comprises four oscillations with two kinds of elongation, one set greater a and the other less b, the values of a and b being moreover given as the semi-distance of the elongations of the same order A and A′, B and B′. We have then

$$q = \frac{b}{a},$$

$$a_0 = a + bq = \frac{a^2+b^2}{\sqrt{ab}}\sqrt{q} = \frac{a^2+b^2}{\sqrt{ab}}e^{-\frac{\lambda}{2}}.$$

If α is the angle which corresponds to the displacement $\frac{a^2+b^2}{\sqrt{ab}}$, and observing that

$$\text{arc tan}\,\frac{\pi}{\lambda} = \frac{\pi}{2} - \text{arc tan}\,\frac{\lambda}{\pi},$$

equation (39) becomes

$$(40) \qquad m = \alpha\frac{\mathrm{H}}{\mathrm{G}}\frac{\tau}{\pi}\sqrt{1+\frac{\lambda^2}{\pi^2}}\,e^{-\frac{\lambda}{\pi}\,\text{arc tan}\,\frac{\lambda}{\pi}}.$$

888. The method of recoil is not convenient when the damping is feeble. Weber and Zöllner* employed in this case a mixed method. At the passage of the needle through zero, successive swings are given in alternately opposite directions, as in the preceding cases; but these swings are sometimes in accordance with, and are sometimes in opposition to, the actual velocity.

At the end of a great elongation A (Fig. 170) on the positive side, the needle, as it passes through zero, receives a positive swing which reduces the following elongation at C′; at the second passage a negative swing gives an elongation B′ of mean magnitude, and the next oscillation BB′ is free; at the fourth passage a positive swing gives a small elongation C; then a negative swing at the fifth passage reproduces a large elongation A′. The vibration $\mathrm{A}'\mathrm{A}_1$ is still free, and the same series recommences.

* W. Weber and Zöllner. *Berichte der Königl. Sachs. Gesellschaft.* Leipzig, 1880.

As long as the permanent regime is not established, three kinds of oscillations are produced; and the form of the experiment gives a period which comprises six oscillations.

If a, c', b', b, c, a', a_1 are the successive distances, we have the ratios

$$
\begin{aligned}
c' &= aq - a_0,\\
b' &= a_0 - c'q,\\
b &= b'q,\\
c &= a_0 - bq,\\
a' &= a_0 + cq,\\
a_1 &= a'q.
\end{aligned}
$$

The damping being determined by a series of free oscillations,

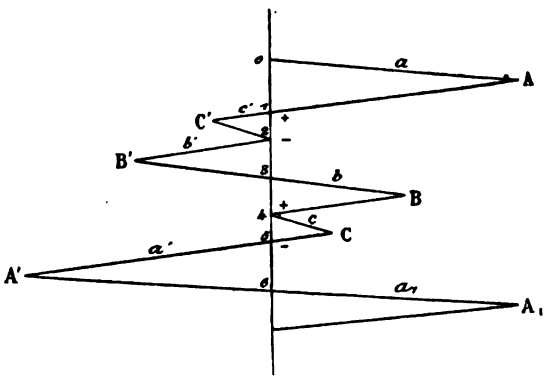


Fig. 170.

we may deduce several values of the amplitude a_0 from the preceding equations. The most advantageous equations are

$$2a_0 = a' + bq + c(1 - q) = (a' + b) - (b - c)(1 - q),$$

$$2a_0 = \frac{(a' + b) + (b' + a_1)}{2} + (a' + b')(1 - q) + (c' - c)q.$$

The first is quite independent of the position of zero; the second only requires a knowledge of zero for terms which are already very small.

If the damping were so great that we could soon attain a final state, the period would contain no more than three vibrations

corresponding to the distances a, b, and c (Fig. 171). We should then have

$$b = aq,$$
$$c = a_0 - bq = a_0 - aq^2,$$
$$a = a_0 + cq = a_0(1+q) - aq^3.$$

These equations give, as an experimental condition,

$$b + c = a.$$

We might take advantage of this relation to determine the position

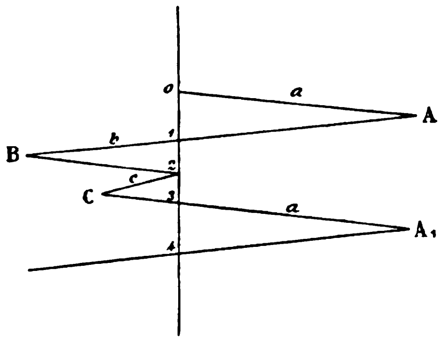


Fig. 171.

of zero. The former equation will give the value of q, and we shall have

$$a_0 = a\,\frac{1+q}{1+q^3}.$$

This method, however, has no interest when the damping is considerable.

889. Influence of the Duration and of the Instant of the Swing.—The use of the ballistic galvanometer in the various cases which precede assumes that the impulse is given to the needle in a very short time, and at the exact moment at which it passes its position of equilibrium. It is important to examine what conditions must be introduced into the formula if these conditions are not exactly realised.*

* O. Chwolson. *Mélanges de Phys. et de Chim.*, p. 403. Petersburg, 1881. E. Dorn. *Wied. Ann.*, Vol. XVII., p. 654. 1882.

We shall assume that the duration of the transient current, as well as the errors made as to the moment at which it starts, are very small compared with the time of oscillation of the needle. It will be sufficient to consider the pendulum movement, which may be represented by the equations

$$(41)\qquad \begin{aligned} u &= u_0 \cos \gamma t, \\ \gamma x &= u_0 \sin \gamma t, \end{aligned}$$

in which x is the distance at the epoch t, u the velocity of displacement, and u_0 the velocity at the time $t=0$, which corresponds to the position of equilibrium, the time T of oscillation being equal to $\dfrac{\pi}{\gamma}$.

We notice, in the first place, that the ratio

$$(42)\qquad u_0^2 = u^2 = +\gamma^2 x^2$$

gives the velocity for the position of equilibrium by the velocity at a given instant and the corresponding distance, whatever be the origin of the time.

Let us assume, as a first case, that at the time t_1 we give the system an instantaneous velocity v_1; we shall have

$$(43)\qquad \begin{aligned} u_1 &= u_0 \cos \gamma t_1 + v_1, \\ \gamma x_1 &= u_0 \sin \gamma t_1. \end{aligned}$$

The motion is pendular, and may be represented by the equations

$$(44)\qquad \begin{aligned} u &= u_0' \cos \gamma t - \gamma x_0' \sin \gamma t, \\ \gamma x &= u_0' \sin \gamma t + \gamma x_0' \cos \gamma t, \end{aligned}$$

in which u_0' and x_0' are the velocity, and the distance for the moment $t=0$, which is no longer the epoch of the real or apparent passage through the position of equilibrium. If we make $t=t_1$, and take count of equations (43), we get

$$(45)\qquad \begin{aligned} u_0' &= u_0 + v_1 \cos \gamma t_1, \\ \gamma x_0' &= -v_1 \sin \gamma t_1. \end{aligned}$$

Equation (42) enables us to calculate the velocity v_0 with which the movable body passes through its position of equilibrium

$$v_0^2 = u_0'^2 + \gamma^2 x_0'^2 = u_0^2 + v_1^2 + 2u_0 v_1 \cos \gamma t_1$$
$$= (u_0 + v_1)^2 - 4u_0 v_1 \sin^2 \frac{\gamma t_1}{2};$$

we get from this, if the time t_1 is very short,

$$(46) \qquad \frac{v_0}{u_0 + v_1} = 1 - \frac{u_0 v_1}{2(u_0 + v_1)^2} \gamma^2 t_1^2 = 1 - \frac{u_0 v_1}{2(u_0 + v_1)^2} \pi^2 \frac{t_1^2}{T^2}.$$

The velocity v_0 is less than the velocity $u_0 + v_1$ which the movable body would have acquired, if the discharge had taken place at the moment $t = 0$.

890. It is easy to generalise these expressions. If we impart to the system any given series of instantaneous impulses $u_1, u_1', u_1'' \ldots$ at the times $t_1, t_1', t_1'' \ldots$ the modified motion may still be represented by equations (44), with the following values for the period $t = 0$.

$$u_0' = u_0 + \Sigma v_1 \cos \gamma t_1,$$
$$\gamma x_0' = -\Sigma v_1 \sin \gamma t_1.$$

The sums $\sum$ should be replaced by the integrals when the impulse is continuous. Suppose, for instance, that the impulse, starting at the time t_0 lasts for a period θ, and gives an acceleration w at the time t; we have then

$$(47) \qquad \begin{aligned} u_0' &= u_0 + \int_{t_0}^{t_0+\theta} w \cos \gamma t \, dt, \\ \gamma x_0' &= -\int_{t_0}^{t_0+\theta} w \sin \gamma t \, dt. \end{aligned}$$

891. We shall first of all apply these formulas to the case of an impulse kept uniform for the time θ, which represents the case of a constant transient current. The acceleration w is then constant; and the total impulse given to a free system, or to a system having a very great period of vibration, will be

$$v = \int_{t_0}^{t_0+\theta} w \, dt = w\theta.$$

If t_1 is the mean time $t+\frac{\theta}{2}$ of the impulse, we have

$$\int_{t_1-\frac{\theta}{2}}^{t_1+\frac{\theta}{2}} w \cos\gamma t\, dt = \frac{2w}{\gamma}\cos\gamma t_1 \sin\frac{\gamma\theta}{2} = v\frac{\sin\frac{\gamma\theta}{2}}{\frac{\gamma\theta}{2}}\cos\gamma t_1,$$

$$\int_{t_1-\frac{\theta}{2}}^{t_1+\frac{\theta}{2}} w \sin\gamma t\, dt = \frac{2w}{\gamma}\sin\gamma t_1 \sin\frac{\gamma\theta}{2} = v\frac{\sin\frac{\gamma\theta}{2}}{\frac{\gamma\theta}{2}}\sin\gamma t_1.$$

If we replace these values in equation (47), and compare the expressions thus obtained with the second members of the equations (45), we see that a uniform impulse during the time θ_1 produces the same effect as if at the time t_1 the instantaneous velocity

$$v\frac{\sin\frac{\gamma\theta}{2}}{\frac{\gamma\theta}{2}} = v\left[1-\frac{1}{6}\frac{\gamma^2\theta^2}{4}\right]$$

had been given to the system

Replacing v_1 by this value in equation (46), we get to the same degree of approximation

$$(48) \qquad \frac{v}{u_0+v} = 1 - \frac{v\gamma^2}{u_0+v}\left[\frac{u_0}{2(u_0+v)}t_1^2 + \frac{1}{6}\frac{\theta^2}{4}\right].$$

If the middle of the impulse corresponds to the position of equilibrium, we must make $t=0$ in this expression.

If, finally, the needle was at rest at the moment of the impulse, we have $u_0=0$, and therefore

$$\frac{v_0}{v} = 1 - \frac{1}{6}\frac{\gamma^2\theta^2}{4} = 1 - \frac{\pi^2}{24}\frac{\theta^2}{T^2} = 1 - 0{\cdot}41123\frac{\theta^2}{T^2}.$$

892. Another particularly interesting case is that of an impulse which has a sinusoidal character, like that which would result from

the current produced by induction when the coil is turned, with a constant velocity, through an angle of 180°, in the terrestrial field. If we neglect the extra current, we may write

$$w = w_1 \sin \pi \frac{t - t_0}{\theta},$$

and the value of the total impulse which the charge would impart to a free system is

$$v = \int_{t_0}^{t_0+\theta} w_1 \sin \pi \frac{t - t_0}{\theta} dt = \frac{2w_1\theta}{\pi}.$$

If, then, t_1 is the mean time of the impulse,

$$\int_{t_1-\frac{\theta}{2}}^{t_1+\frac{\theta}{2}} w_1 \sin \pi \frac{t - t_0}{\theta} \sin \gamma t \, dt = v \frac{\cos \frac{\gamma\theta}{2}}{1 - \frac{\gamma^2\theta^2}{\pi^2}} \cos \gamma t_1,$$

$$\int_{t_1-\frac{\theta}{2}}^{t_1+\frac{\theta}{2}} w_1 \sin \pi \frac{t - t_0}{\theta} \sin \gamma t \, dt = v \frac{\cos \frac{\gamma\theta}{2}}{1 - \frac{\gamma^2\theta^2}{\pi^2}} \sin \gamma t_1.$$

A sinusoidal impulse produces thus the same effect as if we imparted to the needle at the period t_1 the velocity

$$v \frac{\cos \frac{\gamma\theta}{2}}{1 - \frac{\gamma^2\theta^2}{\pi^2}} = v \left[1 - \gamma^2\theta^2 \left(\frac{1}{8} - \frac{1}{\pi^2} \right) \right],$$

and equation (46) gives

$$(49) \quad \frac{v_0}{u_0 + v} = 1 - \frac{v\gamma^2}{u_0 + v} \left[\frac{u_0}{2(u_0 + v)} t_1^2 + \left(\frac{1}{8} - \frac{1}{\pi^2} \right) \theta^2 \right].$$

Finally, if the needle was at first at rest in the position of equilibrium,

$$\frac{v_0}{v} = 1 - \left(\frac{1}{8} - \frac{1}{\pi^2}\right)\gamma^2 6^2 = 1 - \left(\frac{\pi^2}{8} - 1\right)\frac{6^2}{T^2} = 1 - 0{\cdot}2337\,\frac{6^2}{T^2}.$$

893. Correction of Observations.—When a transient current is measured by the swing of a needle originally at rest, the effect of the duration of the discharge is to diminish the angular deflection, which is proportional to the initial velocity, by the fraction $0{\cdot}4112\,\frac{6^2}{T^2}$, or by $0{\cdot}2337\,\frac{6^2}{T^2}$, according as the current is uniform or sinusoidal.

In the method of multiplication, for instance (886), the initial swing for a single impulse is connected with the maximum deflection a_0 by the ratio

$$a_0 = a(1 - q).$$

In order to obtain the influence of the retardation and of the duration of the impulse, substituting the swings for the initial velocities, and supposing the impulse uniform, we must replace v_0 by a, u_0 by aq, and v by a_0 in equation (48). Calling the second member of the equation $1 - \delta$, we get

$$\frac{a}{aq + a_0} = 1 - \delta,$$

or

$$a_0 = a\left(\frac{1}{1-\delta} - q\right) = a(1 - q)\left(1 + \frac{\delta}{1-q}\right).$$

Replacing a_0 by its approximate value $a\ (1 - q)$, and $(aq + a_0)$ by a, we have for the term of correction δ

$$\delta = (1 - q)\gamma^2\left[\frac{q}{2}t_1^2 + \frac{1}{6}\frac{6^2}{4}\right];$$

and therefore

$$a_0 = a(1 - q)\left[1 + \frac{\pi^2}{T^2}\left(\frac{q}{2}t_1^2 + \frac{1}{6}\frac{6^2}{4}\right)\right].$$

With a sinusoidal swing we should have similarly by equation (49)

$$a_0 = a(1-q)\left\{1+\frac{\pi^2}{T^2}\left[\frac{q}{2}t_1^2+\left(\frac{1}{8}-\frac{1}{\pi^2}\right)\frac{\theta^2}{4}\right]\right\}.$$

In the method of recoil (887) the deflection b' for the small impulse is still equal to aq; but to obtain the extent a' of the retarded swing we should replace v_0 in equation (48) by a', u_0 by $-b'q$, and v by a_0; we have then

$$\frac{a'}{a_0 - b'q} = 1 - \delta,$$

which gives

$$a_0 = \frac{a'}{1-\delta} + b'q = (a'+b'q)\left(1+\frac{a'}{a_0}\right).$$

or

$$a_0 = (a'+b'q)\left[1+\frac{\pi^2}{T^2}\left(-\frac{q^2}{2}t_1^2+\frac{1}{6}\frac{\theta^2}{4}\right)\right].$$

For a sinusoidal impulse we should replace in this formula

$$\frac{1}{6}\frac{\theta^2}{4} \quad \text{by} \quad \left(\frac{1}{8}-\frac{1}{\pi^2}\right)\frac{\theta^2}{4}.$$

The mixed method (888) would give rise to analogous corrections; but unless the ratios $\frac{t_1^2}{T^2}$ and $\frac{\theta^2}{T^2}$ cannot be altogether neglected, it seems difficult to arrange experiments in so perfect a manner that these corrections can be made with any certainty.

894. Discharge of a Condenser.—When a condenser of capacity C is put in connection with the earth, or the two armatures are connected with each other by a conductor of resistance R, and whose coefficient of self-induction is L, the charge Q and the intensity of the current I at the period t are given by expressions

$$Q = Ae^{pt} + A'e^{p't}$$
$$I = -\left[Ape^{pt} + A'p'e^{p't}\right],$$

in which ρ and ρ' are the roots of the equation

$$(50) \qquad \rho^2+\frac{R}{L}\rho+\frac{1}{CL}=0.$$

The coefficients A and A′ are defined by the conditions relative to the commencement of the discharge, at the moment t_0

$$Q_0=Ae^{\rho t_0}+A'e^{\rho' t_0},$$
$$0=A\rho e^{\rho t_0}+A'\rho' e^{\rho' t_0}.$$

If there is a galvanometer in the circuit, the acceleration of motion, being proportional to the intensity of the current, is of the form

$$w=-\left[a\rho e^{\rho t}+a'\rho' e^{\rho' t}\right],$$

with the condition

$$\frac{A}{a}=\frac{A'}{a'}.$$

Replacing $\cos\gamma t$ by its value in imaginary exponentials, we have

$$-2w\cos\gamma t=(a\rho e^{\rho t}+a'\rho' e^{\rho' t})\left(e^{\gamma t\sqrt{-1}}+e^{-\gamma t\sqrt{-1}}\right)$$
$$=a\rho\left[e^{(\rho+\gamma\sqrt{-1})t}+e^{(\rho-\gamma\sqrt{-1})t}\right]+a'\rho'\left[e^{(\rho'+\gamma\sqrt{-1})t}+e^{(\rho'-\gamma\sqrt{-1})t}\right],$$

an expression which may be written in the form

$$-2w\cos\gamma t=a\rho(e^{\alpha t}+e^{\beta t})+a'\rho'(e^{\alpha' t}+e^{\beta' t}).$$

We shall find, in like manner,

$$-2w\sqrt{-1}\sin\gamma t=a\rho(e^{\alpha t}-e^{\beta t})+a'\rho'(e^{\alpha' t}-e^{\beta' t}).$$

As the discharge is in strictness null only after an infinite time, in calculating u_0' and x_0' by equations (47) we ought to integrate these expressions between t_0 and the very long time θ. But we have

$$\int_{t_0}^{\theta}e^{\alpha t}dt=\frac{1}{\alpha}(e^{\alpha\theta}-e^{\alpha t_0}).$$

If the values of ρ and of ρ' are real, they should be negative from equation (50); if they are imaginary, the real parts are still negative, and equal to $-\frac{R}{2L}$. The real part of the exponent a is then negative, and the ratio $\frac{e^{a\theta}}{a}$ tends towards zero, when θ tends towards infinity. As this is also the case with other terms, the values reduce to

$$2\int_0^\infty w \cos \gamma t dt = a\rho\left(\frac{e^{at_0}}{a}+\frac{e^{\beta t_0}}{\beta}\right)+a'\rho'\left(\frac{e^{a't_0}}{a'}+\frac{e^{\beta' t_0}}{\beta'}\right),$$

$$2\sqrt{-1}\int_0^\infty w \cos \gamma t dt = a\rho\left(\frac{e^{at_0}}{a}-\frac{e^{\beta t_0}}{\beta}\right)+a'\rho'\left(\frac{e^{a't_0}}{a'}+\frac{e^{\beta' t_0}}{\beta'}\right).$$

If we assume $t_0=0$, which gives $a\rho+a'\rho'=0$, and replace the exponentials by their values, we get

(51)
$$\int_0^\infty w \cos \gamma t dt = \frac{a\rho^2}{\rho^2+\gamma^2}+\frac{a'\rho'^2}{\rho'^2+\gamma^2},$$

$$-\int_0^\infty w \sin \gamma t dt = \gamma\left(\frac{a\rho}{\rho^2+\gamma^2}+\frac{a'\rho'}{\rho'^2+\gamma^2}\right).$$

The total swing imparted by the discharge to a free system will be

$$v=\int_0^\infty w dt = a+a';$$

from this follows

$$a=v\frac{\rho'}{\rho-\rho'},$$

$$a'=v\frac{\rho}{\rho'-\rho}.$$

The value of v is defined by the total discharge $m=Q$.

If a and a' are replaced by their values in equations (51), and p and t_1 are two constants, we may write

$$(52)\qquad \begin{aligned} \int_0^\infty w \cos \gamma t dt &= v \frac{\rho\rho'(\gamma^2 - \rho\rho')}{(\rho^2+\gamma^2)(\rho'^2+\gamma^2)} = pv \cos \gamma t_1, \\ \int_0^\infty w \sin \gamma t dt &= \gamma v \frac{\rho\rho'(\rho+\rho')}{(\rho^2+\gamma^2)(\rho'^2+\gamma^2)} = pv \sin \gamma t_1. \end{aligned}$$

The discharge still produces the same effect as if we gave the needle an instantaneous impulse pv at the time t_1. We have further, by equations (52),

$$p = \rho\rho' = \frac{1}{CL},$$

$$\tan \gamma t_1 = \gamma \frac{\rho+\rho'}{\gamma^2 - \rho\rho'} = \frac{\pi}{T} \frac{R}{\frac{\pi^2 L}{T^2} - \frac{1}{C}}.$$

This result is independent of the nature of the discharge, whether it be continuous or oscillatory.

It may further be remarked that the factor b, by which we must multiply the total impulse to obtain the instantaneous one, is independent of the resistance of the conductor. This factor should be very large. The time t_1, which represents a kind of retardation, is proportional to the resistance, other things being equal.

895. Measurement of a very Short Time.—If the discharge m which traverses a galvanometer arises from a constant current I_0, the circuit of which has been closed for a time θ, very short compared with the time of oscillation of the needle, the angle of swing, corrected for the damping (882) and for the time of the discharge (891), satisfies the equation

$$m = I_0 \theta = \frac{H}{G} \frac{T}{\pi} a.$$

If δ is the deflection which the permanent current I_0 would give, we deduce

$$(53)\qquad \theta = \frac{T}{\pi} \frac{a}{\tan \delta}.$$

Pouillet proposed this method for measuring a very short time, such as that which a ball takes in traversing a gun-barrel. It only applies provided the effects of induction may be neglected—that is to say, with rectilinear and very short circuits; in other cases the extra current on opening and closing must be allowed for. The latter has generally a very slight influence; for it is produced only when the circuit is closed by the layer of air in which the spark passes, and the resistance of which is very great. If we assume that we can disregard the quantity of electricity which corresponds to it, and if R and L are the elements of the circuit (**532**), we have

$$(54) \qquad m=\int_0^\theta \mathrm{I}dt=\mathrm{I}_0\left[\theta-\frac{\mathrm{L}}{\mathrm{R}}\left(1-e^{-\frac{\mathrm{R}\theta}{\mathrm{L}}}\right)\right].$$

When the duration θ is very great compared with the time necessary to start the current, the value of the exponential may be neglected, and this equation gives, at any rate approximately,

$$(53') \qquad \theta=\frac{\mathrm{T}}{\pi}\frac{a}{\tan\delta}+\frac{\mathrm{L}}{\mathrm{R}}.$$

We see that it is sufficient to add to the time calculated by equation (53) a constant term equal to the quotient of the coefficient of self-induction of the circuit by its resistance. This correction is, however, only sufficient for very short times. Formula (54) shows that the impulse decreases then much more rapidly than the time.*

An experimental device enables us to get rid of all difficulties with induced currents. Instead of opening the circuit after the interval θ, the constant electromotive force is suppressed, a metal wire of equal resistance being substituted for it. The two induction currents follow then the same law (**534**); and, as they are in opposite directions, the total quantity of electricity which traverses the galvanometer in the time θ is equal to $\mathrm{I}_0\theta$.

896. When the armatures of a condenser of capacity C, are connected by a conductor of resistance R, which has no coefficient of self-induction, the intensity I of the current at any given time

* HELMHOLTZ. *Pogg. Ann.*, Vol. LXXXIII., p. 505. 1851.—*Wissenschaft. Abhandl.*, Vol. I., p. 529.

is connected to the difference of potentials of the armature, by **Ohm's** law $E = IR$, and we have $Idt + CdE = 0$. It follows that

$$\frac{E}{R} + C\frac{dE}{dt} = 0, \qquad \text{or} \qquad t = RCl.\frac{E_0}{E},$$

E_0 being the difference of potential at the origin of the time.

This is the principle of the method used by the late Mr. Sabine* to determine very short intervals of time, and particularly the velocity of projectiles. The armatures of a condenser are connected on the one hand with a battery of small resistance, which maintains a constant difference of potential; and, on the other, with a wire of very great resistance R. The two wires being at a given distance on the path of a projectile, the first is cut and then after a time t the second. The ratio of the residual difference of potential E to the initial value E_0, is determined by discharges in a ballistic galvanometer.

We may, moreover, graduate the instrument directly by the aid of a rotating commutator, to which a given velocity is suddenly imparted, and which, when once the condenser has been charged and insulated, closes the circuit for an interval of time known by the distance of two stops on the edge of a disc, the velocity of which is known. Intervals of time can thus be estimated which do not exceed 0·0001 of a second.

897. MEASUREMENT OF A DISCHARGE BY THE ELECTRODYNAMOMETER.—When the same current I passes through the two coils of an electrodynamometer (**864**), the couple for a deflection δ is equal to $S'GI^2 \cos\delta$.

K being the moment of inertia of the movable coil, the angular velocity ω_0, imparted by a discharge the total duration of which is very small compared with the duration of the oscillations, is given by equation

$$K\omega_0 = S'G\int_0^\theta I^2 \cos\delta dt.$$

If the deflections are very small, and I^2_m represents the mean square of the intensity of the current, we may write

$$K\omega_0 = S'G_1 I^2_m \theta.$$

* R. SABINE. *Phil. Mag.* [5], Vol. I., p. 337. 1876.

If T is the time of oscillations corrected for damping, and α_1 the angle of swing, we have by known relations (681),

$$T^2 = \pi^2 \frac{C}{K},$$

(55) $$I_m^2 \theta = \frac{K}{S'G} \omega_0 = \alpha_1 \frac{C}{S'G} \frac{T}{\pi} e^{\frac{\lambda}{\pi} \operatorname{arc\,tan} \frac{\pi}{\lambda}},$$

or, for a very feeble damping,

$$I_m^2 \theta = \frac{C}{S'G} \frac{T}{\pi} \left(\alpha_1 + \frac{\alpha_3 - \alpha_1}{4} \right).$$

If the two coils of the electrodynamometer are traversed by different currents, the angular velocity imparted by two discharges which correspond respectively to intensities I and I′, is determined by the equation

$$K\omega_0 = S'G \int II' dt,$$

the integral being merely extended to the time in which the two currents exist simultaneously. It is assumed further that the deflections are very small in this interval. The value of the integral depends on the law of the variation of the two currents. This law is generally the same, not only for two distinct discharges, but also for the same discharge which is shared by the shunt between the two coils. The swing produced in these conditions is a very complex phenomenon.

898. Duration of a Discharge.—The observation of swings produced by the same discharge in the galvanometer and in the electrodynamometer may, as Weber* showed, give an approximate value of its duration and of the quantity of electricity which corresponds to it.

For if we suppose the intensity I of the current constant, we have in the galvanometer

$$m = I\theta = \frac{H}{G} \frac{T}{\pi} \alpha.$$

* W. Weber. *Electrodyn. Maasbestimmungen*, Vol. I., p. 80. 1846.

Similarly, if we accentuate those letters which have an analogous meaning, the electrodynamometer gives

$$I^2\theta = \frac{C}{S'G'}\frac{T'}{\pi}\alpha'.$$

We deduce from these two equations

$$I = \frac{C}{S'H}\frac{G}{G'}\frac{T'}{T}\frac{\alpha'}{\alpha},$$

$$\theta = \frac{S'H^2}{C}\frac{G'}{G^2}\frac{T^2}{\pi T'}\frac{\alpha^2}{\alpha'}.$$

We may eliminate the constants of the electrodynamometer, by observing the deflections δ and δ', which the same current produces in the galvanometer and in the electrodynamometer. We have, in fact,

$$\frac{H^2}{G^2}\tan^2\delta = \frac{C}{S'G'}\tan\delta';$$

and therefore

$$I = \frac{H}{G}\frac{T'}{T}\frac{\tan^2\delta}{\alpha}\frac{\alpha'}{\tan\delta'},$$

$$\theta = \frac{T^2}{\pi T'}\frac{\alpha^2}{\tan^2\delta}\frac{\tan\delta'}{\alpha'}.$$

The latter expression only contains experimental numbers, but in general it will only give a rough valuation of the real duration of the discharge; for the supposition that the current is uniform is often far from being true.

899. Series of Discharges.—Broken Currents.—It is evident that a continuous series of discharges which are identical with each other, and such that the sum of the duration of each discharge and of the interval separating it from the next, is small compared with the time of oscillation of the needle, will produce in the galvanometer the same deflection as a current of constant intensity which sets in motion the same quantity of electricity in each unit of time.

If N is the number of discharges per second, m the quantity of electricity corresponding to each of them, we shall thus have

$$i = Nm = \frac{H}{G}\tan\delta.$$

In this way we may measure discharges which would give separately only very feeble swings.

In order to demonstrate that the action of a current is proportional to the quantity of electricity which passes, Pouillet* sent through a galvanometer the succession of discharges obtained by breaking periodically the circuit of a constant element. The break consisted of a wooden wheel with a metal ring on its contour, which was continuous on one side and cut away at equal intervals on the other. Two elastic springs rest, one on the continuous and the other on the grooved half. When this rests on a metal half the current passes, and is broken when it is on the other. With a short circuit and conductors without appreciable induction, the deflection first increases with the velocity, then becomes constant from a certain velocity, and proportional to the ratio between the width of a tooth and the sum of a solid and of a hollow. In Pouillet's experiments the currents could be broken 1200 times in a second without the intensity undergoing any change; but this law is no longer true if the effects of induction are not negligable, and it is readily seen then that the intensity diminishes as the velocity increases.

Let I_0 be the intensity of the permanent current, N the number of breaks in a second, a the ratio of a solid to the sum of a solid and of a hollow, the value of m determined by equation (54) gives for the mean intensity

$$I = aI_0 \left[1 - \frac{NL}{Ra} \left(1 - e^{-\frac{aR}{NL}} \right) \right],$$

an expression which tends towards zero when N tends towards infinity.

The experiments of Bertin† and of Cazin‡ on broken currents are quite in agreement with this formula. It follows that the quantity of electricity which corresponds to the extra current on opening can really be neglected.

900. Alternating Currents.—When a circuit is traversed by a series of transient currents or of discharges alternately in opposite

* Pouillet. *Comptes rendus*, Vol. iv., p. 787. 1837.

† Bertin. *Ann. de Chim. et de Phys.* [4], Vol. xvi., p. 25. 1869.

‡ Cazin. *Ann. de Chim. et de Phys.* [4], Vol. xvii., p. 385. 1869.

directions, and following each other with sufficient rapidity, the mean intensity of the current is equal to the algebraical sum of the quantities of electricity which pass in each unit of time. It is zero, as a particular case, if the successive discharges are equal and opposite in direction. This would be the case of a sinusoidal current, or more generally of a periodical current of any given form, the intensity of which changes in sign, while retaining the same value, at the end of a half period.

It would likewise be so with an induction coil, such as that of Ruhmkorff, in a closed circuit; for the quantities of electricity which correspond to the two kinds of induced currents are equal (**541**). If, further, the circuit contains a constant electromotive force, the deflection is the same as if this electromotive force alone existed.*

It may, in passing, be observed that this experiment is an indirect verification of the theory. It shows that the resistance of the circuit is not a function of the intensity, for at each instant the current only depends on the algebraical sum of the electromotive forces.

901. Bilateral Deflection. — It happens, especially with astatic galvanometers, that for a current whose mean intensity is zero, the needle is in unstable equilibrium at zero, and that, once deflected, it diverges to 90°. This fact, which was observed by Poggendorff,† and called by him *bilateral deflection*, is due to the temporary magnetism developed in the needle by the current. It has been investigated and examined in a complete manner by Professor Chrystal.‡

When the external field is weak and the alternate current sufficiently strong, the needle leaves its position of equilibrium and deflects through 90° in one direction or the other. This is the phenomenon first observed by Poggendorff, and which Professor Chrystal calls *bilateral deflection*.

If the intensity of the current is diminished, or the intensity of the magnetic field is increased, two cases may present themselves, according to the initial position of the needle: 1. If it is exactly in the plane of symmetry of the coil it is stationary, notwithstanding the passage of the current; 2. If its initial direction

* Schuster. *Phil. Mag.* [4], Vol. xlviii., p. 254. 1874.—Jamin. *Comptes rendus*, Vol. xciv., p. 1616. 1882.

† Poggendorff. *Pogg. Ann.*, Band xlv., p. 353. 1838.

‡ Chrystal. *Phil. Mag.* [5], Vol. ii., p. 401. 1876.

is right or left of the plane of symmetry, the passage of the current always increases the deflection by an amount which is at first proportional to the original deflection. This deflection, which Professor Chrystal calls *unilateral*, is independent of the rapidity with which the alternating currents succeed each other.

Let α and β be the angles which the direction of the needle makes with the mean plane of the windings in its position of rest and at the time t, and let I be the intensity of the current at the same instant. If the magnetism of the current were invariable, the moment of the couple which acts upon it would be

$$C = HM \sin(\beta - \alpha) - GMI \cos\beta = M[H \sin(\beta - \alpha) - GI \cos\beta];$$

but assuming that the magnetic moment varies proportionally to the component of the forces which act parallel to the axis of the needle, we may put

$$M = M_0 + M_0 k_1 = M_0(1 + k_1),$$

with

$$k_1 = k\left[GI \sin\beta + H \cos(\beta - \alpha)\right].$$

The moment of the couple is then, at the time in question,

$$C = \left[1 + k[GI \sin\beta + H \cos(\beta - \alpha)]\right]\left[H \sin(\beta - \alpha) - GI \cos\beta\right]M_0.$$

To obtain the mean value C_m, this expression must be multiplied by dt, integrated from 0 to θ, and the result divided by θ. The terms which do not contain the intensity do not vary; those which contain the first power of I disappear; those which contain the second should be multiplied by the mean square I_m^2 of the intensity. We have thus

$$C_m = \left[\sin(\beta - \alpha) - \frac{G^2 I_m^2}{2H} k \sin 2\beta + \frac{H}{2} k \sin 2(\beta - \alpha)\right] HM_0.$$

As the coefficient k is very small, the third term in the bracket may be neglected in comparison with the two others, and putting

$$A = \frac{G^2 I_m^2}{2H} k,$$

we may simply write

$$C_m = \left[\sin(\beta - \alpha) - A \sin 2\beta \right] HM_0.$$

To account for the effect produced, we may construct the two curves B and B′ (Fig. 172) of equations

$$y = \sin \beta,$$
$$y' = A \sin 2\beta,$$

and superpose them with a difference of phase equal to α. The

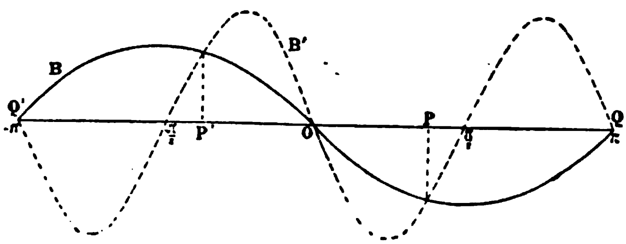


Fig. 172.

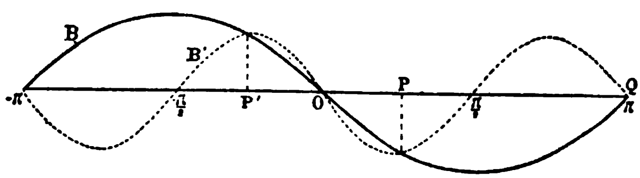


Fig. 173.

abscissæ of the intersections correspond to the positions of rest of the needle.

Suppose first that $\alpha = 0$, that is to say, the initial position of the needle in the plane of symmetry. This is the case of Fig. 172. There are five positions of equilibrium, O, P, P′, Q, and Q′; and the needle may occupy three, O, P, and P′. The first, which corresponds to zero, is unstable; the two others are stable, and correspond to deflections near 90°. When the factor A is < 1, we have the arrangement of Fig. 173, or that of Fig. 174, according

as A is greater or less than 0·5. In the former case the zero is in a position of unstable equilibrium, and there are two adjacent positions P and P′ of stable equilibrium on each side. In the second case the curves only intersect at the point O, and this point is a position of stable equilibrium.

The ratio A being less than 0·5, suppose that we displace the curve $\sin\beta$ by the angle α, which amounts to saying that the initial position of the needle is not in the plane of symmetry, but at O′ (Fig. 175). When the current passes, the needle sets at rest at

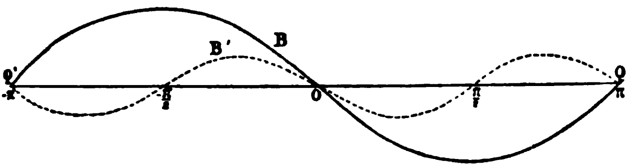


Fig. 174.

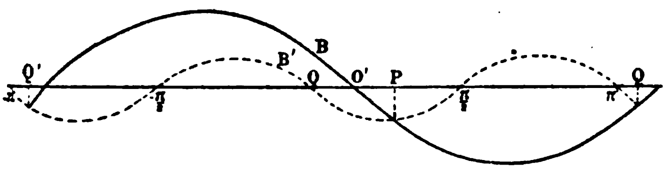


Fig. 175.

the point P, and this position is nearer 90° the greater was the primitive divergence α.

The condition of equilibrium

$$a \sin(\beta - \alpha) = A \sin 2\beta$$

shows that if the original deviation α is very weak, the angle β and the displacement $\beta - \alpha = \delta$ of the needle are both proportional to the initial displacement.

902. Use of the Electrodynamometer with Alternating Currents.—The electrodynamometer (**853**) lends itself especially to the measurement of periodical alternate currents, provided the period is small in comparison with the time of vibration of the movable coil, and that the current passes entirely in the two coils.

The deflection δ gives then the mean square of the intensity by the same expressions as if it were the case of a uniform current.

Such, for instance, would be the case of a sinusoidal current. When a circuit contains a sinusoidal electromotive force of the form

$$E = E_0 \sin 2\pi \frac{t}{T},$$

the current is also sinusoidal, and of the same period T. The intensity may be represented by the equation

$$I = A \sin 2\pi \left(\frac{t}{T} - \phi\right),$$

and the mean square of the intensity is equal to $\frac{A^2}{2}$. It is useful to remember that the resistance, and the coefficient of the electrodynamometer, occur in the expression for the maximum amplitude and of the phase of the current.

903. If the coils are traversed separately by sinusoidal currents of the same period, but different in intensity and in phase

$$I = A \sin 2\pi \frac{t}{T},$$

$$I' = A \sin 2\pi \left(\frac{t}{T} - \psi\right),$$

as in the case in which these coils are shunted in respect of each other (**548**), we should have for the permanent deflection

$$\tan \delta = \frac{S'G}{C} \frac{AA'}{\theta} \int_0^\theta \sin 2\pi \frac{t}{T} \sin 2\pi \left(\frac{t}{T} - \psi\right) dt.$$

The integration being made for any given time θ, which contains a whole number of periods, we have

$$(56) \qquad \tan \delta = \frac{S'G}{C} AA' \left(\frac{1}{2} - \sin^2 \pi\psi\right).$$

If we suppose that the coefficient of mutual induction may be neglected, which is almost the case with the arrangement of the

ordinary electrodynamometer, in which the axes of the two coils are rectangular, the difference of phase is given by the expression

$$\tan 2\pi\psi = \frac{2\pi}{T}\frac{L'r - Lr'}{(r+r')r + \frac{4\pi^2}{T^2}(L+L')L},$$

in which L and L′, r and r' are the coefficients of self-induction and the resistances of the two coils.

The difference of phase is zero if we have

$$\frac{L}{L'} = \frac{r}{r'},$$

that is to say, if the resistances are proportional to their coefficients of self-induction. This condition is realised for two similar coils.

In the general case, as the difference of phase is between 0 and $\frac{1}{4}$, $\sin^2 \pi\psi$ is between 0 and $\frac{1}{2}$; consequently for given intensities of the two currents, the factor in the brackets in equation (56) may vary from 0 to $\frac{1}{2}$, according to the difference of phase, which depends on the resistance and the coefficients of induction of the two coils.

When the resistance r' and the coefficient of self-induction L′ of the movable coil are very great in comparison with those of the fixed coil, we have sensibly

$$\tan 2\pi\psi = \frac{2\pi}{T}\frac{\frac{r}{r'} - \frac{L}{L'}}{\frac{r}{L'} + \frac{4\pi^2}{T^2}\frac{L}{r'}} = \frac{2\pi L'}{Tr}\frac{\frac{r}{r'} - \frac{L}{L'}}{1 + \frac{4\pi^2}{T^2}\frac{LL'}{rr'}}.$$

904. Suppose, finally, that in order to measure a sinusoidal current we join in series the two coils of an electrodynamometer, of which r' and L′ are the total resistance and the coefficient of self-induction, and that we shunt the instrument at two points of the principal circuit, between which is a wire whose elements are r and L, and let M be the coefficient of mutual induction of the

two branch wires. The amplitudes A, A′, and A_0 of the current in the two branches and in the principal circuit satisfy the equations

$$\frac{A^2}{r'^2+\frac{4\pi^2}{T^2}(L'-M)^2}=\frac{A'^2}{r^2+\frac{4\pi^2}{T^2}(L-M)^2}=\frac{A_0^2}{(r+r')^2+\frac{4\pi^2}{T^2}(L+L'-2M)^2}.$$

When the coefficient of self-induction of the wire, which acts as a shunt, is zero, we have $L=0$, $M=0$, and therefore

$$A_0^2=A'^2\left(\frac{r+r'}{r}\right)^2\left[1+\frac{4\pi^2L'^2}{T^2(r+r')^2}\right].$$

If the second term of the bracket is very small, which is the ordinary case, we see that the use of shunts, with the precautions mentioned, may also serve to determine alternate currents by the aid of an electrodynamometer.

If the ratio $\frac{L'}{r+r'}$ had any considerable value, we might eliminate it by two different experiments in which one of the resistances r or r' is varied.

905. Measurement of Alternating Currents by the Electrometer.—The most correct means of measuring alternating currents is that which has been pointed out in (868), and which consists in connecting, with two points A and B of the circuit, the two electrodes of a quadrant electrometer the needle of which is connected with one pair of quadrants.* The deflection of the needle, being proportional to the square of the difference of potential, does not change its sign with this difference. If then currents, which are alternately in opposite directions, succeed each other at very short intervals compared with the time of oscillation of the needle, this, like the movable coil of the galvanometer, takes up a fixed deflection proportional to the mean square of the intensity.

* Joubert. *Comptes rendus*, Vol. xci., p. 161. 1880.

If R is the resistance between the two points A and B, k the constant of the instrument, and δ the deflection observed, the mean square I_m^2 of the intensity is given by the expression

$$I_m^2 = \frac{2\delta}{kR^2}.$$

The only condition to be fulfilled is that the coefficient of self-induction of the interposed resistance R may be disregarded.

The method employed in (869) for measuring the energy expended between the two points A and B also applies to alternating currents. The difference of the deflections α and β gives

$$\frac{\beta-\alpha}{kR} = \frac{1}{Rt}\int_0^t (V_2 - V_1)(V_2' - V_1') = \frac{1}{t}\int_0^t EI\,dt.$$

906. Investigation of Currents in the Variable State.—Ordinary galvanometrical methods enable us to measure variable currents whose variations are slow in comparison with the time necessary for the damping of the needle, but they are not sufciently so in the case of rapid variations like those which accompany the effects of induction. The methods to which we should have recourse depend greatly on the conditions of the experiment. We shall restrict ourselves to a few general conditions and to some examples.

The most direct means of knowing the state of a circuit at a given time t, is to put two points in connection with an electrometer during a very short time θ, and then to insulate the electrometer. The permanent deflection, or the initial swing, of the galvanometer is proportional to the charge of the instrument—that is to say, to the difference of potential E_0 between the points A and B, or to the intensity I of the current at the time t. We may also substitute for the electrometer a condenser of capacity C, the charge of which is then measured by a galvanometer.

The charge, however, which is acquired by the instrument in the latter case is not, strictly speaking, independent of the duration of the contact. If the effects of self-induction can be neglected, the difference of potential E of the armatures at the time θ is given (242) by the formula

$$E = E_0 \frac{R}{R+r}\left[1 - e^{-\frac{\theta(R+r)}{RCr}}\right],$$

in which R is the resistance of the dielectric, and r the resistance of the connecting wires.

As the ratio $\frac{r}{R}$ is generally very small, we have sensibly

$$E = E_0 \left[1 - e^{-\frac{\theta}{Cr}} \right].$$

The change may be considered instantaneous, if the quantity Cr is infinitely small compared with the duration θ of the contact, which ought to be so small that the principal current I has not had time to undergo an appreciable change. It must finally be assumed that the quantity of electricity removed from the principal circuit by the electrometer or the condenser does not appreciably modify the current between the points A and B. These conditions are usually pretty easy to realise.

The connections at A and B should be made and broken simultaneously, unless the conditions of the experiment do not allow of one of the points A being kept at constant potential—by connecting it to earth, for instance. It is sufficient then to arrange the contact at the point B.

The electrometer may still be replaced by a galvanometer. The swing imparted to the needle will be proportional, other things being equal, to the principal current I.

907. The sudden opening of the current in the principal circuit S would produce, in a closed circuit S′ of resistance R′, an induced current the quantity m of which, being proportional to the coefficient M of mutual induction, would satisfy equation (**541**)

$$mR' = MI.$$

From this would be derived a means of determining I by the swing of a ballistic galvanometer situated in the circuit S′. If the principal circuit S contains neither resistances nor capacities of considerable magnitude, and the intensity I of the current is small, the duration of the spark on breaking may be regarded as negligable.

The time t at which a variable current is observed should be counted from a starting-point which is in relation with the nature of the phenomenon. In the case of induced currents, this origin is naturally when the induction begins. The interval t will be deduced from the mechanical arrangements employed, either directly or by the aid of an auxiliary current as in Pouillet's method (**895**).

908. We shall cite, in the first case, as more simple, the method used by Mr. Sabine* for determining the form and velocity of propagation of the electrical wave produced in a cable, one end of which is connected to the earth, and the other A is at a potential which varies according to any given law (**294** *et seqq.*).

The end A of the cable is insulated, and any given point of its length is connected with a condenser the second armature of which is to earth. A rotating commutator (**896**) first connects the point A with a battery, and after a known time t, insulates the condenser from the point M, and discharges it through a galvanometer. The swing of the needle is proportional to the potential acquired by the point at the moment of breaking. In this way the law of variation of the potential during the closing of the circuit is determined.

By arranging the contact studs suitably on the rotating commutator, the length of contact of the point A with the battery may be limited, or it may be successively raised to different potentials of either sign for intervals of known length; so that electrical waves alternately positive and negative may be sent through the cable. If the contact stud relative to the point M is then arranged so that the charge of the condenser is null, it will determine the period at which the point M is in the neutral state between two successive waves.

It is, however, supposed that the presence of the conductor at the point M does not change the original conditions of the experiment, which necessitates that its capacity is very small in comparison with that of the cable.

909. The experiments of Von Helmholtz† on the variable period of the current of closing refers to a more complicated phenomenon.

The total quantity of electricity, which corresponds to the extra current of making (**533**) or of breaking, for a current I in the permanent condition, is equal to $I\frac{L}{R}$.

It may be observed that if the channel of the coil is given as well as the current, this quantity of electricity only depends on the weight of the wire; for the coefficient of self-induction L (**780**) and the resistance R (**726**) are each proportional to the weight of the metal. It is therefore advantageous, in studying the phenomenon, to use a thick metal wire of good conducting material.

* R. SABINE. *Phil. Mag.* [5], Vol. II., p. 321. 1876.

† HELMHOLTZ. *Pogg. Ann.*, Vol. LXXXIII., p. 505. 1851.—*Wissenschaft. Abhandl.*, Vol. I., p. 429.

Von Helmholtz used a coil of thick copper wire, which was made to act on a magnetised needle intended to measure the discharge for a transient current. A key closes the circuit and opens it after a time θ; we have then (895)

$$m = \int_0^\theta \mathrm{I}dt = \mathrm{I}_0 \left[\theta - \frac{\mathrm{L}}{\mathrm{R}}\left(1 - e^{-\frac{\mathrm{R}\theta}{\mathrm{L}}}\right)\right].$$

The experiment is then begun again with this difference, that instead of opening the circuit after a time θ, an equal resistance is substituted for the battery. The new swing corresponds to the quantity of electricity $m_0 = \mathrm{I}_0\theta$. The ratio of the two swings gives the ratio of the mean current during the time θ to its final value. From this is easily deduced the intensity at each instant.

910. In order to make or break circuits at known intervals, we may use bodies which have a motion of translation, as the fall of a weight, or oscillating or rotating apparatus. One of the most convenient is a very heavy pendulum, which is allowed to fall from a given height, and which is allowed to act on contact studs at the time of its greatest velocity.

In the case of a current on closing, for instance, the pendulum first strikes against a lever, which closes the circuit and marks the origin of the phenomenon, and then against another one which opens the circuit. This second lever may be displaced parallel to itself by means of a micrometric screw. The interval which separates the two contacts is deduced from the time of oscillation and the distance of the levers.

911. This arrangement was utilised by Von Helmholtz,* and after him by Schiller,† in investigating the electrical oscillations produced in an open circuit when the current in an adjacent one is broken. Let A be the inducing coil, A′ the induced coil, the ends of the wire in which are connected with the armatures of a condenser. When the inducing current is opened, an electromotive force is developed in the circuit A′, which conveys opposite electricities to the two armatures. The capacity of the wire itself is the charge of each of its halves for unit difference of potential at the two ends. If these communicate with a condenser, the

* HELMHOLTZ. *Verhandl. der Naturhist. Medicin. Vereins zu Heidelberg*, Vol. v., p. 27. 1869.—*Wissenschaft. Abhandl.*, Vol. I., p. 531.

† SCHILLER. *Pogg. Ann.*, Vol. CLII., p. 535. 1874.

capacity of the condenser added to that of the wire forms the total capacity C.

The apparatus thus charged by induction is then left to itself. The discharge through the induced wire is continuous or oscillating (536), according as we have one or other of the conditions

$$L \lesseqgtr \frac{R^2C}{4}.$$

Von Helmholtz used in his experiments a small Ruhmkorff's coil without a soft iron core. In order to determine the condition of the induced wire at the time t, after opening the inducing current, the sciatic nerve of a frog, which is still the most sensitive galvanoscope, was introduced at this moment by means of a key.

The intensity of a discharge current is a maximum when the charge of the condenser passes through zero (538) and is null at the intermediate periods. It is then only that the frog is inactive. The very great resistance which it opposes extinguishes the following oscillations. In this way more than fifty equidistant oscillations have been observed.

Schiller connected the armatures of a condenser with a quadrant electrometer, one of the armatures and the corresponding pair of quadrants being to earth. The first contact of the pendulum breaks the inducing circuit, the second separates the induced wire from the insulated armature of the condenser. The electrometer gives then the difference of potential of the two armatures at the instant of the second contact. The micrometric screw of the second lever enables us to measure intervals of time of 0·000001 of a second.

912. When the variable current is naturally periodic, as that of most machines based on induction, or if by any device we can make the effects periodic, instead of working on a single period, it is more advantageous to arrange the break so as to catch the effect we want to measure at a given, and always the same, time of the successive periods. We may thus substitute the measure of a permanent effect for that of a temporary one.

Such are the experiments of Guillemin* on the current on closing in telegraph lines.

A rotating cylinder of wood has on its periphery a series of studs or plates of copper of different breadths, which by means of elastic plates render it possible to close the current for known

* C. M. GUILLEMIN. *Ann. de Chim. et de Phys.* [3], Vol. LX., p. 385. 1860.

time t, and then set up an instantaneous connection, lasting about 0·0002 seconds, between a galvanometer and two points of the circuit. The needle of the galvanometer experiences a swing at each contact; but if the cylinder turns continuously, it acquires a permanent deflection proportional to the principal current. M. Guillemin found thus that on air lines from 280 to 1000 kilometres in length, the time necessary for setting up definite current varied from 0·004 to 0·028 seconds. The results depend, moreover, on the energy of the battery, and more especially on the condition of the line.

M. Blaserna* used an analogous commutator to observe the extra currents which are produced on opening or closing the battery. The intensity of the principal current at a given time was deduced from the quantity of electricity induced in an adjacent circuit on breaking (907). As the two effects reproduce themselves periodically, the galvanometer of the inducing circuit and the ballistic galvanometer of the induced circuit both gave permanent deflections.

Bernstein† and Mouton‡ applied the same method to the investigation of electrical oscillations produced by induction in an open circuit.

In the apparatus of Bernstein, a disc, rotating about a vertical axis, has four points united in pairs and situated on the same diameter. The former two dip in mercury cups in the form of sectors, close the inducing circuit, and break it again after an interval long enough for the permanent regime to have become established. The ends of the induced wire are connected by a galvanometer of high resistance—on the one hand with a continuous ferrule near the axis, in which dips the third point; and on the other with a stretched iron wire, which the latter point grazes for an instant at each turn of the disc. A micrometric screw, which acts on the wire, enables us to estimate the time which elapses between this contact and the rupture of the inducing circuit. The needle of the galvanometer tends to acquire a permanent deflection when the connections are made. Bernstein observed the first swing, and did not wait for the state of equilibrium.

Mouton made use of three wheels A, B, C, mounted on the same axis and turning with a uniform motion. The first A closes

* BLASERNA. *Journ. des Sc. Nat. et Économ. de Palerme*, Vol. VI., 9. I. 1870.—*Ann. de Chim. et de Phys.* [4], Vol. XXII., p. 500.

† BERNSTEIN. *Pogg. Ann.*, Vol. CXLII., p. 54. 1871.

‡ MOUTON. *Ann. de l'École Normale* [2], Vol. VI., p. 207. 1877.—*Journal de Physique*, Vol. VI., pp. 5, 46.

the inducing current long enough to establish the permanent state, and then breaks it; after a variable interval of time the two wheels B and C, by means of knife-edges with which they are fitted, connect simultaneously, and for a time which does not exceed 0·000025 of a second, a quadrant electrometer with the ends of the wire of the induced coil. The electrometer acquires a permanent deflection which measures the difference of potential of the two ends of the wire at the moment of contact. The advantage of the electrometer is that, whatever be its capacity, it quickly acquires its normal charge although the contact is so short, and this charge does not affect the distribution of potentials in the induced wire; for when the regime is once established, the electrometer is in a permanent state, and the only effect of the successive contacts is to repair the losses which may be rendered negligable.

Observations agree in showing that the oscillations which follow the first are isochronous. Their duration only depends on the induced coil. That of the first oscillation, which is always longer, depends further on the induced coil. It is greatly increased by placing soft iron in the core of the secondary coil.

913. The same method may be applied to the investigation of alternating currents produced by induction machines. The make and break is placed on the axis of the machine itself, and is adjusted so that the contact between the electrometer and any two points of the induced wire, which is kept either open or closed, takes place at a definite phase of the period.

The phenomenon being made permanent, we might also use a galvanometer. It is, however, to be feared that the shunt produced by the galvanometer circuit, on making contact, would alter the distribution of potentials on the induced wire. But this inconvenience may be avoided by interposing in the circuit of the galvanometer an electromotive force which can bring the needle back to zero. It must be observed that the compensation is independent of the manner in which the break acts, for it controls both the current to be measured and that of the battery opposed to it.

Fig. 176 shows the arrangement used. Let R be the resistance between two points A and B of the principal circuit, R_1 a resistance shunting these two points, and AKGE a shunt of given resistance, which, however, is very great, between the point A and a variable point C of the resistance R, forming a resistance r between these two points. This shunt contains the galvanometer G,

* JOUBERT. *Ann. de l'École Normale* [2], Vol. x., p. 145. 1882.

the opposed pile E, and the key K. If the resistances R and R_1 have no induction of their own, then if V is the difference of potential at the two points A and B, I the intensity of the principal current, I′ that of the shunt current on contact, and, finally, E the electromotive force of the opposing battery, we have

$$V = E\frac{R_1}{r} = IR = I_1R_1;$$

and therefore

$$I = \frac{E}{r}\left(1 + \frac{R_2}{R_1}\right).$$

914. Optical Galvanometer.—The action of a magnetic field on polarized light (591) also furnishes a means of measuring the

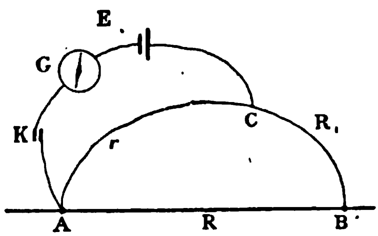


Fig. 176.

intensity of the current. Suppose that a ray of polarized light traverses, in the direction x_1, the thickness $x_1 - x_0 = e$ of a body for which Verdet's constant (594) is ω, the rotation θ of the plane of polarization is expressed as a function of the component of the field, parallel to the direction of the ray, by the formula

$$\theta = \omega(V - V') = \omega\int_{x_0}^{x_1} F_1 dx, \tag{57}$$

or, if F_m is the mean value of this component for the length e,

$$\theta = \omega e F_m.$$

When the magnetic field is produced by a current, the mean magnetic force F_m is equal to the product of the intensity by a more or less complex function of the dimensions of the circuit.

If the luminous ray traverses the axis of a coil for instance, we shall have

$$F_m = IX_m,$$

and the factor X_m will be calculated by the integral of the values (**729**) relative to the different windings of the coil. Equation (43) of (**742**), for example, will give this factor for a coil of great length compared with the thickness e of the body if we suppose it near the centre.

If V and V′ are the potentials for unit current, we have generally

$$(58) \qquad I\omega(V - V') = \theta.$$

Other things being equal, the intensity is proportional to the rotation of the plane of polarization, which will enable us to determine the ratio of two currents.

915. In order to know the absolute value of a current, we must first know the constant ω for the body observed, and the elements of the coil which would enable us to calculate $V - V'$; but we may choose such conditions of experiment as will enable us to get rid of measurements of dimensions. We need only remember that when we pass from one point to another in the field of a current, the change of electromagnetic potential $V - V'$ should be increased by as many times $4\pi I$ as the surface of the circuit has been traversed in a direction opposite that of the field (**452**).

If the ends of the medium in question are on either side of the coil, and are so far that the values of the potentials V and V′ are insensible, the rotation produced by each winding will be $\omega 4\pi I$. A coil of n windings will give then the rotation

$$(59) \qquad \theta = \omega 4\pi n I.$$

This expression depends neither on the shape, nor on the size, nor on the relative direction of the windings, nor, again, on the direction of the radius in respect of their mean plane; it is sufficient to know the number of windings.

916. We may first utilise equation (59) to determine the constant ω relative to a very active body, which would then serve as galvanometric apparatus. Bisulphide of carbon is the best substance.

In strictness it would be necessary to take a tube of infinite length, but we should first calculate by formula (12) of (366) the correction for the potential at the two ends of the tube. This correction could be determined experimentally by the rotation observed with the tubes at each end of the tube actually employed. The maximum action is, in fact, produced on the liquid layers nearest the currents. If the least distance from each end of the tube to a winding is greater than ten times the radius of the winding, the solid angles, which correspond to the values of V and V′, are less than $\pm\frac{4\pi}{400}$, and the corresponding correction does not amount to 0·005.

If the current is reversed, double the rotation of the liquid is observed.

Gordon,* Becquerel,† and Lord Rayleigh‡ have determined the constant of bisulphide of carbon, but their measurements cannot be directly compared; for the first refers to the green ray of thallium, the two others to the ray D of the spectrum; and the experiments were made at different temperatures. According to Bichat, § the rotation diminishes by about 0·0013 for a rise of temperature of 1°. If we correct for temperature, and reduce the results to the same wave length by Verdet's ‖ formula, we find for unit difference of potential (CGS) that the rotation of bisulphide of carbon will be

At zero.	At 18°.	
0·04260	0·04163	Gordon.
0·04626	0·04520	H. Becquerel.
0·04298	0·04200	Lord Rayleigh.

A coil of 5000 turns, with a current of an ampère, will give a rotation of about 275′, or a double rotation of 550′—that is to say, an approximation of 0·002 if the error of reading is 1′.

This optical method has great advantages. It always gives the actual value of the intensity, and there is no perturbation due to

* GORDON. *Phil. Trans. for 1877*, p. 1.

† H. BECQUEREL. *Ann. de Chim. et de Phys.* [5], Vol. XXVII., p. 312. 1882.

‡ Lord RAYLEIGH. *Proceedings of the Roy. Soc.*, No. 232. 1884.

§ BICHAT. *Journal de Physique*, Vol. VIII., p. 204. 1879.

‖ VERDET. *Comptes rendus de l'Acadt de Science*, Vol. LVII., p. 670. 1863.—*Ann. de Chim. et de Phys.* [3], Vol. LXIX., p. 415.

heating the circuit by the passage of the current. The only precaution is to keep the temperature of the bisulphide constant, especially in the most efficacious layers.

917. CALORIMETRICAL MEASUREMENTS.—The calorific energy W, developed in time t, in a conductor of resistance R is

$$W = I^2Rt.$$

If the wire is placed in a calorimeter, the quantity of heat Q imparted to the liquid will be

$$Q = \frac{W}{J} = \frac{I^2Rt}{J};$$

where J is the mechanical equivalent of heat, and therefore

$$I^2 = \frac{JQ}{Rt}.$$

Let M be reduced value of the total mass of the calorimeter, $T_1 - T_0$ the rise of temperature, apart from the correction for losses by radiation or by conductivity; we have $Q = M(T_1 - T_0)$, and therefore

$$I^2 = J\frac{M}{R}\frac{T_1 - T_0}{t}.$$

In order to allow for variations in the resistance of the wire and the specific heat of the liquid, we must assume that these variations are proportional to the temperature, and put

$$R = R_0(1 + \alpha T),$$
$$c = 1 + \gamma T;$$

from this we have, for the temperature T,

$$\frac{I^2R_0}{J}(1 + \alpha T)dt = M(1 + \gamma T)dT,$$

or

$$\frac{dT}{1 + (\alpha - \gamma)T} = \frac{R_0}{JM}I^2dt.$$

Integrating this expression for the time θ, and determining the constant by the condition of having $T = T_0$, for $t = 0$ we get

$$\frac{1}{1-\gamma} l. \frac{1+(\alpha-\gamma)T_1}{1+(\alpha-\gamma)T_0} = \frac{R_0}{JM}\int_0^\theta I^2 dt.$$

The coefficients α and ϕ being very small, we may write

$$\frac{1}{\theta}\int_0^\theta I^2 dt = \frac{JM}{R_0\theta}(T_1 - T_0)\left[1 - (\alpha-\gamma)\frac{T_1 - T_0}{2}\right].$$

If the current is constant, the first member of this equation is equal to I^2. With a variable current the experiment gives the mean square of the intensity. As the phenomenon only depends on the square of the intensity, the method may be used for alternate currents.

We may vary the experimental arrangements in many different ways. One of the simplest is that in which the calorimeter itself, having the shape of a thermometer, gives the temperature directly. If the current passes continuously, the heating of the calorimeter is constantly compensated by the loss of heat. As this latter is sensibly proportional to the excess of temperature of the thermometer over that of the surrounding air, the mean square of the intensity of the current is proportional to this same excess of temperature.

One source of error, which is inherent in the method, is that during the passage of the current the interior of the wire is necessarily at a higher temperature than the external. It follows from this that the real resistance is greater than that calculated for the temperature of the calorimeter. The method would give in general too high a value for the intensity of the current.

In order to determine this intensity in absolute value, we must know the absolute values of R_0 and J. The number J is usually expressed in kilogrammetres, and the quantity of heat in kilogramme degrees. Seeing that the kilogrammetre (612) is equal to 10^5 C.G.S. units, and that Joule's number 425 is generally assumed for the equivalent of heat, it follows that the value of J in C.G.S. units, the heat, being expressed in *gramme degrees*, is about 4.17×10^7.

* JAMIN and AMAURY. *Comptes rendus*, Vol. LXX., p. 661. 1870.

918. MEASUREMENTS OF CURRENTS BY ELECTROLYSIS.—Electrolysis furnishes another means of measuring currents which may be useful in many cases—for instance, in the direct valuation of powerful currents.

It follows from Faraday's laws, that the chemical action of the current, is proportional to the quantity of electricity which passes (955). The ratio of the quantity of electricity decomposed, to the corresponding time, will give the intensity of the current if this is constant, or, in the opposite case, its mean intensity. The bodies which have been most used, and which, in fact, seem to suit best, are acidulated water, a solution of copper sulphate, or a solution of nitrate or chlorate of silver.

With water the volume of gas liberated is measured. The gases should be dried and their temperature and pressure known. In order to avoid these corrections, Bunsen finds it better to weigh the water decomposed. The voltameter is weighed before the experiment, and the gases evolved pass through a drying tube which absorbs the aqueous vapour, and is then again weighed after displacing the residual gases by air. One cause of error arises from the quantity of gas dissolved in water. This might be removed by exhausting the gas with a mercury pump.

The water should be acidulated. When sulphuric acid is used, ozone, peroxide of hydrogen, and other secondary products are formed, and the volume of gas collected is too small. If, however, the temperature of the voltameter is raised to 50°, these oxygenated compounds are formed in negligable quantities.

Experiments are more convenient and more certain with metal salts. The essential condition is to obtain on the negative electrode a coherent deposit which can be easily washed, so as to get rid of all trace of acid, and which does not oxidize in the air.

The form of the deposit depends more particularly on the *density* of the current—that is, the quotient of the intensity by the surface of the electrodes. With copper the deposit is only good when this density is small. If it increases it is rough and botryoidal. With a stronger current it becomes pulverulent. The concentration of the solution has a much smaller influence. The plate covered with copper should be placed for a few moments in boiling distilled water immediately on being taken out of the bath, then wiped and dried with blotting-paper. It would rapidly oxidize if left while moist in the air.

With silver the plate, when once washed with distilled water, may be left to dry spontaneously.

As positive electrode, we may use either a platinum plate or a plate of the same kind as the metal deposited, and which, as it dissolves, reforms the salt decomposed by electrolysis. With a platinum plate the solution becomes poorer, and acid is set free; but this does not affect the deposit provided the reduction of the salt is not carried to the point of materially altering the strength of the solution.

When two plates of the same metal are used, the soluble electrode should theoretically lose all that the other has gained; and it should therefore be a matter of indifference which plate is weighed. This is found to be the case with plates of pure silver in a 15 per cent. solution of nitrate of silver and a suitable density of the current. The alteration in weight is exactly the same in the two plates, and this agreement is an excellent control of the weighings. But general experience shows that, for copper more particularly, the loss on the positive plate is greater than the gain of the negative one. The metal, in becoming disaggregated, falls in particles of extreme minuteness in the liquid, and special precautions must be taken that the particles are not deposited on the negative plate. There are found, moreover, on the soluble plate oxygen compounds, the importance and nature of which depend on the intensity of the current.*

The most recent experiments have given for the chemical action of an ampère,

	Per Second.	
	Silver.	Water.
	Mg.	Mg.
Kohlrausch	1·1183	0·09325
Mascart	1·1156	0·09303
Lord Rayleigh	1·1179	0·09321

From the mean of these numbers the weight of silver deposited by an ampère in an hour is

4·022 grammes.

* See F. and W. KOHLRAUSCH. *Sitz. des Phys. Med. Ges. zu Würzburg.* 1884. —RAYLEIGH. *Phil. Trans.*, Part II., p. 411. 1884.—MASCART. *Journal de Physique* [2], Vol. I., p. 109. 1882. And Vol. III., p. 283. 1884.

CHAPTER III.

COMPARISON OF RESISTANCES.

919. The resistance of a linear conductor is the quotient of the difference of potential at the two ends by the strength of the current which traverses it. The measurement of a resistance in absolute values requires then the knowledge of an electromotive force and of a current; but in order to compare two resistances, it is sufficient to compare the currents which correspond to the same electromotive force, or the electromotive forces which correspond to the same current.

We shall only concern ourselves for the present with comparative measurements. They are of special importance; for the resistance is the only electrical quantity which it is possible to represent by standards of great fixity and of easy use, as for lengths and weights. Resistances are comparable with masses. A current is as necessary for the comparison of two resistances, as a force such as gravity is for comparing two masses. In either case the ratio obtained is independent of the intensity of the action put in play, provided it is the same for both terms.

920. Unit of Resistance.—Legal Ohm.—The standard of resistance may be chosen arbitrarily. Jacobi* proposed to use a copper wire of known dimensions, and, in order to avoid errors arising from the unequal purity of the metal, he proposed to distribute specimens of this wire among physicists.

For a long time telegraphists took as unit a kilometre or a mile of copper wire of known diameter, but at present more exact measurements are required for industrial purposes; for the least trace of foreign substances, and physical changes such as tempering or twisting, so greatly modify the conductivity of a metal that the nature and dimensions of a wire are insufficient to define the resistance; the temperature, moreover, has a considerable influence.

* Jacobi. *Comptes rendus*, Vol. XXXIII., p. 277. 1851.

Pouillet,* who observed these different causes of variation, referred all measurements of conductivity from 1837 to that of distilled mercury. He took as a standard for comparison the column of mercury comprised within a cylindrical tube. The diameter was determined by weighings of mercury, and the ends terminated in two flasks of large aperture.

Werner Siemens† has supplied for industrial purposes a great number of standards which represent very approximately a column of mercury at 0°, a metre in length and a millimetre in cross section.

This is still an arbitrary unit. While retaining mercury as standard metal, it is more rational to choose a column the resistance of which is in a determinate ratio with the absolute unit. The International Commission on Electrical Units, assembled at Paris in 1884, adopted as *practical* unit, under the name of the *legal ohm*, the resistance of a column of mercury a square millimetre in cross section and 106 cm. in length at the temperature of melting ice. From numerous experiments made by different methods, this unit only differs by a few thousandths from the value, 10^9 absolute C.G.S. units, which represents the theoretical definition of the ohm (613).

921. Construction of the Standard.—The construction of a standard which shall agree with its legal definition, is an operation which involves calibrating a tube and weighing the mercury which it contains at the temperature of zero. It is of little importance that the shape of the tube is in exact conformity with the definition, provided we know the dimensions and the column of mercury which fills it; but, in order to facilitate calculations, the resistance of the standard must not differ greatly from its theoretical value.‡

The calibration should be made by the methods in use for accurate thermometers. We shall assume that the tube, carefully selected from among those with the most regular section, has first been divided into parts of equal length. Let a be the number of any given division, $a+\alpha$ the length of the cylindrical column of section equal to the mean section of the tube, which has a volume equal to that comprised between the division a and the

* Pouillet. *Elements de Physique*, 3rd Edition, Vol. I., p. 586. 1837.

† Siemens. *Pogg. Ann.*, Vol. cx., p. 1. 1860. *Works*, p. 229.

‡ Mascart, de Nerville, and Benoît. *Résumé d'Expériences sur la Détermination de l'Ohm.* Paris, Gauthier-Villars 1884.

zero of the scale. The point is to determine the correction α for each division.

A column of mercury is taken which occupies the nth part of the tube, and its length is measured in n successive parts of the scale. Let a_0 and a_1, a'_1 and a_2, a'_2 and $a_3 \ldots\ldots a'_{n-1}$ and a_n be the divisions corresponding to the end of the column, α_0, α_1, α_2, $\alpha_3 \ldots\ldots \alpha_n$ the various terms of correction. As a first approximation we may assume that these corrections are respectively the same for adjacent divisions a_1 and a'_1, a_2 and $a'_2 \ldots\ldots$ The corrected length l of the column—that is to say, the length which it would occupy in a tube of mean section—is expressed by a series of values such that

$$l = a_1 + \alpha_1 - (a_0 + \alpha_0) = a_1 - a_0 + \alpha_1 - \alpha_0.$$

If

$$(1) \qquad \begin{aligned} \delta_1 &= a_1 - a_0, \\ \delta_2 &= a_2 - a'_1, \\ &\vdots \\ \delta_n &= a_n - a'_{n-1}, \end{aligned}$$

are the various lengths observed, and assuming that the volume of mercury is the same—that is, that the temperature has not varied—

$$(2) \qquad \begin{aligned} \alpha_0 - \alpha_1 + l &= \delta_1, \\ \alpha_1 - \alpha_2 + l &= \delta_2, \\ \alpha_2 - \alpha_3 + l &= \delta_3, \\ &\vdots \\ \alpha_{n-1} - \alpha_n + l &= \delta_n. \end{aligned}$$

As the divisions a_0 and a_n correspond to the ends of the scale, the corrections α_0 and α_n are null. If we eliminate l between these n equations, we have

$$(3) \qquad \begin{aligned} \alpha_1 - \alpha_2 + \alpha_1 &= \delta_2 - \delta_1, \\ \alpha_1 - \alpha_3 + \alpha_2 &= \delta_3 - \delta_1, \\ \alpha_1 - \alpha_4 + \alpha_3 &= \delta_4 - \delta_1, \\ &\vdots \\ \alpha_1 + \alpha_{n-1} \quad &= \delta_n - \delta_1. \end{aligned}$$

Adding these latter, we get

$$(4) \qquad n\alpha_1 = \Sigma(\delta - \delta_1),$$

the expression $\delta - \delta_1$ denoting the excess of any given measure of length over the first.

The correction a_1 being known, all the others will be successively determined by equations (3). A further control would be obtained by repeating the same series of observations with columns of different lengths. The curve of the corrections obtained for a certain number of points is finally traced, and from it we deduce the values for all intermediate divisions.

922. In order to obtain the best results from the control experiments, they should be carried out systematically.

As the first column occupies almost the nth part of the total length L of the scale, its extremities will always be near the $n+1$ points of division of the scale in n equal parts; these are the *principal points* of the calibration. If λ_n is the excess of the reduced length of the column of mercury over the nth part of the length L, and $\delta'_1, \delta'_2, \delta'_3 \ldots \delta_n$ the excess of the different lengths observed over the same quantity, putting

$$(1)' \qquad \begin{aligned} \lambda_n &= l - \frac{L}{n}, \\ \delta'_1 &= \delta_1 - \frac{L}{n}, \\ \delta'_2 &= \delta_2 - \frac{L}{n}, \\ \vdots \quad & \quad \vdots \quad \vdots \\ \delta'_n &= \delta_n - \frac{L}{n}; \end{aligned}$$

we might replace the n equations by the following

$$(2)' \qquad \begin{aligned} a_0 - a_1 + \lambda_n &= \delta'_1, \\ a_1 - a_2 + \lambda_n &= \delta'_2, \\ a_2 - a^3 + \lambda_n &= \delta'_3, \\ \vdots \qquad & \quad \vdots \\ a_{n-1} - a_n + \lambda_n &= \delta'_n, \end{aligned}$$

in which $\delta'_1, \delta'_2 \ldots$ are quantities of the order of the corrections given by the readings, and λ_n a constant relative to this first calibration.

A second column of double length $2l$, which is measured in the same way, by placing its ends successively near the principal points, will give $n-1$ equations of the form

$$(2)'' \qquad \begin{aligned} a_0 - a_2 + \lambda_{n-1} &= \delta''_1, \\ a_1 - a_3 + \lambda_{n-1} &= \delta''_2, \\ a_2 - a_4 + \lambda_{n-1} &= \delta''_3, \\ \vdots \qquad \vdots \qquad &\vdots \\ a_{n-2} - a_n + \lambda_{n-1} &= \delta''_{n-1}. \end{aligned}$$

A column of triple length $3l$ will give $n-2$ equations,

$$(2)''' \qquad \begin{aligned} a_0 - a_3 + \lambda_{n-2} &= \delta'''_1, \\ a_1 - a_4 + \lambda_{n-2} &= \delta'''_2, \\ \vdots \qquad \vdots \qquad &\vdots \\ a_{n-3} - a_n + \lambda_{n-2} &= \delta'''_{n-2}. \end{aligned}$$

Continuing in this way with increasing columns, the column of length equal to $(n-2)\ l$ will give 3 equations,

$$(2)^{(n-2)} \qquad \begin{aligned} a_0 - a_{n-2} + \lambda_3 &= \delta_1^{(n-2)}, \\ a_1 - a_{n-1} + \lambda_3 &= \delta_2^{(n-2)}, \\ a_2 - a_n \quad + \lambda_3 &= \delta_3^{(n-2)}, \end{aligned}$$

and the last column two equations,

$$(2)^{(n-1)} \qquad \begin{aligned} a_0 - a_{n-1} + \lambda_2 &= \delta_1^{(n-1)}, \\ a_1 - a_n \quad + \lambda_2 &= \delta_2^{(n-1)}. \end{aligned}$$

We are thus led to a system of $n\dfrac{(n+1)}{2} - 1$ equations with $2n$ unknowns, or rather $2(n-1)$ unknowns, since a_0 and a_n are zero, which determine the corrections of all the principal points with the same accuracy. The particular symmetry of this system enables us to resolve equations under different symmetrical forms. It will be more correct to consider the different values of λ_1 as the unknown, but the probable error is not sensibly increased if we eliminate these quantities by the subtraction of two consecutive equations of each group.*

* M. Thiesen. *Carl. Repertorium*, Vol. XV., pp. 285 and 678. Munich, 1879.

Putting

$$\text{(5)}\qquad \begin{aligned} \Delta'_1 &= \delta'_2 - \delta'_1, \\ \Delta'_2 &= \delta'_3 - \delta'_2, \\ &\vdots \\ \Delta''_1 &= \delta''_2 - \delta''_1, \\ &\vdots \\ \Delta_1^{(n-1)} &= \delta_2^{(n-1)} - \delta_1^{(n-1)}, \end{aligned}$$

we obtain a new system of equations, which we may write

$$\text{(3)}'\qquad \begin{cases} a_1 - a_0 = a_2 - a_1 + \Delta'_1, \\ a_2 - a_1 = a_3 - a_2 + \Delta'_2, \\ \vdots \\ a_{n-1} - a_{n-2} = a_n - a_{n-1} + \Delta'_{n-1}; \end{cases}$$

$$\text{(3)}''\qquad \begin{cases} a_1 - a_0 = a_3 - a_2 + \Delta''_1, \\ a_2 - a_1 = a_4 - a_3 + \Delta''_2, \\ \vdots \\ a_{n-2} - a_{n-3} = a_n - a_{n-1} + \Delta''_{n-2}; \text{ etc.} \end{cases}$$

The same subtraction taken inversely will give the same equations with the opposite signs. The whole of the two systems of equations may be represented synoptically by a table the symmetry of which is evident.

$$\begin{array}{c|cccccc} & a_1 - a_0 & a_2 - a_1 & a_3 - a_2 & & a_{n-1} - a_{n-2} & a_n - a_{n-1} \\ & = & = & = & & = & = \\ a_1 - a_0 & 0 & -\Delta'_1 & -\Delta''_1 & \ldots & -\Delta_1^{(n-2)} & -\Delta_1^{(n-1)} \\ a_2 - a_1 & +\Delta'_1 & 0 & -\Delta'_2 & \ldots & -\Delta_2^{(n-3)} & -\Delta_2^{(n-2)} \\ a_3 - a_2 & +\Delta''_1 & +\Delta'_2 & 0 & \ldots & -\Delta_3^{(n-4)} & -\Delta_3^{(n-3)} \\ \vdots & \vdots & +\Delta''_2 & +\Delta'_3 & \ldots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ a_{n-1} - a_{n-2} & +\Delta_1^{(n-2)} & +\Delta_2^{(n-3)} & +\Delta_3^{(n-4)} & \ldots & 0 & -\Delta'_{n-1} \\ a_n - a_{n-1} & +\Delta_1^{n-1} & +\Delta_2^{(n-2)} & +\Delta_3^{(n-3)} & \ldots & +\Delta'_{n-1} & 0 \end{array}$$

The sum of the terms of the first vertical column being zero, the sums of successive vertical columns give

$$\text{(6)}\qquad \begin{aligned} n(a_1 - a_0) &= \Sigma_1 \Delta, \\ n(a_2 - a_1) &= \Sigma_2 \Delta, \\ &\vdots \\ n(a_n - a_{n-1}) &= \Sigma_n \Delta. \end{aligned}$$

As the corrections α_0 and α_n are zero, we have the values of α_1 and α_{n-1} directly, and therefore those of all the other principal points.

The calculation presents several verifications on which it is useless to insist here. The curve constructed with these different values of α will show by its continuity whether the subdivision of the tube has been carried so far that we could deduce the corrections for the intermediate points. For more details we must refer to the memoir published by M. Benoît on this subject.

923. The mean capacity v of one division of the tube at zero is then determined by observing the number n of divisions corrected for errors of calibration, and for the spherical curvature of the ends, which a weight p of mercury occupies in melting ice, and the experiment will be repeated as a control with columns of different lengths. If d is the specific gravity of mercury, and ϵ the length of a division at zero, we have

$$v = \frac{p}{n\epsilon d}.$$

If, lastly, ρ is the specific resistance of mercury—that is to say, the resistance of a column of unit length and of unit section, s the section of the tube corresponding to any given division, the total resistance of the tube is

$$R = \rho\epsilon \Sigma \frac{1}{s}.$$

It is not really necessary to calculate the section at each point. If we consider the column comprised between two divisions a and b, the corrections for which are α and β, the mean section of this column is

$$s = \frac{v(b-a+\beta-\alpha)}{\epsilon(b-a)} = \frac{v}{\epsilon}\left(1 + \frac{\beta-\alpha}{b-a}\right),$$

and the corresponding resistance

$$r = \rho\frac{\epsilon(b-a)}{s} = \rho\frac{\epsilon^2}{v}\frac{(b-a)^2}{b-a+\beta-\alpha}.$$

* J. R. Benoît. *Travaux et Mémoires du bureau international des poids et mesures.* Vol. II., Part I., p. C. 35, 1883.

Suppose we divide the tube into a series of equal lengths by the divisions $a, b, c \ldots\ldots l, m$, except the last which is arbitrary, and the corrections of which are $\alpha, \beta, \gamma, \ldots\ldots \lambda, \mu$; putting

$$b-a=c-b=\ldots\ldots=C,$$

we get an approximate value for the resistance of the tube between the divisions a and m, by the expression

$$R=\rho\frac{\epsilon^2 C^2}{v}\left[\frac{1}{C+\beta-\alpha}+\frac{1}{C+\gamma-\beta}+\ldots\right]+\rho\frac{\epsilon^2}{v}\frac{(m-l)^2}{m-l+\mu-\lambda},$$

and this value is more exact as the total number of divisions is greater.

In order to ascertain if the subdivision is sufficient, a series of analogous calculations is repeated by giving C values which gradually decrease, and we stop when the differences of the successive results become inappreciable.

If the extreme divisions a and m are chosen, so that the ratio $\frac{R}{\rho}$ given by this expression is equal to $\frac{1{\cdot}06^2}{(0{\cdot}1)^2}$, the resistance of the column of mercury comprised between them at the temperature of zero will be exactly an *ohm*.

In comparative experiments it is necessary that the ends terminate in vessels of large diameter, also containing mercury. Theory and experiment show that to the special resistance of the tube must be added that of the portions of mercury which are near its ends in the vessels, and that the correction for this double communication is obtained with sufficient accuracy if we add to the length of the tube a fraction of its diameter equal to 0·82. After having calculated the length of the tube which corresponds to an ohm, 0·82 of the diameter is subtracted, and it is cut off to this new length.

In this way M. Benoît produced four standards, the difference of whose resistance did not exceed 0·0002.*

924. A straight mercurial standard is inconvenient. Copies may be constructed by means of a narrow tube of any given form, containing mercury, and whose ends are connected with reservoirs of large diameter. Fig. 177 represents a standard, the tube of which, twisted in the form of a double spiral, has been filled with mercury in vacuum, and is connected with external reservoirs by platinum

* BENOÎT. *Comptes rendus*, Vol. XCIX., p. 864. 1884.

wires. The capillary part of the tube is immersed in a bath, the temperature of which may be determined by a thermometer T; a tube A at the side enables us to agitate the liquid by a current of air. We determine, by experiment, either the correction which must be made to bring the resistance to the temperature zero, or the temperature at which the resistance is equal to the legal ohm. The

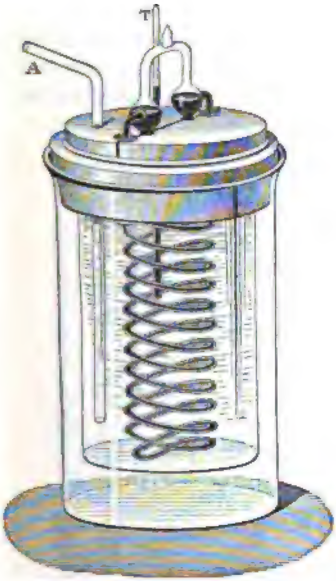


Fig. 177.

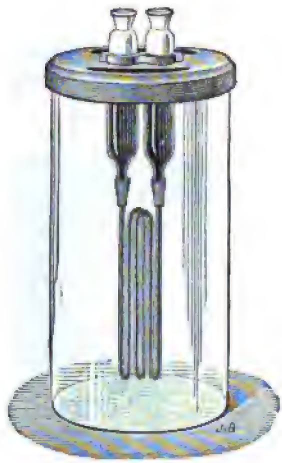


Fig. 178.

change of resistance of the mercury in a glass tube, as a function of the temperature, may be calculated by the formula

$$R = R_0 (1 + 0{\cdot}0008649 t + 0{\cdot}00000112 t^2).$$

It may, however, be feared that the platinum in the mercury in time partially dissolves, and thus modifies the conductivity. In his recent researches on the practical construction of the legal ohm, M. Benoît adopted a slightly different form (Fig. 178). The ends of the capillary tube terminate in large open tubes. The filling also takes place in a vacuum, but the mercury can be easily renewed.

Glass instruments being fragile, the practical standards are usually constructed with metal wires; alloys are to be preferred to pure

metals, for their resistance varies less with the temperature. While the coefficient of variation is 0·0039 for copper, it is only 0·00444 for argentan, and 0·00031 for an alloy of two-thirds silver and one-third platinum.

Fig. 179 represents the form of standard which was adopted by the committee of the British Association in 1865. The wire is an alloy of platinum-silver; it is covered with a double layer of silk, and varnished with shellac. After having been doubled, it is folded as a spiral, which is then placed between two concentric cylinders of brass, and immersed in a mass of paraffine filling the interval of the two cylinders. The two ends of the wire are soldered to two stout

Fig. 179.

copper wires bent twice at right angles, and the amalgamated ends of which dip in mercury cups.

The cylinder may be placed in ice or in water; a thermometer placed in the central tube gives then the temperature. Although this arrangement gives a large surface of contact with water, the bad conductivity of paraffine always leaves great uncertainty as to the true temperature of the wire, unless the temperature of the bath has been kept constant for several hours.

The form adopted by Siemens is represented in Fig. 180. The wire, which is of argentan, and covered with silk, is coiled as a

double spiral on the surface of a wooden cylinder, and ends in two large metal rods. It is protected by a metal case, which leaves sufficient space for the circulation of air, and the whole is enclosed in a wooden box. A thermometer can be placed in a cavity in the centre of the coil, but the determination of the temperature is more difficult, since the instrument cannot be immersed in liquid. The best plan is to surround it with a thick layer of wadding.

925. There is at present no very exact information as to the degree of permanence of standards of resistance. With mercury the only changes to be feared are those arising from the deformation of the glass; the imperfect purity met with in commercial specimens of mercury has no appreciable influence; a fresh filling of the tube does not change the results, and, except as regards the facility of the

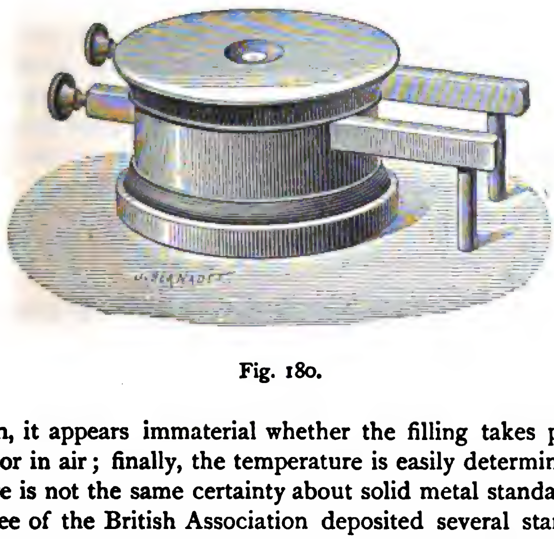


Fig. 180.

operation, it appears immaterial whether the filling takes place in a vacuum or in air; finally, the temperature is easily determined.

There is not the same certainty about solid metal standards. The committee of the British Association deposited several standards at Kew Observatory, either of platinum, or of alloys of platinum and silver, of platinum and iridium, or of gold and silver. Matthiesson and Hockin compared these standards in 1867, and determined the temperatures at which they then had the same resistance.*

A fresh comparison made in 1876 by Chrystal and Saunder,† with very elaborate precautions as to the exact determination of temperature, did not give the same results; but it might be thought

* *British Association Report for 1867.* Dundee. Reprint, p. 145.

† *British Association Report for 1876*, p. 13. Glasgow.

that these differences arose from the determination of the temperatures. Fleming* resumed this investigation in 1881; he found considerable differences from the preceding determinations, and the deviations sometimes amounted to 0·0011. It seems difficult to decide whether such large differences are due to a real alteration of the standards of resistance, or simply to the difficulties of exactly determining the temperatures.

926. RESISTANCE BOXES.—In comparative experiments it is necessary to have a series of resistances, the values of which increase regularly. They usually consist of coils placed in the same box, and provided with keys by which they can be introduced into a circuit.

In order to prevent any action of these coils on galvanometers, and to reduce the effects of induction to a minimum (760), the wire, after being doubled, is wound on the coil. These wires should be carefully insulated, and when once the coil is constructed, it is well to surround it with a layer of paraffine.

The two ends of the wire are soldered to stout pieces of brass, with a small interval between them. They are cut away on each side, so as to allow of the introduction of a plug, which thus connects them by a conductor, the resistance of which may be neglected. This mode of connection is practical and convenient.

The resistance of the plugs cannot always be disregarded, and experiment shows that it may amount to as much as 0·0001 of an ohm.†

The coils are arranged in succession, so that each piece of brass connects two successive pieces. When a plug is removed, the resistance of the corresponding coil is brought into the circuit; by putting in the plug it is removed.

If, instead of soldering the wires directly to the pieces of brass themselves, they are joined by the intervention of small copper rods, the wire of two successive coils should not be soldered to the same rod, but each wire should have a rod of its own, so that the resistance introduced by removing two plugs is exactly the same as is obtained by removing each plug in succession. These rods, as usually employed, have a certain resistance which cannot be neglected in accurate measurements.

In verifying these coils, it is desirable that each of the pieces of brass should have a hole, in which a special plug, provided with a binding screw, can be introduced. This is a very convenient plan

* *British Association Report for 1883*, p. 41. Southport.

† DORN. *Wiedemann's Annalen*, Vol. XXII., p. 558. 1884.

for independently bringing into the circuit any of the resistances of which the box is made up.

927. Many different subdivisions may be adopted for graduating the values of the resistances. The most economical would be to use a series of coils, the resistances of which vary as the terms of the progression $1, 2, 2^2, 2^3 \ldots\ldots 2^n$. With $n+1$ coils we should have all the resistances from 1 to $2^{n+1}-1$. By taking twelve coils, the first of which is an ohm, we may obtain all resistances from 1 ohm to 8191 ohms.

Thus, to get a given resistance—107, for instance—it is sufficient to write the number in the primary system $2^6+2^5+2^3+2+1$, or 1101011, and to leave open all the coils which correspond to the ciphers 1, closing by plugs those which correspond to the zeros.

It is desirable to add to the series a supplementary coil equal to unity, which would enable us to verify by comparison the relative values of the various coils.

The small calculation required by the preceding arrangement makes its use very inconvenient.

The coils may be combined like boxes of weights in taking for the series the values

1; 1, 2, 2, 5; 10, 10, 20, 50; 100, 100, 200, 500;
1000, 1000, 2000, 5000.

We have thus a total of 10,000 units, and the means of verifying all the coils.

The coils are most frequently arranged in a linear series, and all those which are not to be used are plugged; but this arrangement is so far inconvenient that a variable number of plugs is used for each combination, and therefore a variable number of contacts, the effect of which cannot always be neglected.

A better arrangement is that of a dial box (Fig. 181). Each dial consists of nine equal coils, connected by brass plates numbered from 0 to 9, there being no connection between the plates 9 and 0. In the centre is a copper disc, connected with the plate 0 of the next dial by plates or bars of brass L, L′, L″. The plugs are placed between the disc and the plates of the segments; as their number is fixed, one for each dial, and always at work, we may consider that the resistance they introduce into the circuit is constant.

The boxes usually contain four dials corresponding to the *units*, *tenths*, *hundreds*, and *thousandths*, as seen in the lower part of Fig. 181. With a supplementary unit placed inside the box, the

ends of which terminate in two lateral binding screws v and v', the total resistance is 10000 ohms.

Special plugs furnished with binding screws, and which are placed in holes in the segments, render it possible to remove any resistance from the set of dials.

The wire used in resistance coils is usually very fine; it would be dangerous to pass through it powerful currents, which would endanger the insulating substance, or even burn the metal. Less delicate resistances are obtained with rods of carbon, such as are used for the electric light. These carbon resistances vary very little

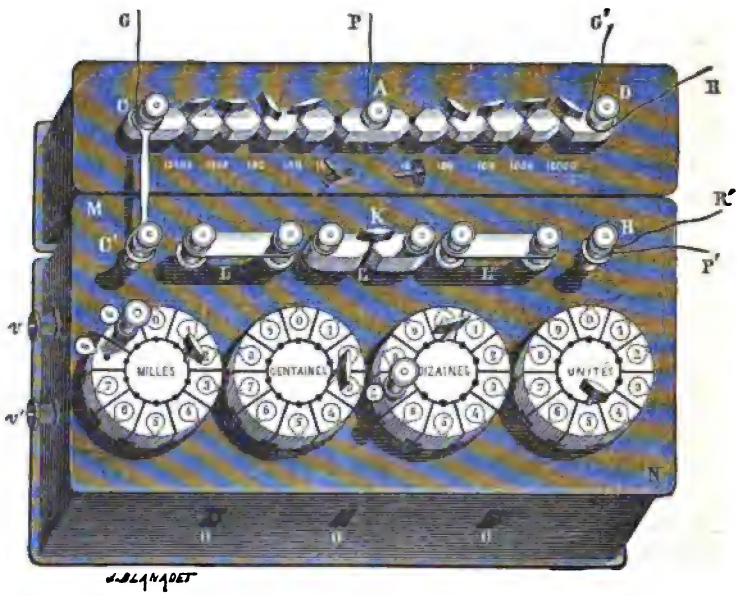


Fig. 181.

with the temperature; the connections are effected by means of copper mounts.

Very large and practically convenient resistances are obtained by means of black lead lines ruled on ebonite, or better in a polished groove in the ebonite. The ends of the line are connected with binding screws, and the line itself is then covered with varnish.*
It is necessary to control these resistances from time to time;

* PHILLIPS. *Phil. Mag.*, Vol. XL., p. 41. 1870.

experiment shows that they may alter in the course of time, but they are very little affected by changes of temperature.*

928. Conductivity Boxes.—Sir W. Thomson has given this name to a system of coils arranged so as to compare directly the inverses of resistance—that is to say, conductivities.†

When several resistances r_1, r_2,.... r_n, are arranged in multiple arc between two points (209) the conductivity of the system, or the inverse of its resistance R, is equal to the sum of the conductivities of each of the arcs.

$$\frac{1}{R}=\frac{1}{r_1}+\frac{1}{r_2}+\dots+\frac{1}{r_n}.$$

Consider, for instance, a series of coils the resistances of which vary as powers of 2. All the coils are connected at one end with the

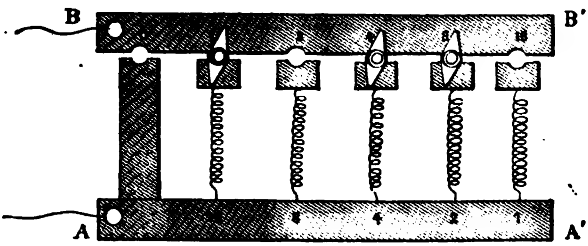


Fig 182.

same bar of brass, A A′ Fig. 182, while the other end terminates in a copper block. These blocks are at a small distance from a second similar bar, B B′, parallel to the first, and with which they may be connected by plugs. In this way we introduce between the two bars, and therefore between the points A and B of the circuit, as many coils in parallel arc as there are shunts.

The figures of the top bar represent the resistances of each coil, and the lower numbers represent the conductivities multiplied by 16. With the plugs as arranged in the figure, the conductivity of the system would be $\frac{22}{16}$, and its resistance $\frac{16}{22}$.

* Werner Siemens. · *Reproduction of the Unit of Resistance.* 1882.

† Sir W. Thomson has proposed to designate by *mho*, which is the word *ohm* reversed, the conductivity of a body whose resistance is an ohm. A *mhometre* is an apparatus for measuring conductivities.

929. Correction for Temperature.—The coils of resistance boxes have the value assigned to them only at a special temperature generally given by the constructor; a correction is necessary if we work at a different temperature.

Let t_0 be the standard temperature, t the actual temperature, and α the coefficient of variation; we have

$$R = R_0 [1 + \alpha (t - t_0)].$$

The boxes should be arranged so that we may know the temperature of the coils. The best arrangement consists in letting the inside of the box communicate freely with the outer air by the apertures O (Fig. 181) in which are thermometers the bulbs of which are near the coils. A difference of temperature of t_0 corresponds to a relative error of 0·0003 or 0·0004, according as the wire is an alloy of platinum-silver or of argentan.

In the resistance boxes constructed by Elliott Brothers (Fig. 181), the supplementary unit v v' enables us to get the temperature by a very sensitive electrical method. This unit is made of a copper wire wound on an ebonite cylinder and occupying the entire length of the box. The variation in the resistance of the copper being about 0·0039 for 1°—that is to say, eleven times that of the wire of the coil—the measurement of this resistance by the box itself, with a relative approximation of 0·0001, will indicate the temperature to within a thirtieth of a degree.

The variations of temperature most to be feared are those which arise from the passage of the current itself; it may be well to give an idea of this by a numerical example.

The quantity of heat developed per second in each unit of length of a wire of section ω and specific resistance σ, at a difference of potential E for unit length, is equal in mechanical units to $EI = E^2 \dfrac{\omega}{\sigma}$, or in thermal units to $\dfrac{E^2\omega}{J\sigma}$, where J is the mechanical equivalent of heat. If p is the specific gravity of the metal, c its specific heat, and therefore $\gamma = cp$ the thermal capacity for unit volume, the rise of temperature $d\theta$ corresponding to the time dt is given by equation

$$\omega\gamma d\theta = \frac{E^2}{J} \frac{\omega}{\sigma} dt, \qquad \text{or} \qquad \frac{d\theta}{dt} = \frac{E^2}{J} \frac{1}{\gamma\sigma}.$$

The rapidity of heating $\frac{d\theta}{dt}$ is then independent of the diameter, and the relative increase of resistance for unit time is

$$\frac{dr}{dt}=a\frac{d\theta}{dt}=\frac{E^2}{J}\frac{a}{\gamma\sigma}.$$

For a copper wire we have $p=8{\cdot}85$, $c=0{\cdot}095$, $\gamma=0{\cdot}840$, and $a=0{\cdot}0039$; if, further, $E=n$ volts $=n\ 10^8$, $\sigma=1615$, and $J=4.2.10^7$, we have

$$\frac{d\theta}{dt}=\frac{n^2}{57}10^7.$$

This is the number, in centigrade degrees, by which, apart from losses, the temperature of a wire would be raised in each second for a fall of potential of n volts per centimetre.

For argentan we have $\alpha'=\alpha\times 0{\cdot}11$ and $\sigma'=13\ \sigma$, the value of γ being virtually the same; it follows therefore that, other things being equal, the heating would be 120 times less for argentan than for copper.

These effects are diminished by allowing the current to pass for as short a time as possible through the coils.

930. Rheostate.—Before boxes of coils were introduced, a greater or less length of wire was brought into the circuit in order to vary the resistance. Pouillet* used a platinum wire 132 metres in length stretched backwards and forwards on a board. Wheatstone constructed a more convenient form of the apparatus, and called it the *rheostate*.

Wheatstone's rheostate consists of two parallel cylinders of the same diameter turning in the same direction, and with the same velocity; one is of brass with a smooth surface, the other of glass or of hard wood with a continuous helicoidal grove. A wire of argentan winds itself on either of the cylinders according to the direction of the winding. All that portion of the wire which is in the grooves of the wood is insulated and acts as resistance; that which is on the brass forms part of it, and is as it were suppressed.

This ingenious arrangement has many practical inconveniences. The wire being drawn first in one direction, and then in another, is modified or even loses its shape; its resistance can no longer be

* Pouillet. *Comptes rendus*, Vol. iv., p. 785. 1837.

† Wheatstone. *Bakerian Lecture for 1843.* *Phil. Trans.*, Vol. v., 133, p. 303. *Scientific Papers*, p. 105.

considered as regular or proportional to the length. Finally, the point of contact with the brass cylinder is very badly defined; the uncertainty is much greater than with the usual methods of measuring.

Jacobi's rheostate* has in some respects great advantages over that of Wheatstone. There is only one insulating cylinder, on which the wire is permanently coiled, and which rotates about its axis; a roller which moves parallel to the axis is pressed against the wire by a spring and makes contact. The motion of the roller is defined by that of the cylinder, and moves through one turn of the wire when the cylinder makes a turn. In this way any deformation of the wire is avoided, and the point of contact with the wire is more strictly defined, but the resistance at the point of contact is still very variable. This is the principal drawback to the use of rheostates as measuring instruments; they are however very convenient if we wish to vary the resistance continuously, without wanting to know the exact value.

These cylindrical rheostates may be replaced by the wire rheostates of Pouillet† and of Poggendorff.‡ Two platinum wires are stretched parallel to each other; a sliding contact connects the two wires at any point, and allows the introduction of a given length into the circuit. The double contact is generally obtained by means of a solid piece of metal in which are hollowed out two mercury cups; each of the wires passes through a fine hole drilled in the mercury cups, capillarity being sufficient to prevent the mercury from flowing out. Nevertheless with accurate measurements it is difficult to ensure that the connection of the wire with the mercury always takes place at exactly the same point of the sliding contact.

Part of these inconveniences may be avoided by stretching the two wires in an upright glass tube, which is more or less filled with mercury from a vertical reservoir. By means of suitable binding screws the two wires can be brought into the circuit either separately or in multiple arc, so as to obtain very unequal degrees of sensitiveness. We may thus get the temperature of the wire by immersing them in a badly conducting liquid, such as petroleum.

931. Comparison of Resistances by the Ratio of Currents or of Electromotive Forces.—The simplest method, at any rate in theory, for comparing two resistances, is to compare the currents which the same electromotive force E gives in the circuits of which they successively form part.

* Jacobi. *Pogg. Ann.*, Vol. liv., p. 340. 1841. Vol. lix., p. 145. 1843.

† Pouillet. *Elements de Phys. Experim.*, Third edition, Vol. i., p. 585. 1837.

‡ Poggendorff. *Pogg. Ann.*, Vol. lii., p. 511. 1841.

Let r_0 be the resistance of the battery and of the galvanometer, including the connecting wires, r and r_1 the two resistances to be compared, I_0 the intensity of the current with the resistance r alone, I and I′ the intensities obtained when the resistances r and r_1 are successively introduced into the circuit. We have

$$(7)\qquad I_0 r_0 = I(r_0+r) = I'(r_0+r') = E,$$

whence

$$(8)\qquad \frac{r}{r'} = \frac{I_0 - I}{I_0 - I'}\frac{I'}{I};$$

the ratio of the resistances only depends on the ratio of the intensities, and it is sufficient to use a graduated galvanometer. The corresponding variations dr and dI of the resistance r and of the current I give, from equation (8),

$$-\frac{dr}{r} \doteq \frac{I_0}{I(I_0 - I)}\,dI.$$

For the same absolute sensitiveness dI of the galvanometer, the relative error for the value of r is a minimum, when the product $I(I_0 - I)$ is a maximum, that is when $I_0 = 2I$, or $r = r_0$, which gives

$$-\frac{dr}{r} = 2\,\frac{dI}{I};$$

the relative error for the resistance r is twice the relative sensitiveness of the galvanometer.

If the resistances r and r' are first connected end to end, then in parallel, the corresponding intensities I_1 and I_2 of the current still give the ratios

$$(7)'\qquad I_0 r_0 = I_1(r_0 + r + r') = I_2\left(r_0 + \frac{1}{\frac{1}{r} + \frac{1}{r'}}\right) = E.$$

Comparing with equations (7), we deduce

$$(8)'\qquad \frac{r}{r'} = \frac{I_0 - I}{I - I_1}\frac{I_1}{I_0} = \frac{I_2 - I}{I_0 - I_2}\frac{I_0}{I},$$

so that the experiment gives several controls.

932. When the resistances r and r' are very great in reference to the circuit, and very different from each other, we may facilitate the comparison by the use of shunts.

If ρ is the resistance of the battery up to the binding screws of the galvanometer, g that of the galvanometer, m and m' the shunts placed in the galvanometers with the resistances r and r', i and i' the corresponding resistances, we have

$$(9)\qquad E = mi\left(\rho + \frac{g}{m} + r\right) = mir\left(1 + \frac{\rho + \frac{g}{m}}{r}\right),$$

or sensibly

$$E = mir = m'i'r';$$

consequently

$$(10)\qquad \frac{r}{r'} = \frac{m'i'}{mi}.$$

If one of the resistances is so great compared with the other that the use of shunts is not sufficient to get deviations which are measurable, and within the limits of the scale, the electromotive power is modified by taking a variable number of identical couples. The numbers of the couples in the two cases being n and n', we have

$$\frac{n}{n'} = \frac{mir}{m'i'r'}, \qquad \text{or} \qquad \frac{r}{r'} = \frac{n}{n'}\frac{m'i'}{mi}.$$

In observing the large resistance the shunt is usually suppressed, and only one couple is taken for the small one, this gives $n' = 1$, $m = 1$; in this case the ratio of the resistances is given by the simple expression

$$(11)\qquad \frac{r}{r'} = nm'\frac{i'}{i}.$$

This is the way in which the resistance of the insulating envelope of a telegraph cable is usually measured; the cable being immersed in a vessel of water, one of the ends is insulated and the other is connected to the water through the battery of the galvanometer.

933. When the resistances are so great that the fall of potential from one end to the other can be directly measured, the same current

may be passed through them, and their ends be alternately connected with the quadrants of an electrometer, or with the galvanometer of very high resistance; the ratio of the electromotive forces observed is equal to that of the resistances.

The electrometer may be used with alternate currents, if the coefficient of self-induction is not appreciable, and if the circuit is sufficiently insulated that the condition is not altered when one point is put to earth. If this is not the case, the case of the electrometer must be insulated.

934. Resistance of a Galvanometer or of a Battery.—This method enables us to determine the resistance of the battery and that of the galvanometer itself, without having recourse to another galvanometer. Equation (8) gives, in fact,

$$(12) \qquad r_0 = \frac{I}{I_0 - I} r.$$

Knowing r, the second member represents sensibly the resistance of the battery, or that of the galvanometer, when one is very small compared with the other.

The resistance of the battery, and that of the galvanometer, may be determined separately by making use of shunts. The current i is first measured with a shunt s, of power m on the galvanometer; then removing the shunt a resistance r is added, and the total current I is observed.

We have then

$$(13) \qquad E = mi\left(\rho + \frac{g}{m}\right) = I(\rho + g + r).$$

This second equation between ρ and g combined with the preceding (12), which gives $r_0 = \rho + g$, enables us to calculate the two values sought.

If the resistance r has been chosen so that the two intensities i and I are equal, we get simply

$$\rho = \frac{r}{m-1} = \frac{rs}{g},$$

or

$$(13)' \qquad \rho g = rs.$$

A galvanometer of great resistance may directly give the resistance of the battery.* The total current I is first observed, and then the current i obtained by inserting between the poles, a shunt s of the same order of resistance as that of the battery. We have then

$$E = I(\rho + g) = i\left(\rho\frac{g+s}{s} + g\right).$$

If the ratios $\frac{\rho}{g}$ and $\frac{s}{g}$, are very small, we may write

$$Ig = ig\left(\frac{\rho}{s} + 1\right);$$

consequently

(14) $$\rho = s\frac{I - i}{i}.$$

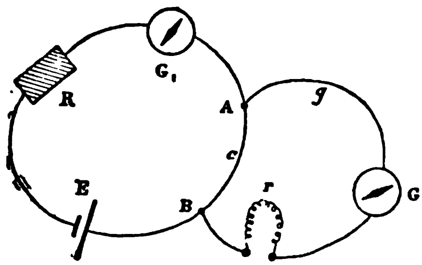


Fig. 183.

935. Use of Two Galvanometers.—In the preceding methods the electromotive force must be constant, which is not always the case (especially with polarisable batteries) when they are traversed by currents of very different strengths.

This source of error may be eliminated by using two galvanometers. The battery circuit contains a rheostate R (Fig. 183) and a galvanometer G_1; at two points A and B of the circuit, separated by a resistance r, are connected the ends of a wire which contains a second galvanometer G; in this branch AGB, of resistance g, the two resistances r and r' are successively interposed.

By means of the rheostate the deflections of either of the two galvanometers in the three experiments may be kept constant.

* Sir W. Thomson. *Journal of the Society of Telegraph Engineers*, Vol. I., p. 399. 1873.

Suppose first that the intensity I_0 is constant in the principal circuit, and let i_0, i, and i' be the three intensities observed in the galvanometer G, we have

$$I^{\circ}=\left(\frac{g}{a}+1\right)i_0=\left(\frac{g+r}{a}+1\right)i=\left(\frac{g+r'}{a}+1\right)i',$$

whence

$$\frac{r}{r'}=\frac{i_0-i}{i_0-i'}\frac{i'}{i}. \tag{15}$$

The conditions are the same as in the first case (931).

If the intensity i_0 is kept constant in the branch circuit, the intensities I_0, I, I′ of the principal current give the ratios

$$i_0=I_0\frac{a}{g+a}=I\frac{a}{g+r+a}=I'\frac{a}{g+r'+a},$$

from which is deduced

$$\frac{r}{r'}=\frac{I-I_0}{I'-I_0}. \tag{15'}$$

This latter method, which was described by Bosscha,* can only give exact results provided the differences $I-I_0$ and $I'-I_0$ are large enough—that is to say, if the resistances are of the same order as that of the galvanometer.

When the resistance of the galvanometer, which serves for the readings, is very large, it may be placed as a shunt on a constant part of the branch circuit, and the formulæ do not alter.

936. Transient Currents.—An instantaneous current, such as those produced in induction, may also be used provided the duration of the current is very small compared with that of the oscillations of the galvanometer needle; in the methods based on the comparison of two currents it is sufficient to replace the deflections of the needle by the angle of swing. W. Weber,† for instance, displaced a magnet between fixed limits in the interior of a coil. The resistances to be compared, r and r', will be introduced in the circuit as before (931); but the experiment may also be arranged differently. The circuit is first closed by a galvanometer of resistance y, and then the galvanometer is shunted by the resistances r and r' in succession, then by the same resistance in

* Bosscha. *Pogg. Ann.*, Vol. cx., p. 452. 1860.

† W. Weber. *Electrodyn. Maasbestim.*, p. 209. 1863.

parallel, and finally by the two resistances in series; if a_0, a, a', a_1 and a_2 are the swings which correspond to the various experiments, and ρ the resistance from the principal circuit to the galvanometer, we have

$$a_0(\rho+g)=a\left[\left(\frac{g}{r}+1\right)\rho+g\right]=a'\left[\left(\frac{g}{r'}+1\right)\rho+g\right]$$
$$=a_1\left\{\left[g\left(\frac{1}{r}+\frac{1}{r'}\right)+1\right]\rho+g\right\}=a_2\left[\left(\frac{g}{r+r'}+1\right)\rho+g\right].$$

From this is deduced, for the ratio of the resistances, the different values

$$(16)\qquad \frac{r}{r'}=\frac{\frac{a_0}{a'}-1}{\frac{a_0}{a}-1}=\frac{1-\frac{a_1}{a}}{1-\frac{a_1}{a'}}=\frac{\frac{1}{a_1}+\frac{1}{a_2}-\frac{1}{a}-\frac{1}{a'}}{\frac{1}{a}-\frac{1}{a_2}}.$$

937. Differential Galvanometer.—A differential galvanometer may be used in various way to measure resistances. The battery, of electromotive force E, and of resistance ρ, is closed by two branches A and B; one of them contains the resistance r and one of the coils g of the galvanometer, the other the coil g', and a standard resistance r'. The principal current I and the currents i and i' in the two branches, give the ratios

$$\frac{i}{\frac{1}{g+r}}=\frac{i'}{\frac{1}{g'+r'}}=\frac{\mathrm{I}}{\frac{1}{g+r}+\frac{1}{g'+r'}}=\mathrm{E}-\mathrm{I}\rho.$$

If k and k' are the factors respectively proportional to the galvanometric constants of the two coils, the deflection δ of the needle may be expressed by the formula

$$\delta=ki-k'i',$$

whence putting

$$\mathrm{D}=(g+r)(g'+r')+\rho(g+g'+r+r'),$$
$$\delta=\frac{\mathrm{I}}{g+g'+r+r'}[k(g'+r')-k'(g+r)]=\frac{\mathrm{E}}{\mathrm{D}}[k(g'+r')-k'(g+r)].$$

If we adjust the resistance r' so that the needle stops at zero, we get

$$\frac{k}{k'}=\frac{g+r}{g'+r'}.$$

When the differential galvanometer is adjusted, we have $k=k'$ and $g=g'$; consequently

$$r=r'.$$

It is easy to get rid of errors of adjustment by substitution, as in a double weighing; a variable resistance which is not standardised is used for r'; r is then replaced by a standard resistance r_1 which re-establishes equilibrium, and we have

$$r=r_1.$$

If, instead of bringing the needle to zero in each experiment, it was observed at the same point corresponding to a deflection δ, the method of substitution would only be exact provided the principal current I, and therefore the electromotive force, remained unchanged.

938. The absolute sensitiveness of the method is determined experimentally by ascertaining what deflection is produced by a known variation of one of the resistances.

Suppose that, the differential galvanometer being adjusted, the two resistances r and r' differ by dr.

The deflection δ is given by the equation

$$\delta=k(i-i')=\frac{k\mathrm{E}}{\mathrm{D}}dr,$$

and the absolute sensitiveness is measured by the ratio

$$\frac{\delta}{dr}=\frac{k\mathrm{E}}{\mathrm{D}};$$

for a given electromotive force, it is proportional to the factor $\frac{k}{\mathrm{D}}$.

The resistances r and r' being very near, we have sensibly

$$D=(g+r)^2+2\rho(g+r)=(g+r)^2\left(1+\frac{2\rho}{g+r}\right),$$

so that, if the resistance of the battery is very feeble compared with that of the galvanometer, the sensitiveness is proportional to $\frac{k}{(g+r)^2}$.

We may examine what should be the resistance of the galvanometer with a given channel so that the sensitiveness is a maximum. If we replace the wire used, by a wire of another diameter m times as small, we should replace the constant k by m^2k (731) and g by m^4g, which amounts to replacing

$$\frac{k}{(g+r)^2} \quad \text{by} \quad k\left(\frac{m}{m^4g+r}\right)^2.$$

But this latter expression is a maximum for

$$m^4g=\frac{r}{3},$$

that is to say, when the resistance of each of the coils is the *third of the resistance to be measured.**

When this condition is realised, we have

$$D=\left(\frac{4}{3}r\right)^2, \qquad \delta=\left(\frac{3}{4}\right)^2 kE\frac{dr}{r^2}.$$

The deflection Δ which the branch containing the resistance r would of itself give, the other being cut, is

$$\Delta=k\frac{E}{\rho+g+r}=\frac{3}{4}\frac{kE}{r};$$

consequently

$$\frac{dr}{r}=\frac{4}{3}\frac{\delta}{\Delta}.$$

939. When the resistances to be compared are very small, they are put successively as shunts on the coils of the differential

* WEBER. *Zur Galvanometrie.—Göttinger Memoiren*, Vol. x., p. 65. 1862.

galvanometer;* it is then advantageous to pass the current alternately and in opposite directions in two coils, and we have

$$I=\frac{g+r}{r}i=\frac{g'+r'}{r'}i'=\frac{E}{\rho+\frac{gr}{g+r}+\frac{g'r'}{g'+r'}}.$$

Putting

$$D_1=\rho\,(g+r)\,(g'+r')+gr\,(g'+r')+g'r'\,(g+r),$$

we get

$$\delta=ki-k'i'=\frac{E}{D_1}\Big[kr\,(g'+r')-k'r'\,(g+r)\Big].$$

As before, we have $r=r'$ for $\delta=0$, when the galvanometer is adjusted, and we might eliminate any defects of adjustment by substitution.

With an adjusted galvanometer, the deflection δ_1 which a difference dr produces between the two resistances is

$$\delta_1=\frac{kEg}{D_1}dr.$$

Comparing with the value obtained in the preceding arrangement, we have

$$\frac{\delta_1}{\delta}=\frac{Dg}{D_1}=\frac{(g+r+2\rho)\,g}{\rho\,(g+r)+2gr};$$

it is seen that the second method is more sensitive than the rest, when

$$(g+r+2\rho)\,g>\rho\,(g+r)+2gr,$$

or

$$r<g.$$

If the ratio $\frac{r}{g}$ is very small, we have sensibly

$$D_1=g^2\,(\rho+2r),$$

* HEAVISIDE. *Journal of the Society of Telegraph Engineers*, Vol. II., p. 115. 1873.

and consequently

$$\delta_1 = \frac{kE}{g(\rho + 2r)} dr.$$

The deflection Δ_1 produced by a single coil is

$$\Delta_1 = \frac{kEr}{g(\rho + r)};$$

it follows that

$$\frac{dr}{r} = \frac{\rho + 2r}{\rho + r} \frac{\delta_1}{\Delta_1}.$$

940. The differential galvanometer also enables us to compare very different resistances, provided we shunt the coil corresponding to the smallest resistance r'. Let m' be the multiplying power of the shunt, i and i' the intensities; the needle being at zero, we have

$$ki = k'i'.$$

$$i(r+g) = m'i'\left(r' + \frac{g'}{m'}\right);$$

consequently

$$k(m'r' + g') = k'(r+g),$$

and if the galvanometer is adjusted

$$r = m'r'.$$

When the shunt is insufficient, the two coils are put in separate circuits, one with the resistance r and n couples, the other with the resistance r', a single couple and the shunt. This method is also often used in measuring the resistance of an insulator; we have, in that case,

$$r = nm'r'.$$

941. We shall give further the method of Sir C. W. Siemens* and that of Fleeming Jenkin,† as allied to the use of the differential galvanometer.

* C. W. Siemens. *British Association Report.* 1867. Reprint, p. 142.
† Jenkin. *British Association Report.* 1867. Reprint, p. 144.

In the apparatus of Siemens, the two coils of the differential galvanometer are removed from each other by a fixed quantity; they are displaced parallel to each other in respect of the needle, until this comes to zero. An empirical graduation gives the ratio of the intensities of the two currents.

This ratio is also given by the two rectangular frames of Fleeming Jenkin (**854**), when the system is turned through an angle such that the needle remains in the meridian. If ϕ is the angle through which it has been necessary to turn the system from the meridian, we have

$$\tan\phi = \frac{ki}{k'i'} = \frac{k}{k'}\frac{g'+r'}{g+r}.$$

942. WHEATSTONE'S BRIDGE.—The arrangement known as *Wheatstone's Bridge** was first devised by Christie,† and applied

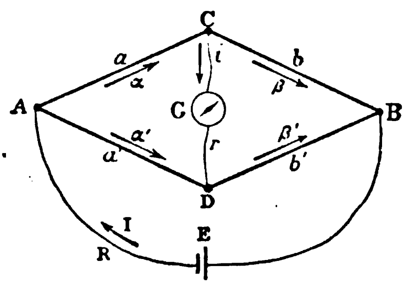


Fig. 184.

by him to the measurement of resistances, as long ago as 1833. It is an arrangement of six wires, which may be represented as the four sides and two diagonals of a quadrilaterial. From the original form of Wheatstone's apparatus, in which the wires terminated in the four corners of a lozenge, it is sometimes called the *parallelogram* of resistances. One of the diagonals contains the battery, the other a galvanometer; the experiment consists in

* WHEATSTONE. *An Account of Several New Instruments and Processes for Determining the Constants of a Voltaic Circuit. The Bakerian Lecture for 1843. Phil. Trans.*, Vol. CXXXIII., pp. 303, 327. *Scientific Papers*, p. 127.

† CHRISTIE. *Experimental Determination of the Law of Magneto-Electric Induction. Phil. Trans. for 1833.*

adjusting the resistances of the four sides so that no current passes in the galvanometer.

Let A and B (Fig. 184) be the two summits connected by the battery, C and D those which connect the galvanometer; a, b, a', and b', the resistances of the four sides AC, CB, AD and DB; α, β, α', β', the currents which traverse them respectively; R the resistance of the diagonal of the battery, r that of the diagonal of the galvanometer or of the *bridge*; I and i the intensities of the current in the two diagonals.

In order that no current shall pass in the galvanometer, it is necessary and sufficient that the two points C and D shall be at the same potential. If V and V′ are the potentials of the two summits A and B, and suppose that the wire CD is cut; the fall of potential from A to C is

$$(V - V')\frac{a}{a+b} = (V - V')\frac{1}{1+\frac{b}{a}},$$

and the fall from A to D in like manner is

$$(V - V')\frac{1}{1+\frac{b'}{a'}}.$$

The potentials are equal at C and D, and no current will pass in a wire interposed between these two points, if

$$(17)\qquad \frac{b}{a} = \frac{b'}{a'}, \qquad \text{or} \qquad ab' = ba'.$$

This condition of equilibrium is that which it is usually attempted to produce.

943. GENERAL PROPERTIES OF A NETWORK OF CONDUCTORS.— We shall consider that a net work of linear conductors forms a *complete* system, when any two points may be connected with each other by a closed circuit taken within the network. The resistances being given, as well as the electromotive forces they contain, the

* Compare POGGENDORFF. *Ann. de Chim. et de Phys.* [3], Vol. XVIII., p. 489. 1846.—BOSSCHA. *Pogg. Ann.*, Vol. CIV., p. 460. 1858.—LUCIEN DE LA RIVE. *Archiv. de Genève*, Vol. XVII., p. 105. 1863.—J. RAYNAUD. *Journal de Physique*, Vol. II., p. 161. 1873.

intensities of the currents may be determined by Kirchhoff's equations.

The form of the equations, from the algebraical point of view, enables us to establish several important properties; but we shall rather attempt to deduce these properties from considerations drawn from the nature of the phenomena.

Suppose that the network contains n conductors, and let m be the number of summits—that is to say, points at which at least three conductors terminate.

The condition

$$\Sigma i = 0 \tag{18}$$

applied to the summits will give rise to $m-1$ distinct equations. For consider the two extremities A and A′ of a conductor, and apply this law successively to all the summits met in going from the point A to the point A′ by a path external to the conductor AA′; we shall thus obtain a series of different equations, for each of them contains at least one new current; but these equations imply the condition that the sum of the currents which traverse any given plane P, cutting the entire system of conductors, including the first, is equal to zero; this in particular is the case for the sum of the currents which terminate at A′, so that the equation relative to this point is already implicitly contained in the preceding ones.

Let p be the minimum number of conductors which must be removed in order to suppress every closed circuit. These p conductors form what we shall call a system of *necessary* wires, and which may in general be chosen in several different ways.

The condition relative to closed circuits,

$$\Sigma (ir - e) = 0, \tag{19}$$

gives rise to p different equations. In fact, we shall first show that the addition of a wire in any given network only introduces a new equation of the second order.

Let A and A′ be two points already connected by several paths, V and V′ their potentials. The difference of potential $V - V'$ is equal to any given expression $\Sigma(i_1 r_1 - e_1)$, $\Sigma(i_2 r_2 - e_2)\ldots.$ relative to the different paths which we may follow in going from A to A′. If we join these two points by a new conductor r, containing an electromotive force e, and traversed by the current i, we shall also have

$$V - V' = ir - e = \Sigma(i_1 r_1 - e_1) = \Sigma(i_2 r_2 - e_2) = \ldots.;$$

the addition of the conductor r brings into the system a new equation, and one only.

When a system of p necessary wires is suppressed, the network is entirely open and cannot give rise to any equation of the latter form. The successive addition of p necessary wires, which reproduces the original network, introduces p distinct equations, which demonstrates the proposition.

As the network contains n different conductors, and the physical phenomenon is defined, the sum total of the equations should be equal to the sum n of the intensities to be determined; from this follows the condition

$$p+m-1=n,$$

or

$$p=n-m+1.$$

The least number p of necessary conductors is thus defined by the number of summits, and the number of sides of the network.

944. The intensities being multiplied by their respective resistances in the p equations (19) for the closed circuits, and by ± 1 in the $m-1$ equations (18) for the summits, the common denominator Δ determined by the ordinary rule, comprises the combinations p and p together of the different resistances.

We shall obtain, moreover, the numerator of the fractions which expresses the value of the intensity i_k, if we replace in the denominator Δ the coefficient $\pm r_k$ of i_k in each of the equations by the known corresponding term. The numerator contains these electromotive forces multiplied respectively only by sums of the combinations $p-1$ and $p-1$ together of the resistances.

Every combination $r_1, r_2 \ldots r_p$ of p necessary wires enters into the denominator. For when the resistances are made infinite, which is equivalent to suppressing the corresponding wires, there is no closed circuit, and the equations should be satisfied if the values for the intensities are zero. But if we divide the two terms of the fraction which gives an intensity i, by the product $r_1, r_2 \ldots r_p$ the numerator is null because it only contains combinations of resistances $p-1$ and $p-1$ together; as the denominator cannot be null, it must contain the combination $r_1\, r_2 \ldots r_p$.

Conversely, if a combination $r_1, r_2 \ldots r_p$ does not correspond to a system of necessary wires, and we repeat the same reasoning, the fraction should appear in the form of $\frac{0}{0}$ for some of the

currents, for these are still closed circuits; hence the denominator does not contain the combination in question.

Thus the common denominator of the equations, resolved in respect of the intensities, contains all the combinations of necessary wires, and them only.

Finally, all the combinations are of the same sign. For if we suppress $p-1$ necessary wires, the network only contains one closed circuit. The fraction which gives the intensity of the current in this circuit then appears in the form $\frac{0}{0}$, but after suppressing a common factor it should give

$$i=\frac{\Sigma e}{\Sigma r}.$$

All the resistances which form the residual circuit enter the denominator in terms of the same sign; hence all the terms are of the same sign.

It may be observed that all the conductors which terminate at the same summit do not at the same time form part of a system of necessary wires, for if they are all suppressed except one, it is clear that the latter wire remains open.

945. Lastly, there is a remarkable correlation between the elements of the two wires of the network. Let r_1 and r_2 be any two wires, e_1 and e_2 the electromotive forces they contain; the corresponding intensities i_1 and i_2 will be determined by equations of the form

$$i_1=\frac{A_1^1e_1+A_2^1e_2+\ldots}{\Delta}=\frac{N_1}{\Delta},$$

$$i_2=\frac{A_2^2e_2+A_1^2e_1+\ldots}{\Delta}=\frac{N_2}{\Delta}.$$

The numerator is obtained by replacing the factor $\pm r_1$ in each of the combinations which contains Δ by the second member of the corresponding equation. This numerator contains then the combinations $p-1$ and $p-1$ together of the resistances which leave a simple closed circuit of which r_1 forms part.

On the other hand the terms of N_1 which contain e_2 result themselves from equations in which r_2 enters, and therefore of simple circuits of which r_2 forms part.

The coefficient A_2^1 of e_2 in the value of i_1 contains then simply the combinations $p-1$ and $p-1$ together of conductors which leave the simple circuits of which r_1 and r_2 form part at the same time. The coefficient A_1^2, in the expression of i_2 obviously contains the same combinations.

Moreover, these combinations are respectively of the same sign, for in any residual circuit, the portion of the current i_1 which arises from e_2 is of the same sign as that portion of the current which results from e_1; hence $A_1^2 = A_2^1$.

Thus when a network of linear conductors is complete, the intensity of the current sent through a branch r_1, by the electromotive force of another branch r_2, is equal to that of the current which will be sent through the conductor r_2 by the same electromotive force placed at r_1.

In particular, if the coefficients A_1^2 and A_2^1 are zero, the current in each of the conductors r_1 or r_2 is independent of the electromotive forces which the other contains. These two conductors, as well as the corresponding sides of the network, are then said to be *conjugate.*

The electrical conditions of two conjugate conductors are independent of each other; if we change, for example, the electromotive force, or the resistance of the conductor r_1, or even if the system is suppressed, the general distribution of the currents in the network is modified, but the current in the conductor r_2 does not change, at any rate if the regime is stationary.

In the case of a variable regime, on the contrary, changes in the intensity of the current in the other branches give rise to electromotive forces of induction, the reaction of which is felt on the conjugate conductor of that which has been modified.

946. We may mention here a remarkable property which has been demonstrated by M. Thévenin.* In any system of conductors traversed by permanent currents, let us consider two points A and A′ the potentials of which are V and V′. If these two points are connected by a new conductor r, the difference of potential tends to produce a current in this conductor, but the original equilibrium is restored by introducing at the same time an electromotive force $-E$ in the contrary direction equal to $V - V_1$ in absolute value, and the current is zero in the conductor r. If ρ is the total resistance of the original system between the points A and A′, if we now introduce in the conductor r an electromotive

* THÉVENIN. *Comptes rendus*, Vol. XCVII., p. 159. 1883.

force $+E$, which annuls the preceding, a fresh distribution of currents is produced; but from the principle of the superposition of the conditions of equilibrium (202) the strength of the current which traverses the conductor r is defined by the equation

$$E = V - V' = i(\rho + r).$$

The two points A and A′ of the primitive system behave then in reference to a new conductor by which they are connected, as a single conductor of resistance ρ, equal to that which at first existed between them, and containing an electromotive force equal to the difference of potential in the primitive state.

This important relation might be utilised for determining either the resistance between two points, or the differences of their potentials,

947. Problem of Wheatstone's Bridge.—In Wheatstone's arrangement the six conductors present the same relations of

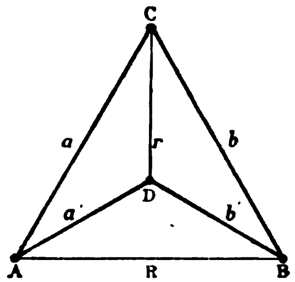


Fig. 185.

position as the six edges of a triangular pyramid (Fig. 185), for each conductor is adjacent to four others and opposed to the sixth. As the general case, it might be assumed that all the sides contain electromotive forces; we shall denote them by the same letter E_a, $E_{a'}$...., affected by an index which indicates the resistance of the side in which it is placed.

We observe, in the first place, that two opposite sides R and r are conjugate, if the four others satisfy equation (17). Further, if two couples of opposite sides R and r, a and b' are respectively conjugate, the two others are so likewise, for the equations

$$ab' = ba',$$
$$Rr = ba',$$

give also

$$Rr = ab',$$

and all the opposite sides are conjugate in pairs.

From what has been said above (943) the four summits will form three distinct equations, and the closed circuits three other equations—that is to say, that the number of necessary conductors is equal to three. We might take, for instance, the six equations,

$$(20) \quad \begin{aligned} I &= \alpha + \alpha', \\ I &= \beta + \beta', \\ \alpha &= \beta + i; \\ a\alpha + ri - a'\alpha' &= E_a + e - E_{a'}, \\ b\beta - b'\beta' - ri &= E_b - E_{b'} - e, \\ RI + a\alpha + b\beta &= E + E_a + E_b. \end{aligned}$$

It is useless to solve these equations in their greatest generality.

In the case of a permanent regime it is sufficient to consider a single electromotive force for the ordinary practical case. Moreover, from the principle of the superposition of permanent states (202), we shall get the strength of the current in any given side by simply adding the currents relative to each of the electromotive forces taken separately.

The problem is more complicated for variable electromotive forces, but we shall suppose then that two of the resistances, r and R for instance, are conjugate. This case corresponds to that which is carried out in practice in comparing coefficients of induction.

The general remarks (944) relative to the properties of a complete network, enable us to find directly the common denominator of the equations (20). The six resistances give 20 combinations 3 by 3, but as we must deduct the four combinations of three conductors terminating at the same summit, there only remain 16 terms in the denominator, and we may write it in the form

$$\Delta = Rr(a + a' + b + b') + R(a + a')(b + b') + r(a + b)(a' + b') + aa'bb'\left(\frac{1}{a} + \frac{1}{a'} + \frac{1}{b} + \frac{1}{b'}\right)$$

The numerators are directly deduced by the known rule. Putting

$$D=(a+a')(b+b')+r(a+a'+b+b'),$$

we have

$$
(21)\qquad
\begin{aligned}
I&=E\frac{D}{\Delta},\\
i&=E\frac{a'b-ab'}{\Delta},\\
\alpha&=E\frac{a'(b+b')+r(a'+b')}{\Delta},\\
\alpha'&=E\frac{a(b+b')+r(a+b)}{\Delta},\\
\beta&=E\frac{b'(a+a')+r(a'+b')}{\Delta},\\
\beta'&=E\frac{b(a+a')+r(a+b)}{\Delta}.
\end{aligned}
$$

If ρ is the resistance of the whole set of conductors, a, b, a', b', and r, comprised between the points A and B, the intensity I of the total current may be written

$$I=\frac{E}{R+\rho};$$

from this follows

$$R+\rho=\frac{\Delta}{D}=R+\frac{r(a+b)(a'+b')+ab(a'+b')+a'b'(a+b)}{D},$$

or

$$\rho=\frac{r(a+b)(a'+b')+ab(a'+b')+a'b'(a+b)}{r(a+a'+b+b')+(a+a')(b+b')}.$$

If, for shortness sake, we represent by (a, b) the resistance $\dfrac{1}{\frac{1}{a}+\frac{1}{b}}=\dfrac{ab}{a+b}$ of the two branches a and b arranged in parallel, this value of ρ may then be written

$$\rho=\frac{r+(a,b)+(a',b')}{\dfrac{r}{(a+b),(a'+b')}+\dfrac{(a,b)(a',b')}{(a,a')(b,b')}}.$$

When the diagonal r is open, or $r=\infty$, this expression becomes

$$\rho=(a+b),(a'+b')=\frac{1}{\frac{1}{a+b}+\frac{1}{a'+b'}},$$

which was evident.

The resistance ρ has still the same value when the bridge is in equilibrium, the diagonals r and R being conjugate, for the current is then null and the resistance r does not come into the equations.

948. If we indicate by the index zero the values corresponding to the equilibrium of the bridge, and putting for brevity

$$\mathrm{M}=b(a+a')+r(a+b),$$
$$\mathrm{N}=\mathrm{R}(a+a')+a'(a+b),$$

we have

$$\mathrm{D}_0=\frac{a+a'}{a}\mathrm{M},$$
$$\Delta_0=\frac{\mathrm{D}_0}{a+a'}\mathrm{N}=\frac{\mathrm{MN}}{a};$$

from this follows

$$\text{(22)}\qquad \begin{aligned} i_0&=0,\\ \mathrm{I}_0&=\mathrm{E}\frac{a+a'}{\mathrm{N}},\\ \alpha_0=\beta_0&=\mathrm{E}\frac{a'}{\mathrm{N}},\\ \alpha'_0=\beta'_0&=\mathrm{E}\frac{a}{\mathrm{N}}. \end{aligned}$$

949. Suppose now that there are electromotive forces in all the branches, but that the diagonals R and r are conjugate; we may put

$$\mathrm{P}=(a+a')\mathrm{E}+a'(\mathrm{E}_a+\mathrm{E}_b)+a(\mathrm{E}_{a'}+\mathrm{E}_{b'}),$$
$$\mathrm{Q}=ar(\mathrm{E}_a-\mathrm{E}_{a'}+\mathrm{E}_b-\mathrm{E}_{b'}).$$

Equations (20), or the sums of equations (21) relative to all the electromotive forces considered separately, give then, if we

eliminate the resistance b' by the equation of the equilibrium of the bridge,

$$
(23) \qquad
\begin{aligned}
\mathrm{I} &= \frac{\mathrm{P}}{\mathrm{N}}, \\
i &= \frac{b\,(\mathrm{E}_a - \mathrm{E}_{a'} + e) - a\,(\mathrm{E}_b - \mathrm{E}_{b'} - e)}{\mathrm{M}}, \\
\alpha &= \frac{a'}{a+a'}\frac{\mathrm{P}}{\mathrm{N}} + \frac{\mathrm{Q}}{(a+a')\mathrm{M}} + \frac{b(\mathrm{E}_a - \mathrm{E}_{a'} + e)}{\mathrm{M}}, \\
\alpha' &= \frac{a}{a+a'}\frac{\mathrm{P}}{\mathrm{N}} - \frac{\mathrm{Q}}{(a+a')\mathrm{M}} - \frac{b(\mathrm{E}_a - \mathrm{E}_{a'} + e)}{\mathrm{M}}, \\
\beta &= \frac{a'}{a+a'}\frac{\mathrm{P}}{\mathrm{N}} + \frac{\mathrm{Q}}{(a+a')\mathrm{M}} + \frac{a(\mathrm{E}_b - \mathrm{E}_{b'} + e)}{\mathrm{M}}, \\
\beta' &= \frac{a}{a+a'}\frac{\mathrm{P}}{\mathrm{N}} - \frac{\mathrm{Q}}{(a+a')\mathrm{M}} - \frac{a(\mathrm{E}_b - \mathrm{E}_{b'} - e)}{\mathrm{M}}.
\end{aligned}
$$

It will be seen that the intensities I and i are respectively independent of the electromotive forces and of the resistances of the conjugate sides, as should be the case (945).

950. When the electromotive force E is the only permanent one, and the coefficients of mutual induction of the various wires may be neglected, the other electromotive forces only depend on the effects of self-induction due to variations of the currents. If L_a, $\mathrm{L}_{a'}$..... are the respective coefficients of self-induction of the branches the resistances of which are a, a'...., we have (518)

$$
\mathrm{E}_a = -\mathrm{L}_a \frac{d\alpha}{dt}, \qquad \mathrm{E}_{a'} = -\mathrm{L}_{a'} \frac{d\alpha'}{dt} \ldots\ldots
$$

These values, substituted in the preceding equations (23), will give the intensities at each instant.

For the branch r in particular, which contains the galvanometer, we have

$$
(24) \quad i = \frac{a\left(\mathrm{L}_b \dfrac{d\beta}{dt} - \mathrm{L}_{b'} \dfrac{d\beta'}{dt} - \mathrm{L}_r \dfrac{di}{dt}\right) - b\left(\mathrm{L}_a \dfrac{d\alpha}{dt} - \mathrm{L}_{a'} \dfrac{d\alpha'}{dt} + \mathrm{L}_r \dfrac{di}{dt}\right)}{\mathrm{M}}.
$$

If the total variation takes place during a time which is very short compared with the time of oscillation of a needle, this

receives a swing proportional to the integral $\int i dt$ extended to the whole duration of the change.

Each of the terms of this integral which does not depend on i, such as $L_a \int \frac{d\alpha}{dt} dt$, is equal to $L_a(\alpha_2 - \alpha_1)$, α_1 and α_2 being the initial and final intensities. This term is zero when the intensity is the same at the two limits.

This would particularly be the case for terms relating to all branches of the network if, instead of a constant electromotive force E, we introduce in the branch R a transient electromotive force like that obtained in closing and then quickly opening the circuit of a battery, or by the displacement of an adjacent magnet, or, lastly, by the rotation of a circuit. The condition of equilibrium of the bridge being realised, the needle remains stationary in the two cases, for the intensity i is null at the two limits.

951. When a constant electromotive force E is introduced in the branch R, the diagonals being always conjugate, all the currents are at first zero at the moment the circuit is closed, and finally, from equations (22), acquire the values

$$\alpha_2 = \beta_2 = \alpha_0 = E \frac{a'}{N},$$

$$\alpha'_2 = \beta'_2 = \alpha'_0 = E \frac{a}{N}.$$

The swing of the needle is proportional to

$$\int i dt = E \frac{aa'}{MN} \left[a \left(\frac{L_b}{a} - \frac{L_{b'}}{a'} \right) - b \left(\frac{L_a}{a} - \frac{L_{a'}}{a'} \right) \right].$$

Taking into consideration that $ab' = ba'$, this expression may be written

$$\int i dt = E \frac{aba'}{MN} \left[\left(\frac{L_b}{b} - \frac{L_{b'}}{b'} \right) - \left(\frac{L_a}{a} - \frac{L_{a'}}{a'} \right) \right].$$

Breaking the circuit produces the same effect, but in the opposite direction. In both cases the needle will be stationary if we have

$$(25)\qquad \frac{L_b}{b}-\frac{L_{b'}}{b'}=\frac{L_a}{a}-\frac{L_{a'}}{a'}.$$

Induced currents are not, however, generally of such short duration that the needle has not time to move a little in the direction of the first effect. If we want to balance the bridge so that the current in the galvanometer is always zero, the numerator of the equation must be always zero—that is to say, that we have identically

$$(26)\quad a\left(L_b\frac{d\beta}{dt}-L_{b'}\frac{d\beta'}{dt}\right)-b\left(L_a\frac{d\alpha}{dt}-L_{a'}\frac{d\alpha'}{dt}\right)-(a+b)L_r\frac{di}{dt}=0.$$

From this follows first $\frac{di}{dt}=0$, which gives, for any given period,

$$\alpha=\beta \qquad \text{and} \qquad \alpha'=\beta',$$

or

$$\frac{d\alpha}{dt}=\frac{d\beta}{dt} \qquad \text{and} \qquad \frac{d\alpha'}{dt}=\frac{d\beta'}{dt}.$$

Equation (26) then becomes

$$ab\left[\left(\frac{L_b}{b}-\frac{L_a}{a}\right)\frac{d\alpha}{dt}-\left(\frac{L_{b'}}{b}-\frac{L_{a'}}{a}\right)\frac{d\alpha'}{dt}\right]=0.$$

As the currents α and α' are independent, it follows, from the ratio $ab'=ba'$, that

$$(27)\qquad \frac{L_a}{a}=\frac{L_b}{b} \qquad \text{and} \qquad \frac{L_{a'}}{a'}=\frac{L_{b'}}{b'}.$$

Besides the ordinary condition of conjugate diagonals, the complete equilibrium of the bridge requires then, both for variable and for constant currents, that the coefficients of self-induction of the four branches of the bridge are respectively proportional to their resistances.

952. When the diagonals r and R are not conjugate, the induced currents obey more complex laws. The swing of the needle, when the branch R containing the battery is closed, may be of the opposite sign to the permanent deflection. In order to get over this difficulty, which makes the observations much longer, care is taken to close first the battery and then the galvanometer.* The needle remains then at zero, whatever are the coefficients of induction, if the equilibrium of the bridge for the permanent regime is established, and it is always displaced in the direction of the ultimate deflection.

This result is obtained by working two independent keys, one on the branch which contains the battery and the other on that of the galvanometer. This double operation may be performed

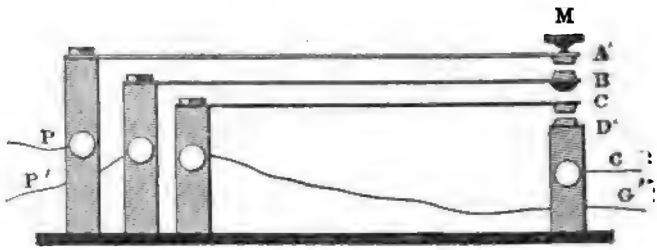


Fig. 186.

automatically by means of a special key with two successive contacts (Fig. 186) formed of three superposed and parallel elastic plates. When the knob M is depressed, the contact of A and B closes the battery circuit; a moment after, the contact of C and D closes the galvanometer circuit.

953. Conditions of Sensitiveness of Wheatstone's Bridge. —The value of i given by equation (18)

$$i = E\frac{a'b - ab'}{\Delta}$$

shows that the sensitiveness of the method, other things being equal, is inversely as Δ (947).

It may first of all be asked whether it is immaterial in which of the diagonals the battery and the galvanometer are placed.

* Sir W. Thomson. *Phil. Mag.* [4], Vol. xxiv., p. 149. 1862.

If the two diagonals are interchanged, as the battery and the galvanometer each brings its own resistance, the denominator Δ acquires a new value Δ', such that

$$(28) \qquad \Delta' - \Delta = (R - r)\left[(a+b)(a'+b') - (a+a')(b+b')\right]$$
$$= (R - r)(a - b')(a' - b).$$

Suppose that the product $(a - b')(a' - b)$ is positive, so that we have at once

$$a > b' \qquad \text{and} \qquad a' > b,$$

or

$$a < b' \qquad \text{and} \qquad a' < b.$$

As the ratio $ab' = a'b$ may be considered as almost satisfied, these two conditions amount to supposing that the four resistances a, a', b, b' are arranged in order of increasing or decreasing magnitudes—that is to say, that the two summits A and B (Fig. 184) are the points of junction, the one of the two largest resistances, and the other of the two smallest.

In this case, if $R > r$, $\Delta' > \Delta$, and the first arrangement is better than the second; the contrary would be the case for $R < r$. Hence this rule:

The maximum sensitiveness is obtained when in Wheatstone's bridge the diagonal which connects the summits where the two greatest resistances and the two least separately terminate, is formed of that one of the branches, the battery or the galvanometer, which itself has the greatest resistance.

In practice, the galvanometer has generally a higher resistance than the battery.

954. Let b be the resistance to be measured, b' the standard resistance which is compared with it, and suppose we adjust so that a balance is obtained, a and a' being arbitrary resistances. If $a = a'$, $b' = b$; but we may make any ratio we like between a and a', so that the condition of equilibrium corresponds to the same ratio between the branches b and b'.

If we follow out the analogy of resistances and of weights, we may call Wheatstone's arrangement a *balance*, while the resistances a and a' are the two arms of the *beam*. It must, however, be observed that the conditions of weighing are here reversed; the resistances to be compared b and b' are here proportional to their respective arms when equilibrium is attained.

955. When near equilibrium,* the current i, which corresponds to an error ϵ in the value of b', is expressed by

$$i = \mathrm{E}\frac{a}{\Delta}\epsilon.$$

We may replace Δ by the value Δ_0, corresponding to the equilibrium of the bridge (**928**), which gives

$$i = \mathrm{E}\frac{a}{\Delta_0}\epsilon = \mathrm{E}\frac{\epsilon}{\mathrm{K}}.$$

Eliminating the resistance a' from the expression of Δ_0 by the condition of equilibrium, we have

$$\mathrm{K} = \frac{\Delta_0}{a} = \left[\mathrm{R}\left(1+\frac{b'}{b}\right) + b'\left(1+\frac{a}{b}\right)\right]\left[r\left(1+\frac{b}{a}\right) + b\left(1+\frac{b}{b}\right)\right];$$

if we put

$$\mathrm{S} = \mathrm{R}b + bb' + \mathrm{R}b\ ,$$
$$\mathrm{T} = r + b + b',$$

the value of K may be written

$$\mathrm{K} = \frac{\mathrm{ST}}{b} + b'r + \mathrm{T}\frac{b'}{b}a + \frac{\mathrm{S}r}{a}. \tag{29}$$

The minimum of absolute error ϵ is that value which corresponds to the feeblest current i, which the galvanometer will measure. The absolute error is then

$$\epsilon = \frac{i}{\mathrm{E}}\mathrm{K},$$

and the relative error

$$\frac{\epsilon}{b'} = \frac{i}{\mathrm{E}}\frac{\mathrm{K}}{b'} = \frac{i}{\mathrm{E}}\mathrm{H},$$

representing by H the function $\frac{\mathrm{K}}{b'}$.

956. It is interesting to investigate the conditions of adjustment which correspond to the maximum sensitiveness.

* See SCHWENDLER, *Phil. Mag.* [4], Vol. XXXI., 1866, and Vol. XXXIII., 1867; HEAVISIDE, *Phil. Mag.* [4], Vol. XLV., p. 114, 1873; GRAY, *Phil. Mag.* [5], Vol. XII., p. 283, 1881.

Of six quantities a, a', b, b', r, R, the resistance b to be measured is the only one given *a priori;* R and r are completely arbitrary; the three other resistances a, a', and b' are connected to b by the condition of equilibrium.

Suppose that b', r, and R are given with b; the resistance a is the only unknown quantity, the ratio of a' to a being known. The value of a, which corresponds to the minimum of the values of K and H is defined by the condition

$$\frac{dK}{da}=0;$$

equation (29) and the condition of equilibrium of the bridge give directly

$$(30)\qquad a^2=\frac{b}{b'}\frac{rS}{T}, \qquad a'^2=\frac{b'}{b}\frac{rS}{T}.$$

The least values of K and of H are then

$$(31)\qquad K_1=\frac{ST}{b}+b'r+2\sqrt{b'r\frac{ST}{b}}, \qquad H_1=\frac{1}{b}\left[\frac{ST}{b'}+br+2\sqrt{br\frac{ST}{b'}}\right].$$

957. If b, R, and r are alone given, which in practice is the usual case, it remains to choose a, and the ratio of the two arms of the beam. Whatever b' is, it is clear that the best values of a and of a' should first satisfy the preceding conditions, which give the minimum K_1 and H_1. Hence we may regard b' as the only independent quantity, and investigate that value which makes the quantities K_1 and H_1 a minimum.

The expression of K_1 shows that the absolute error is smaller the smaller is b', but in practice this fact is only of secondary importance.

Putting

$$\frac{ST}{b'}=y,$$

we may write

$$H_1=\frac{1}{b}\left(y+br+2\sqrt{bry}\right),$$

and if y' is the differential of y in respect of b', the condition of the minimum is

$$y'\left(1+\sqrt{\frac{br}{y}}\right)=0.$$

As the factor within the bracket cannot be zero, it follows that $y'=0$, or

$$(32)\qquad b'^2=\frac{Rb}{R+b}(r+b).$$

Formulæ (30) give then

$$(33)\qquad \begin{aligned} a'^2&=Rr,\\ a^2&=rb\frac{R+b}{r+b},\\ \frac{a^2}{a'^2}&=\frac{b}{R}\frac{R+b}{r+b}=\frac{1+\frac{b}{R}}{1+\frac{r}{b}}.\end{aligned}$$

958. It will be seen that if R and r are very large compared with b, we have

$$\begin{aligned} b'^2&=br,\\ a'^2&=Rr,\\ a^2&=bR,\end{aligned}$$

and the ratio $\frac{a}{a'}$ of the arms of the bridge is very small.

This ratio is also very small if the resistance r alone is very great, which gives appreciably

$$\begin{aligned} b'^2&=Rb\frac{r}{R+b},\\ a'^2&=Rr,\\ a^2&=b(R+b).\end{aligned}$$

As the latter equation may be put in the form

$$\frac{a^2}{b^2}=\frac{R+b}{b},$$

it shows that $a>b$.

If we are desirous of realising the condition of maximum sensitiveness with equal arms a and a', equations (33) will give

$$b^2=Rr=a^2,$$

and therefore

$$a=a'=b=b'=\sqrt{Rr}.$$

In this case the four sides of the bridge should have equal resistances.

959. The same kind of discussion no longer applies to the resistances of the galvanometer and the battery, for the galvanometer constant and the electromotive force must be brought in. If the galvanometer and the battery are given, it is clear that any auxiliary resistance placed in either of the diagonals diminishes the sensitiveness.

Let us assume the battery given, as well as the resistances of the four sides, and the channel of the galvanometer coil; the conditions of maximum sensitiveness are realised (731) when the resistance of the coil is equal to the resistance ρ of the total network between the two extremities C and D (Fig. 184) of the diagonal in which is the galvanometer. For if V and V′ are the potentials of these two points when the diagonal r is open, the current i is the same as if the points C and D were the ends of a single conductor of resistance ρ, containing an electromotive force equal to $V-V'$.

On the other hand, given a certain number of identical elements, we get the maximum current when they are arranged so that the resistance of the battery is equal to the resistance of the circuit.*

In the present case, when the balance has been established, as the two diagonals are conjugate, the external resistance to be considered as that of the battery (947) is that of the two branches

* If n is the number of cells, e the electromotive force, and r the resistance of each; if they are coupled up in series of p cells, and if the $\frac{n}{p}$ series thus obtained are joined in parallel order, the strength I of the current in an external resistance ρ, which joins the two poles, is

$$I=\frac{pe}{\frac{p}{n}pr+\rho}=\frac{e}{\frac{r}{n}p+\frac{\rho}{p}}.$$

The value of p, which makes this expression a maximum, is given by the value

$$\frac{r}{n}p=\frac{\rho}{p}, \quad \text{or} \quad \frac{p^2}{n}r=\rho;$$

that is to say, when the resistance $\frac{p^2}{n}r$ of the battery is equal to the external resistance.

ACB and ADB arranged in parallel, and for the galvanometer that of the two branches CAD and CDB.

The conditions of maximum sensitiveness relative to the battery and the galvanometer are then

$$\frac{1}{R}=\frac{1}{a+b}+\frac{1}{a'+b'},$$

$$\frac{1}{r}=\frac{1}{a+a'}+\frac{1}{b+b'},$$

or, allowing for the condition of equilibrium,

$$(34)\qquad \begin{aligned} R&=b'\frac{a+b}{b+b'},\\ r&=a\frac{b+b'}{a+b}; \end{aligned}$$

it follows that

$$(35)\qquad Rr=ab'=ba'.$$

All the branches are then conjugate in pairs.

If we wish to satisfy at the same time all conditions of the maximum, both for the battery and galvanometer, as well as for the branches of the bridge, equations (32), (33), and (35) give

$$Rr=ab'=ba'=a'^2, \qquad \text{or} \qquad b=a'=\sqrt{Rr},$$

$$b'^2=b\frac{Rr+Rb}{R+b}=b^2;$$

from this follows

$$a=a'=b=b'=\sqrt{Rr},$$

that is to say, that the four branches of the bridge are equal.

Equations (34) give also

$$R=r=a=a'=b=b';$$

the six resistances are then equal to each other.

960. Use of the Electrometer or Electrodynamometer.—The electrometer might be substituted for the galvanometer in using Wheatstone's bridge. The needle should remain at zero

when the balance is made. This arrangement is of especial value when the resistances are considerable, for the intensity of the current in a galvanometer will then be very feeble.

The use of the electrodynamometer in ordinary conditions will be evidently disadvantageous. The indications of the instrument, being proportional to the square of the current strength, could only be utilised for very weak currents. Moreover, as the deflection is always in the same direction, there is no guide in regulating the resistances.

These drawbacks* are overcome by placing the fixed coil in the battery circuit, and the movable one in the bridge. The deflection is then, like that of the galvanometer, proportional to the current i, and changes its sign with it.

In order to get the best conditions of sensitiveness, we must examine the expression

$$GS'Ii,$$

in which S′ is the surface of the movable coil; as we have (**947**)

$$I = E\frac{D}{\Delta} \quad \text{and} \quad i = I\frac{a'b - ab'}{D},$$

it follows that

$$Ii = I^2\frac{a'b - ab'}{D} = E^2\frac{D}{\Delta^2}(a'b - ab').$$

If we suppose equilibrium almost established, we may replace the expressions D and Δ by their approximate values (**948**), which gives

$$Ii = E^2\frac{a(a+a')(a'b - ab')}{\left[b(a+a') + r(a+b)\right]\left[R(a+a') + a'(a+b)\right]^2}.$$

We might also investigate how the wire of the electrodynamometer should be arranged so as to make the expression GS′Ii a maximum, but this discussion would have no interest, as the sensitiveness of the electrodynamometer is much less than that of the galvanometer.

961. RESISTANCE OF A GALVANOMETER.—The condition of equilibrium of the bridge enables us to determine directly the resistance of a galvanometer, when we cannot dispose of a second galvanometer which enables us to use the first as a mere

* F. KOHLRAUSCH. *Pogg. Ann.*, Vol. CXLII., p. 427. 1871.

resistance.* The galvanometer is put in a lateral branch b of the parallelogram, while the bridge which usually contains the galvanometer is closed by a wire with a key K (Fig. 187). The needle of the galvanometer is deflected by the passage of the current β; the resistance b' is so adjusted that the deflection is the same whether the bridge is opened or closed. If the resistances of the four limbs satisfy the relation of ordinary equilibrium, the two ends of the bridge are at the same potential, there is no current in the wire joining them, and the suppression or introduction of

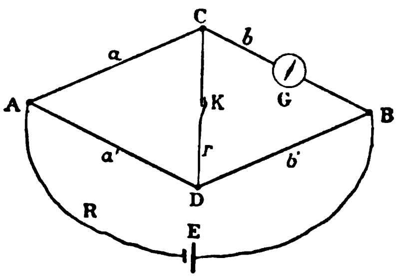


Fig. 187.

this wire does not at all modify the currents in the lateral branches.

We see, in fact, in equations (22), that the currents in the lateral branches are independent of the resistance r of the bridge.

In order to estimate the sensitiveness of this method, we may suppose the bridge formed by a very short wire, and by raising or lowering the key the resistance r is made to vary from zero to infinity. The balance being nearly established, we may replace the denominator Δ by the approximate value

$$\Delta_0 = \frac{\mathrm{MN}}{a} = \frac{\mathrm{N}}{a}\left[b(a+a') + r(+b)\right].$$

Equations (21) give then, for $r=0$,

$$\beta_1 = \mathrm{E}\frac{a}{\mathrm{N}}\frac{b'}{b},$$

* Sir W. THOMSON. *Proceedings of the Roy. Soc.*, Vol. XIX., p. 253. Jan., 1871.

and, for $r=\infty$,

$$\beta_2 = E\frac{a}{N}\frac{a'+b'}{a+b};$$

from this we get

$$\beta_1 - \beta_2 = E\frac{a}{N}\frac{b'a-ba'}{b(a+b)} = E\frac{a}{N}\frac{a\epsilon}{b(a+b)},$$

ϵ being the error of adjustment in the branch b'.

Supposing $b=b'$ and $a'=a=mb$, we get

$$\beta_1 - \beta_2 = \frac{\epsilon}{b'}\frac{E}{\left(1+\frac{1}{m}\right)\left[2R+b(1+m)\right]}.$$

The denominator is a minimum for $m^2=\frac{2R+b}{b}$, and the difference is greater for a given relative error as to b', the feebler is the resistance R of the battery.

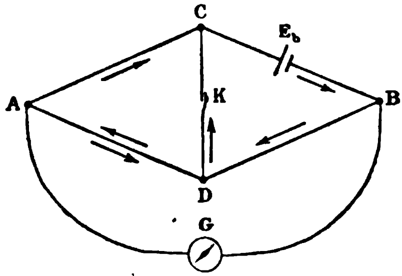


Fig. 188.

962. Resistance of a Battery. Mance's Method.—Let us suppose the resistance to be measured is at the same time the seat of an electromotive force (in the case of a voltaic couple, for instance), we may again* place this resistance in a lateral branch b (Fig. 188), the galvanometer on the diagonal R in the place of the battery, and finally substitute a break K for the galvanometer in the bridge r.

The resistance b' is adjusted so that the deflection is constant, when the key is opened or closed; the relations $a'b=ab'$ is then satisfied. For it is only in this case that, the two branches r and

* Mance. *Proceedings of the Roy. Soc.*, Vol. xix., p. 248. 1871.

R being conjugate, the changes of resistance of the former could have no influence on the current which traverses the second.

If we wish to discuss the experiment more completely, we need only retain the electromotive force E_b in the general formulas; we have then

$$P = a'E_b, \qquad Q = arE_b,$$

and we get

$$
(36) \quad
\begin{aligned}
i &= -E_b \frac{a}{M}, \\
I &= E_b \frac{a'}{N}, \\
\alpha &= \frac{E_b}{a+a'}\left[\frac{a'^2}{N} + \frac{ar}{M}\right], \\
\alpha' &= E_b \frac{a(a'b - Rr)}{MN}, \\
\beta &= \frac{E_b}{a+a'}\left[\frac{a'^2}{N} + \frac{a(r+a+a')}{M}\right], \\
\beta' &= E_b \frac{a\left[(a+a'+r)R + aa'\right]}{MN}.
\end{aligned}
$$

According to the conventions as to the signs made above (**937**), the currents in the different branches have the directions indicated by the arrows in Fig. 188; the only ambiguity is as to the current of the branch AD, the direction of which corresponds to the upper arrow or to the lower one, according as $a'b \gtrless Rr$, and it may happen that this branch is not traversed by any current.

It will be seen that α' changes its direction according as $r=0$ or $r=\infty$—that is to say, when the key is opened or closed. The current being modified in all the branches, except that of the galvanometer, by the breaks in the bridge, the needle can only be stationary provided the effects of induction are negligable or are counteracted; we should then, in general, wait until it comes to rest after each operation.

The changes in the battery current may have the effect of modifying the electromotive force E_b, which we have supposed to be constant. On the other hand there is no object in endeavouring to attenuate these variations, for on them the sensitiveness of the method depends.

The variation of resistance necessary to produce an appreciable change in the intensity of the current is determined experimentally.

Another defect of the method in the case of the resistance of an element, is that delicate galvanometers cannot be used unless they are shunted on the diagonal R.

963. Professor Lodge* gets rid of part of the inconveniences of Mance's method by inserting a condenser Q in the galvanometer circuit (Fig. 189). This arrangement amounts to making $R=\infty$ and therefore $I=0$.

As soon as the condition of equilibrium of the bridge is satisfied, the two diagonals are conjugate; the changes in resistance in one of them CD, are without effect on the difference of potential at the ends A and B of the second, and therefore on the charge of the condenser. The needle of the galvanometer

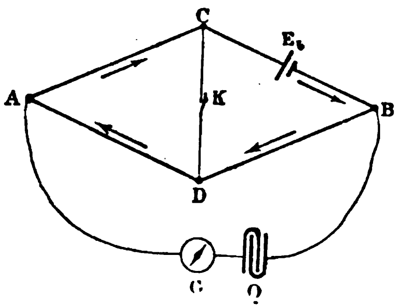


Fig. 189.

remains therefore at zero when the key is worked; but it is necessary, this time, that the effects of induction be entirely eliminated, for no permanent deflection is observed.

The general equations give then

$$i = -E_b \frac{a+a'}{D},$$

$$a = E_b \frac{r}{D} = -a',$$

$$\beta = E_b \frac{a+a'+r}{D} = -\beta'.$$

* Lodge. *Phil. Mag.* [5], Vol. III., p. 515. 1877.

The difference of potential between the summits A and B—that is, between the armatures of the condenser—is

$$\delta V = a'\alpha' + b'\beta' = -E_b \frac{r(a'+b')+b'(a+a')}{(a+a')(b+b')+r(a+a'+b+b')};$$

for $r=0$ it becomes

$$\delta V_1 = -E_b \frac{b'}{b+b'},$$

and for $r=\infty$

$$\delta V_2 = -E_b \frac{a'+b'}{a+a'+b+b'}.$$

The change of potential is then

$$\frac{\delta V_2 - \delta V_1}{E_b} = \frac{ab' - ba'}{(b+b')(a+a'+b+b')} = \frac{a\epsilon}{(b+b')(a+a'+b+b')}.$$

If we assume $b=b'$ and $a=a'=mb$, we get

$$\frac{\delta V_2 - \delta V_1}{E_b} = \frac{m}{4(1+m)}\frac{\epsilon}{b} = \frac{1}{4\left(1+\frac{1}{m}\right)}\frac{\epsilon}{b}.$$

In the actual case the sensitiveness is greater, the greater is the ratio m, and the relative variation of potential to be observed tends to become the quarter of the error in respect of the resistance.

964. Coil Bridge.—The two forms of the Wheatsone electrical balance now most used are the *coil bridge* and the *wire bridge*.

The resistance boxes are ordinarily arranged so as to give at the same time the elements of a Wheatstone's Bridge. Fig. 181 represents Elliott's pattern. MN is the resistance box described above; a series of coils C A D form the two arms a and a' of the balance. The resistance b to be measured is connected to the binding screws D and B by the wires R and R′, and the series of dials represents the standard resistances; the binding screws form thus the four summits of a parallelogram. The battery is interposed between the two summits A and B by the wires P and P′, and the galvanometer between the summits C and D by the wires G and G′. The two circuits, those of the

galvanometer and of the battery, may be opened or closed at pleasure, either by two distinct keys, or by a double contact key. By means of a special plug K the circuit of the dials may be opened, and thus a resistance be introduced which is practically infinite.

The galvanometer is usually one of high resistance with an astatic needle. It is provided with a shunt, and the sensitiveness may be modified by an auxiliary magnet.

The beam CAD is a strong bar of copper of about 2 centimetres in the side, divided in several pieces, which may be connected by plugs, and between which the coils are placed symmetrically in pairs. In order that the two coils of the same order shall be at the same temperature, the corresponding wires are coiled simultaneously on the same core. The resistances of these coils are respectively equal to 10, 100, 1000, and 10,000 ohms. By removing a plug on each side, equal resistances are brought into the two arms if the plugs are symmetrical, or, if not, resistances are introduced in the ratio of one of the values 10, 100, 1000. If all the plugs are left, the special resistance of the beam would be so slight that the galvanometer would give no indication. If only one plug is removed, the resistance of one branch would be virtually zero compared with the other.

The box gives directly resistances in whole numbers from 1 to 10,000 ohms; but by the play of the beam by which the two arms a and a' may be put in the ratio 1, 10, 100, and 1000, or their reciprocals, we may measure all resistances from 0·001 to 10,000,000.

The balance is only arranged to about a unit with a number of four figures given by the four dials. But we may push the approximation further. We may observe the displacements δ and δ' on either side (permanent deflections or swings) which the needle experiences for two successive numbers n and $n+1$ of the box between which is comprised the value corresponding to the resistance sought. The displacements of the needle being proportional to the error of adjustment, we may take for x the closely approximate value

$$x = n + \frac{\delta}{\delta + \delta'}.$$

In the case of mean resistances, we begin by getting a balance to within a unit, with equal arms and shunting the galvanometer to $\frac{1}{1000}$; then the ratio of the arms of the balance is multiplied

10, 100.... -fold by passing a gradually increasing fraction of the current until all the dials are utilised; finally, when equilibrium is almost obtained, the shunt is suppressed, and the deflections δ and δ' on either side of zero are obtained to get the final correction.

It is important only to allow the current to pass during the time strictly necessary for observing the galvanometer, so as to avoid heating the wires.

In order to get rid of any want of exactitude in the ratio of the arms, we may work by substitution. The resistances b and b_1, successively introduced between the binding screws C and B, being balanced by the values n and n_1 of the box, their ratio is

$$\frac{b}{b_1}=\frac{n}{n_1}.$$

965. We should be certain, especially with very delicate instruments, that the circuit of the galvanometer does not contain any incidental electromotive force e (the result, for instance, of a thermoelectric effect), which would considerably disturb the measurements.*

If there were only the electromotive forces E and e, the general equations (14) give

$$i=\frac{e(a+a'+b+b')+\mathrm{I}(a'b-ab')}{\mathrm{D}}.$$

If the condition of equilibrium is almost realised, we may replace D by D_0 (948), and the intensity by its approximate value $\frac{a+a'}{\mathrm{N}}\mathrm{E}$; we have then

$$i=\frac{a}{\mathrm{M}}\left[e\left(1+\frac{b+b'}{a+a'}\right)+\mathrm{E}\frac{a'b-ab'}{\mathrm{N}}\right],$$

or sensibly

$$i=\frac{a}{\mathrm{M}}\left[e\left(1+\frac{b}{a}\right)+\mathrm{E}\frac{ab}{\mathrm{N}}\left(\frac{a'}{a}-\frac{b'}{b}\right)\right].$$

* GLAZEBROOK. *Phil. Mag.* [5], Vol. XI., p. 291. 1881.

There is no current in the galvanometer when the quantity within brackets is null. We deduce

$$\frac{b'}{b}=\frac{a'}{a}+\frac{e}{E}\frac{N}{a}\left(\frac{1}{b}+\frac{1}{a}\right);$$

or

$$\frac{b'}{b}=\frac{a'}{a}+\frac{e}{E}R\left(\frac{1}{a}+\frac{1}{b}\right)\left(1+\frac{a'}{a}\right)\left(1+\frac{a'}{R}\frac{a+b}{a+a'}\right).$$

It will be seen that the influence of the electromotive force e may be considerable when the resistance R of the branch containing the battery is large compared with a and b. Moreover, the term of correction changes its sign with E. If we reverse the battery current, and observe the values b' and b'_1, which establish the balance, then, if δ is the term of correction,

$$\frac{b'}{b}=\frac{a'}{a}+\delta, \qquad \frac{b'_1}{b}=\frac{a'}{a}-\delta,$$

or

$$\frac{\frac{b'+b'_1}{2}}{b}=\frac{a'}{a}.$$

We shall take for the desired resistance, the mean of the values corresponding to the two directions of the battery which give zero current.

If the balance is unchanged on reversing the current of the battery, we conclude that the electromotive force e is null. This would also be shown by closing the circuit of the galvanometer.

When the condition $a'b=ab'$ is exactly satisfied, the electromotive force e of the galvanometer not being zero, the deflection of the needle is independent of the battery. It does not change when the current is reversed, or when the battery circuit is alternately opened and closed.

Hence, if we wish to get rid of all corrections, we should first close the galvanometer circuit, observe the deflection of the needle, and then adjust the resistance until the deflection does not change when the battery is closed. This method, however, is not rigorous unless the effects of induction are insensible.

966. Wire Bridge.—This modification of Wheatstone's bridge is due to Kirchhoff.* It is especially suitable for measuring small resistances and for comparing standards.

In the ordinary parallelogram (Fig. 190), one of the summits C is replaced by a straight wire A'B', along which a movable contact may be displaced. Instead of varying one of the resistances, the balance is made by a suitable displacement of the point C along the wire.

If l be the length A'B' of the wire supposed to be homogeneous and regular, x the distance A'C, then expressing the resistances

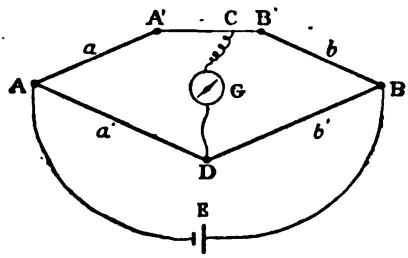


Fig. 190.

a and b of the sides AA' and BB' in units of length of the wire, the condition of equilibrium will be

$$\frac{a'}{b'} = \frac{a+x}{b+l-x}.$$

The wire is usually about a metre in length and 1·5 mm. to 2 mm. in diameter. It is of brass or of argentan, or better of platinum-iridium (85 platinum and 15 iridium). This alloy has the advantage of not being oxidizable and of not amalgamating.

The wire forms one side of an elongated rectangle (Fig. 191), and the three other sides are formed of broad copper strips, the resistance of which may be neglected. These strips have breaks at P, Q, P' and Q', which may be closed, either by thick copper strips or by resistances a, b, a', b'. The two points A and B are connected by the battery, the galvanometer wire is attached at D and to the movable contact C. By means of a divided rule the position of this contact may be determined.

* Kirchhoff. *Pogg. Ann.*, Vol. c., p. 177. 1857.

Contact at C is effected by means of a kind of rounded knife-edge, usually of platinum, which is applied perpendicularly on the wire. This mode of contact would be very defective in the case of a rheostate, but does very well here; for its resistance does not come into play, and it is only necessary to determine exactly the position of the point touched.

Great care must be taken to avoid any alteration or deformation of the wire. With this view, great attention must be paid to the mechanical arrangements for making contact. For the same reason the battery must not be placed between the points D and C, in order not to alter the surface of the wire by sparks on breaking contact.

Fig. 192 represents a bridge of a very careful construction, made by M. Carpenter, for reproducing the ohm. The scale is of brass, and is divided in millimetres. It serves to put the movable contact in connection with the galvanometer, which dispenses with the use

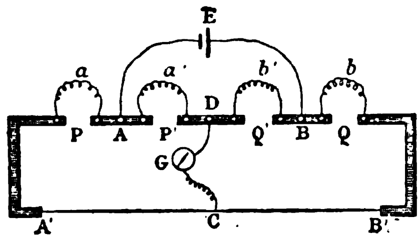


Fig. 191.

of a flexible wire. The contact is made by a steel knife-edge, which is raised in its ordinary position. By pressing an ebonite key M, the knife itself is not acted on, but it is left to the action of a small spring which produces on the wire a slight pressure, always equal and independent of the operator. This knife-edge is supported by a roller which moves along the scale, and can be fixed by a clamping screw and adjusted by a two-way screw. A vernier on the roller gives fractions of division of the scale. The wires P and P′ go to the battery, and the wire G and G′ to the galvanometer.

The resistances *a* and *b*, which are represented on the figure by a copy A of the ohm, and a coil B contained in a copper cylinder, are introduced into the circuit by means of copper mercury cups fixed on lateral bands. They can thus be very easily replaced for each other. The resistances *a′* and *b′* are also enclosed in the same cylinder C. The contact is effected in the

mercury cups, two of which are of ebonite, and by a commutator D. Their function may be interchanged in the equilibrium of the bridge without displacing them.

967. With the ordinary arrangement of wire bridges, the resistance of the copper bands which serve as junctions may be neglected, and the ratios $\frac{a}{a'}$ and $\frac{b}{b'}$ of the resistances to be compared should be so close that the difference might be compensated by two portions of the wire. As these resistances are very small—about one ohm, for instance—a galvanometer with a high resistance would, so to speak, be short-circuited. In this

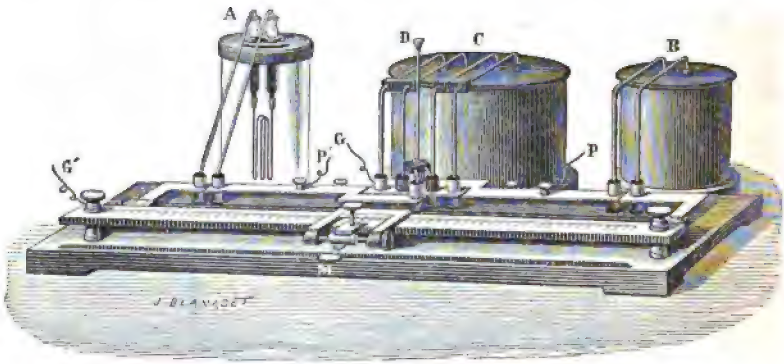


Fig. 192.

case a galvanometer of small resistance, satisfying the conditions of (959), should be used.

If the breaks P and Q (Fig. 191) are themselves formed of resistances which can be neglected, the equation of equilibrium reduces to

$$\frac{a'}{b'}=\frac{l-x}{x}.$$

As the second member of this equation may have all positive values, any given resistances might be compared by this method; but it is easy to see that the sensitiveness diminishes as the point of equilibrium comes near one or other end of the wire. If we call Δx the displacement of the contact corresponding to a variation $\Delta b'$ of b', we have

$$\frac{\Delta b'}{b'}=\frac{l}{x(l-x)}\Delta x,$$

which shows that the absolute or relative sensitiveness is greatest at the centre of the wire and is null at the extremities.

Thermoelectrical currents are greatly to be feared in wire bridges, and we should carefully avoid touching with the fingers any of the metal pieces of the apparatus.

968. The wire bridge may also be employed in two different ways, either by substitution or by the comparison of ratios.

The method of substitution is the most exact, and it requires no knowledge of auxiliary resistances. The two resistances to be compared b and b_1 are placed successively at Q, and the positions x and x_1 of the roller which correspond to equilibrium are determined. We have then

$$\frac{a'}{b'}=\frac{a+x}{b+l-x}=\frac{a+x_1}{b_1+l-x_1}=\frac{x-x_1}{b-b_1-(x-x_1)}.$$

This equation gives the difference $b-b_1$ as a function of the difference $x-x_1$ of the readings, and of the ratio $\frac{a'}{b'}$.

If we wish to eliminate this ratio, it is sufficient to interchange the resistances a' and b'. The two new readings y and y' give then

$$\frac{x-x_1}{b-b_1-(x-x_1)}=\frac{b-b_1-(y-y_1)}{y-y_1},$$

or

$$b-b_1=(x-x_1)+(y-y_1).$$

If the resistance a be chosen, so that the positions x and y are near the ends of the scale, we might measure a difference twice the resistance of the wire.

When the difference $b-b_1$ is greater than the length of the wire, it may be determined by a series of experiments with intermediate resistances $r_1, r_2, \ldots r_n$, such that the successive differences of two of them are directly measurable, as well as those of the first with b and of the last with b_1, and we shall have

$$b-b_1=(b-r_1)+(r_1-r_2)+\ldots+(r_n-b_1).$$

969. In order to compare the values it is necessary to know the values of a and b as a function of the unit of length of the wire, of which they form in some sort the prolongation. Let us count

these resistances respectively from the points A and B to the extreme positions of the contacts at the corresponding ends of the wire. Resistances a' and b' are then introduced at P′ and Q′, the ratio p of which is known, and the position of equilibrium of contact is observed, and then the position x' after inverting the resistances a' and b'. We have

$$\frac{a'}{b'}=p=\frac{a+x}{b+l-x}=\frac{b+l-x'}{a+x'};$$

from this follows

$$a=\frac{px'-x}{1-p}, \qquad b=\frac{p(l-x)-(l-x')}{1-p}.$$

If the ratio p is equal to unity, it follows that $x=x'$, and experiment gives simply the difference

$$a-b=l-2x.$$

By closing the breaks P and Q by strips of copper without resistance, we thus determine the resistances α and β of the ends AA′ and BB′, measured to the position of contact of the extreme divisions.

If the resistance interposed b is less than twice that of the wire, we may work by substitution, inserting and suppressing the resistance b.

970. The resistances a and b, including their complements α and β, being known, the resistances to be compared a' and b' will be placed at P′ and Q′. It is still advantageous to interchange these resistances, which gives two readings x and x', and we have

$$\frac{a'}{b'}=\frac{a+x}{b+l-x}=\frac{b+l-x'}{a+x'}=\frac{a+b+l+(x-x')}{a+b+l-(x-x')}.$$

If the ratio is near unity, we only utilise a very small portion of the wire $(x-x')$. We may then write

$$\frac{a'}{b'}=1+2\frac{x-x'}{a+b+l}, \qquad \text{or} \qquad \frac{a'-b'}{b'}=2\frac{x-x'}{a+b+l}.$$

The difference $a'-b'$ is proportional to the distance of the two positions x and x'; moreover, for the same value of this difference,

the distance $x - x'$ is greater as the auxiliary resistances themselves are greater.

This observation is utilised in reproducing standards of resistance. A series of resistances a and b, a_1 and b_1, a_2 and b_2 are prepared, each pair being equal, so as to keep the readings in the centre of the wire, and the values of which increase in about tenfold order. The resistance b' being the standard, the copy a' is adjusted with the resistances a and b, almost to a single division of the scale. a and b are replaced by a_1 and b_1, and a_1 is again adjusted to within a division, and so forth as long as the sensitiveness of the galvanometer allows. The error made is given by the preceding formula. If the adjustment has been made to to within a division with coils of the order a_n, the error of the copy is $\pm \frac{1}{a_n + \frac{l}{2}}$.

This method of comparison, applied to resistances which are not very small, enables us easily to attain an approximation less than 0·00001, which for a copper wire corresponds to a change in temperature of about $\frac{1}{400}$ of a degree centigrade.

971. When the ratio of the resistances to be compared differs too much from unity, a series of intermediate resistances $r_1, r_2, \ldots r_n$ is used, such that their successive ratios and those of the extremes with a' and b' are directly measurable. We have, in that case,

$$\frac{a'}{b'} = \frac{a'}{r_1} \cdot \frac{r_1}{r_2} \cdots \frac{r_n}{b'}.$$

For very unequal resistances, it would sometimes be advisable to choose the auxiliary resistances so that, when arranged in series, they are almost equal to a', and that, as parallel wires, their conductivity is near that of b'. If p and q are ratios thus determined experimentally, the ratio of the resistances a' and b' will be deduced from equations

$$a' = (r_1 + r_2 + \ldots + r_n)p,$$
$$\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n} = \frac{1}{b'q}.$$

If the different resistances $r_1, r_2, \ldots r_n$ had the same value r, these equations would reduce to

$$a' = nrp,$$
$$nb'q = r,$$

and would give

$$\frac{a'}{b'} = n^2 pq.$$

If the resistances $r_1, r_2, \ldots r_n$, without being equal, are merely near each other, then, if r is their mean value, and β_1, β_2, and β_n very small quantities the sum of which is null,

$$r_1 + r_2 + \ldots + r_n = r(n + \beta_1 + \beta_2 + \ldots + \beta_n) = rn;$$

neglecting quantities of the order of β^2, it follows that

$$\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n} = \frac{n}{r}\left(1 - \frac{\Sigma\beta}{n}\right) = \frac{n}{r},$$

and therefore

$$\frac{a'}{b'} = n^2 pq,$$

When the square of the corrections β may be neglected, it is not necessary to determine them.

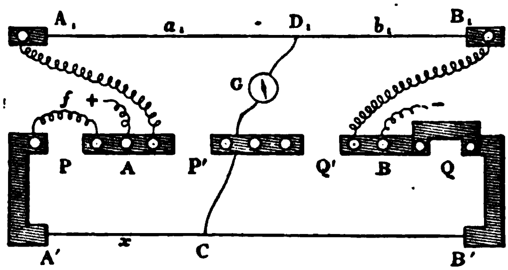


Fig. 193.

972. Calibration of the Wire.—The exactitude of the preceding methods assumes the perfect homogeneity of the wire, if not throughout its entire length, at any rate in the part used for the readings. This condition is usually realised when the wire has been carefully prepared with good alloys and drawn with care, but it is necessary to control it by calibration.

The first method consists in reproducing the same operations as for the calibration of a thermometer tube, and for the construction of the standard (921). An auxiliary resistance is taken which is equivalent to the nth part of the wire, and it is measured by substitution, the lateral resistances being so chosen as to utilise the n parts of the wire. An auxiliary resistance is then used

double or treble that of the first, so that the portions of the wire utilised are near the principal points of the calibration.

The following arrangement will make the operations easier:—

The resistances a' and b' are replaced by a second auxiliary wire A_1B_1 (Fig. 193) like the first. The breaks P′ and Q′ being open, the intervals P and Q are closed—one by a plate without resistance, and the other by an auxiliary resistance f equivalent to the nth part of the wire A′B′. Lastly, the connections of the galvanometer are both variable, the one at D_1 and the other at C.

The contact D_1 being placed at any given point of the wire A_1B_1, let a_1 and b_1 be the two resistances AA_1D_1 and BB_1D_1, which need not be known, x and x' the readings on the wire A′B′ when the

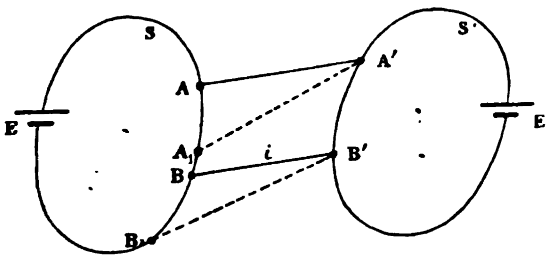


Fig. 194.

resistance f is successively inserted at P and at Q, d the total length of the wire including the terminal branches. We have

$$\frac{a_1}{b_1} = \frac{f+x}{l-x} = \frac{x'}{f+l-x'};$$

and therefore

$$f = x' - x.$$

By changing the position of the point D_1 on the second wire, we might measure the resistance f successively by different portions of the wire comprised between the principal points of the calibration. A series of similar operations, with different auxiliary resistances, will enable us to determine the correction ξ for each division x.

973. The idea of the following method is due to Von Helmholtz.* Let us imagine two circuits S and S′ (Fig. 194) containing electromotive forces E and E′; two points A and B of the former circuit are joined respectively to the two points A′ and B′ of the second; and let i be the intensity in one of the auxiliary wires, BB′ for example.

* GIESE. *Wied. Ann.*, Vol. XI., p. 440. 1880.

If the points A and B are displaced to A_1 and B_1 so that the resistance A_1B_1 is equal to AB, the intensity i does not change, and the resistance of the portion AA_1 is the same as that of BB_1. And in particular, if the point A_1 is come to the point B, the resistances

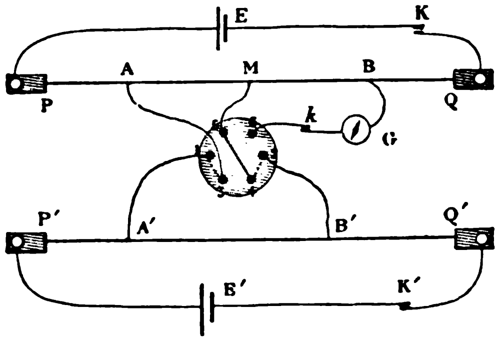


Fig. 195.

AA_1 and BB_1 are equal. This gives a method of dividing a given resistance into two equal parts.

Let PQ be the wire to be calibrated (Fig. 195), and AB the portion which it is wished to divide in two equal parts at the point M, A′ and B′ two points of any given wire P′Q′. Let 1, 2, 3, 4, 5, 6, be the cups of a commutator,* the connections of the different parts

* This is a slight modification of *Pohl's Gyrotrope (Kästner's Archiv.*, Vol. XIII., p. 49, 1828), which is very frequently used as a commutator or *current reverser* (Fig. 196).

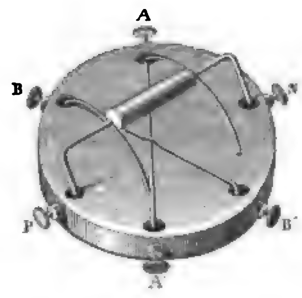


Fig. 196.

The six cups A, B, A′, B′, P, and N are filled with mercury, and are connected with the corresponding binding screws. The cups AA′, BB′ are connected cross wise by two thick insulated copper wires. The two cups

of the two wires are shown by the figure, the cups 1 and 2 serve as axis for the commutator.

According to the direction in which the rocker is inclined, the cups 1 and 3 and 2 and 4, or 1 and 5 and 2 and 6 are connected. In the first case the connections of the two circuits are AA′ and and MB′; in the second, MA′ and BB′, the latter of the two combinations always containing the wire of the galvanometer.

The rocker being in the first position, the contact B′ is displaced so that the current is very weak in the galvanometer. The rocker is reversed, and the constant M is displaced until the play of the rocker leaves the needle of the galvanometer stationary. The resistances AM and MB are then equal.

The connections are made by fine wires provided with small weights at their free ends. These weights serve to stretch the wires, which are simply placed at A and B.

By means of the make and break keys K, K′, and *k*, the currents in the two principal circuits, and then in the galvanometer, are only closed at the time of observation.

The current may, moreover, pass continuously in the two circuits, the heating of the wire having no influence on the exactitude of the method. The only condition is that the currents be constant.

974. Verification of a Resistance Box.—In accurate experiments it is necessary to know the exact ratio of the coils in a resistance box, and a well-constructed instrument should furnish the means of making this verification. It is necessary by a suitable system of contacts to be able to take the different coils separately. We have seen that in Elliot's box (Fig. 181) the cavities in the plate which form the top of each dial enable us by plugs furnished with binding screws to take the resistance of any intermediate coil. These coils may then be investigated separately or by the aid of an auxiliary bridge.

We first compare the different coils of the same set with each other, one by one, two by two, etc., which furnishes a great number of equations of condition. A supplementary unit—for instance, that in the interior of the box at *vv*′—enables us then to form with the first dial ten units, which are compared with the twelves. These

P and N serve as axis for the rocker, which consists of two conducting arcs insulated from each other, and communicating respectively with the cups P and N.

The terminals of the battery being connected at P and at N, the screw B is positive for the position of the rocker shown in the figure, and negative if the rocker is reversed so that the extremities of the arcs dip in the cups A′ and B′. For the preceding experiment one of the cross wires has been suppressed.

two dials and the supplementary unit form thus 100 units, which will serve to compare the tens; and in the same manner with the coils of 100 units.

In determining two sets of coils b and b', care must be taken to determine the resistances of the connecting wires β and β'. As the resistances b and b' are almost equal and the values of β and β' very weak, then, if p is the ratio obtained directly, which is very near unity,

$$p=\frac{b+\beta}{b'+\beta'}=\frac{b}{b'}\left[1+\frac{\beta-\beta'}{b}\right],$$

or

$$\frac{b}{b'}=p\left[1+\frac{\beta'-\beta}{b}\right]=p+\frac{\beta'-\beta}{b}.$$

This precaution is particularly important in the comparison of units. A table of the values of the coils is then made as a function of one of them or of the mean value.

975. THOMSON'S DOUBLE BRIDGE.—Sir W. Thomson has devised a modification of Wheatstone's bridge, which resembles the

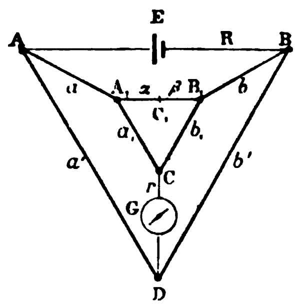


Fig. 197.

wire bridge in that equilibrium is established by means of sliding contacts. This arrangement (Fig. 197) comprises nine conductors, which may be regarded as forming the nine edges of a truncated triangular pyramid. The galvanometer being inserted in the branch CD, the current is null when the two points C and D are at the same potential. This condition cannot be realised unless a certain point C_1 of the branch A_1B_1 is at the same potential as the two points C and D. There would then be no current in a conductor connecting the points C_1 and C.

When equilibrium is established, the resistances a_1 and b_1 are respectively proportional to the resistances α and β of the portions A_1C_1 and B_1C_1 measured from the points A_1 and B_1 to the equipotential surface which passes through the points C and C_1. If γ is the total resistance $\alpha+\beta$ of the conductor A_1B_1, we have first

$$(37) \qquad \frac{a_1}{\alpha}=\frac{b_1}{\beta}=\frac{a_1+b_1}{\gamma}.$$

The same rule applied to the resistances measured from the summits A and B to the point D, on the one hand, and to the surface CC_1 on the other, gives thus

$$\frac{a'}{b'}=\frac{a+\dfrac{1}{\dfrac{1}{a_1}+\dfrac{1}{\alpha}}}{b+\dfrac{1}{\dfrac{1}{b_1}+\dfrac{1}{\beta}}}=\frac{a+\dfrac{a_1}{1+\dfrac{a_1}{\alpha}}}{b+\dfrac{b_1}{1+\dfrac{b_1}{\beta}}},$$

or, replacing the ratios $\frac{a_1}{\alpha}$ and $\frac{b_1}{\beta}$ by their values from equation (37),

$$(38) \qquad \frac{a'}{b'}=\frac{a+\dfrac{a_1}{1+\dfrac{a_1+b_1}{\gamma}}}{b+\dfrac{b_1}{1+\dfrac{a_1+b_1}{\gamma}}}.$$

The current will evidently be null in the galvanometer if it is connected to the points D and C_1. We may then consider this arrangement as an indirect means of determining the ratio of the resistances α and β—that is to say, the point of the conductor A_1B_1 which corresponds to the position of equilibrium, by the ratio of resistances a_1 and b_1, relative to an auxiliary conductor of larger dimensions.

As a particular case, if the sum of the resistances a_1 and b_1 is constant, and is taken equal to γ, the equation reduces simply to

$$(39) \qquad \frac{a'}{b'}=\frac{a+\frac{a_1}{2}}{b+\frac{b_1}{2}}.$$

Let us consider the nine branches arranged as in Fig. 198; a' and b' are the two resistances to be compared; AB and PQ two cylindrical and homogeneous wires; the end C of the wire of the galvanometer may slide along PQ; two contacts insulated from each other, but kept at a constant distance, and communicating respectively with the two ends of PQ, may be displaced along the

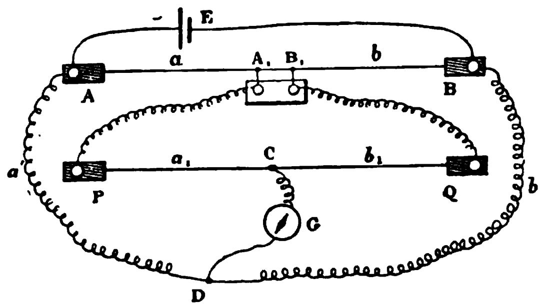


Fig. 198.

wire AB. The slider A_1B_1 has a double contact and is placed first in a position such that the needle of the galvanometer is almost at zero, and the adjustment is effected by displacing the point C. Formula (38) gives the ratio of the two resistances a' and b', and it reduces to formula (39), if the resistance γ of the portion of the wire comprised between the contacts A_1 and B_1, is equal to the entire resistance of the wire PQ.

976. This arrangement has been realised by Sir W. Thomson and Mr. Varley in the form represented by Fig. 199. Each of the wires AB and PQ is replaced by a series of equal coils, connected end to end, and arranged in a circle on two dials denoted by the same letters. The first series is formed of 101

coils of 1000 ohms; the second of 100 coils of 20 ohms. The sliders rest on knobs which connect the successive coils, and are moved by handles. The two contacts of the slider A_1B_1 have between them coils of 1000 ohms, and therefore a resistance equal to 100 coils in the other dial.

The handle of the second dial being on the first contact, the handle of the first is moved until the resistance introduced is sufficient, to within a coil, to establish equilibrium. Moving then the handle of the second dial, we seek, to within a coil of

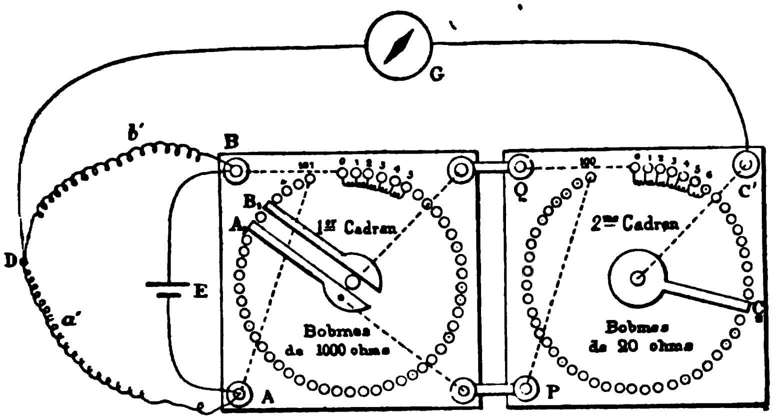


Fig. 199.

the second series, the contact which gives the closest approximation to equilibrium.

If the contact A_1 stops at the Nth coil, and the contact C on the nth, we shall have

$$a = 1000N; \qquad b = (101 - N - 2)\,1000,$$
$$a_1 = 20n, \qquad b_1 = (100 - n);$$

and therefore

$$\frac{b'}{a'} = \frac{(101 - N - 2)\,1000 + (100 - n)\,10}{1000N + 10n} = \frac{10000}{100N + n} - 1.$$

When the ratio sought is near unity, N is nearly 50; an error of one unit in the value of n only produces an error of $\frac{1}{50000}$ in the ratio of the resistances. The approximation is less

as we approach the ends of the first dial, but by observing the deflections of the galvanometer for the two contacts n and $n+1$ which comprise the position of equilibrium, we may, by a proportion, find for a a more exact fractional value.

977. Measurement of Very Small Resistances.—The use of the bridge in its various forms serves very well for mean resistances, but the method ceases to give good results whenever very large or very small resistances are to be compared.

The special difficulties presented by the measurement of very small resistances are mainly due to the relative importance of the resistances at the points of junction. On the other hand a conductor with a very small resistance cannot be compared to a linear wire, two dimensions of which may be neglected in comparison with the third.*

We have defined above what is to be understood by *resistance* of a conductor of three dimensions. Let us consider in the mass, two infinitely small equipotential surfaces—two small spheres, for instance—the one traversed by electricity which enters, and the other by electricity which emerges; if these two spheres serving as electrodes are respectively at the potentials V_1 and V_2, and that I is the total current which traverses them, the resistance R is given by the formula

$$R = \frac{V_1 - V_2}{I}.$$

This resistance may be calculated by dividing the portion of the medium comprised between the two surfaces of the electrodes into tubes of flow, and applying to these tubes the properties of branch currents.

In order to measure the resistances of a conductor, four electrodes are in general necessary: two of them, 1 and 4, put the conductor on the path of a current; two others, 2 and 3, connect two points of the conductor with the galvanometer, or any other measuring apparatus. Let V and I_1 with an index equal to the number of the electrode, denote the potential and the current for each of the electrodes; we have, for the currents,

$$(40) \qquad \begin{aligned} I_1 &= -I_4, \\ I_3 &= -I_2; \end{aligned}$$

* Kirchhoff. *Über die Messung Elekt. Leitungsfähigkeit.—Berliner Monatsbericht, 1880; Gesammelte Abhandlungen*, p. 66.

as the potentials are linear functions of the currents, we may also write

$$
(41) \qquad
\begin{aligned}
V_1 &= C + a_{11}I_1 + a_{12}I_2 + a_{13}I_3 + a_{14}I_4, \\
V_2 &= C + a_{21}I_1 + a_{22}I_2 + a_{23}I_3 + a_{24}I_4, \\
V_3 &= C + a_{31}I_1 + a_{32}I_2 + a_{33}I_3 + a_{34}I_4, \\
V_4 &= C + a_{41}I_1 + a_{42}I_2 + a_{43}I_3 + a_{44}I_4.
\end{aligned}
$$

The quantity C and all the coefficients a are constants; there are relations of condition between these n^2 coefficients, and they may be referred to $\frac{n(n-1)}{2}$ independent coefficients.

Taking into consideration equations (40), we deduce

$$V_2 - V_3 = (a_{21} - a_{31} - a_{24} + a_{34})\, I_1 + (a_{22} - a_{32} - a_{23} + a_{33})\, I_2,$$

an expression which we put in the form

$$(42) \qquad V_2 - V_3 = xI_1 + yI_2.$$

The factor x represents the difference of potential $V_2 - V_3$, when $I_2 = 0$, that is, when the circuit of the two electrodes 2 and 3 is not closed, and that $I_1 = i$; this then is the resistance of the conductor between these two electrodes. It will be seen that two equations like the preceding are sufficient to determine the value of x; experiment giving the difference $V_2 - V_3$ and the currents I_1 and I_2.

It has been supposed that the electrodes are equipotential surfaces. This condition is realised if contacts are made by points so fine that the surface of contact may be considered infinitely small in comparison with a small sphere, which itself is infinitely small compared with the dimensions of the conductors.

In the case of a cylindrical bar, we might take the electrodes 1 and 4 on the two bases, then the electrodes 2 and 3 on a generating line, at such a distance from the ends that the equipotential surfaces may be regarded as normal sections in the interval of the points 2 and 3. In these conditions, the measured resistance x is precisely that of a cylinder comprised within two perpendicular sections passing through the points of contact 2 and 3.

978. According to Prof. Kirchhoff,* in the case of a homogeneous bar in the shape of a parallelopipedon with a square base of length l, and side a, if we take the four electrodes at the summits which correspond to the same lateral face, each of them may be considered as bounded by the eighth of a sphere; if σ is the specific resistance of the bar, and γ a coefficient equal to 0·7272, the resistance measured is expressed by

$$x = \frac{\sigma}{a^2}(l - \gamma a).$$

This resistance is equal then to that of a bar of the same section, the length of which is merely $l - \gamma a$. It has in fact been supposed in the calculation that the ratio $\frac{a}{l}$ is infinitely small; but

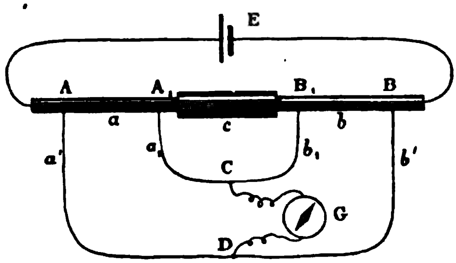


Fig. 200.

even if this ratio were equal to 0·5, the error does not amount to 0·0003.

The preceding discussion defines accurately the conditions within which we must work. All the experimental artifices should be directed to diminishing the surfaces of contact of the electrodes, and eliminating the corresponding resistances.

979. Sir W. Thomson† has used the double bridge mentioned in **975** for measuring small resistances.

Let us suppose that the object is to compare the resistances a and b (Fig. 200), the first comprised between the points A and A_1, and the second between the two points B and B_1. The

* KIRCHHOFF. *Berliner Monatsbericht, 1880; Gesammelte Abhandl.*, p. 71. —GREENHILL. *Proc. Camb. Phil. Soc.*, p. 293. 1879.

† Sir W. THOMSON. *Phil. Mag.* [4], Vol. XXIV., p. 149. 1862.

two conductors are joined by a third wire c of very small resistance; the resistances a', b', a_1, and b_1 are very great. The two extreme points C and D of the branch of the galvanometer are displaced until the current is null. If γ is the intermediate resistance A_1B_1, the ratio of the resistances a' and b' is given by the equation (38). If the resistance γ were null, we should have the usual ratio

$$\frac{a}{b}=\frac{a'}{b'}; \tag{43}$$

if the resistance γ is merely very small, we may write

$$\frac{a'}{b'}=\frac{a+a_1\dfrac{\gamma}{a_1+b_1}}{b+b_1\dfrac{\gamma}{a_1+b_1}}=\frac{a}{b}\left[1+\left(\frac{a_1}{a}-\frac{b_1}{b}\right)\frac{\gamma}{a_1+b_1}\right].$$

As this difference $\frac{b_1}{b}-\frac{a_1}{a}$ is very near zero, it will be seen that

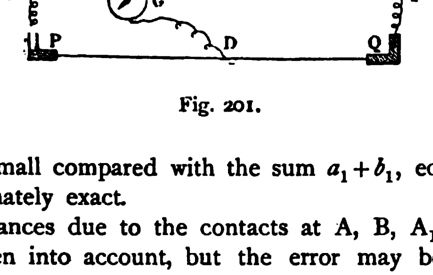


Fig. 201.

if γ is very small compared with the sum a_1+b_1, equation (43) is very approximately exact.

The resistances due to the contacts at A, B, A_1 and B_1 have not been taken into account, but the error may be neglected if the auxiliary resistances a' b', a_1 and b_1 have been sufficiently large.

980. The following modification, due to Hockin and Matthiessen,* eliminates this source of error. X and Y (Fig. 201) are known

* MAXWELL. *Electricity and Magnetism*, Vol. I., p. 444.

resistances in the form of coils which can be put on either side, without their sum being changed; PQ is the wire of a bridge. The contacts are made on each of the conductors by means of two points, or of two knife edges mounted on an insulating plate, and at a constant distance and measured exactly. Each of the points communicates with a small mercury cup in which dips one of the galvanometer wires, the other end of which is at D. The resistances X and Y and the point D are adjusted so that the needle of the galvanometer is at zero. If a is the resistance MA, S the total resistance from M to N, l the resistance of the wire PQ, x that of PD, and T the sum $X+l+Y$. If contact be made successively at A and A′, the values X and X′, x and x' relative to the two equilibriums, give

$$\frac{S}{T}=\frac{a}{X+x}=\frac{a+a}{X'+x'}=\frac{a}{X'-X+x'-x}.$$

If in like manner the contacts are at B′ and B, the values X_1' and X_1, x_1' and x_1 give similarly

$$\frac{S}{T}=\frac{b}{X_1-X_1'+x_1-x_1'};$$

consequently

$$\frac{a}{b}=\frac{X'-X+x'-x}{X_1-X_1'+x_1-x_1'}.$$

As the resistances of the points of contact only act in the branch of the galvanometer, they do not come into play in the equations of equilibrium; the only condition to be fulfilled is that the resistances S and T are unchanged.

981. Use of the Differential Galvanometer.—Professor Tait* has used a differential galvanometer with great advantage in comparing the resistances of stout metal rods. One of the coils of the galvanometer is connected with the points A and A′ of the first conductor, the other to the points B and B′ of the second, while the two conductors placed in succession in the circuit are traversed by the same current. The points are chosen so that the needle is at zero; if the galvanometer is adjusted, we have $a=b$.

* Tait. *Trans. Roy. Soc. Edinburgh*, Vol. xxviii., p. 717. 1877-78.

The way in which the contacts are made has no influence provided the resistance of the galvanometer is considerable.

The use of the differential galvanometer in the conditions pointed out in **939** appears still the most exact method.

The two conductors are traversed by the same current; the ends of the wires of the two coils form for each of them the electrodes 2 and 3 (**977**). The galvanometer being regulated so that the galvanometric constants of the two coils are equal, the resistances g and g' of the two circuits, are adjusted so that the needle remains at zero.

The expression for the difference of potential between the two electrodes A and A′ is

$$V_2 - V_3 = xI_1 + yI_2;$$

on the other hand we have

$$V_2 - V_3 = gI_2,$$

and therefore

$$xI_1 = (g - y)I_2.$$

The other coil, connected with the points B and B′ of the second conductor, gives also

$$x'I_1 = (g' - y')I_2;$$

from this follows

$$\frac{x}{x'} = \frac{g - y}{g' - y'}.$$

A fresh experiment is then made by giving new values g_1 and g'_1, very different from the first, to the resistances of the galvanometer, and we have

$$\frac{x}{x'} = \frac{g_1 - y}{g'_1 - y'} = \frac{g - y}{g' - y'} = \frac{g_1 - g}{g'_1 - g'}.$$

982. Measurement of Very Great Resistances.—The methods given in **902** and **940** enable us to compare very high resistances with the standard resistances with which we ordinarily work. We may attain the same results by other arrangements.

The two ends of a battery consisting of a great number of identical elements arranged in series are closed by the two resistances to be compared. Between the point of junction of the two resistances, and a point of the battery, a bridge containing a galvanometer is introduced, and the second end of the wire is displaced until the needle of the galvanometer is brought to zero.* Of the n couples which form the battery, the wire leaves p on one side, and $n-p$ on the other. As there is no current in the bridge, the intensity I is the same in the two branches; if E is the electromotive force of an element, ρ its resistance, R and R′ the two resistances to be compared, which correspond respectively to the numbers p and $n-p$ of elements, we have

$$I = \frac{pE}{R+p\rho} = \frac{(n-p)E}{R'+(n-p)\rho},$$

from which

$$\frac{R}{R'} = \frac{p}{n-p}.$$

983. When the resistance to be measured is that of the insulating envelope of a cable, the current is very strong when connection is made with the battery; it then progressively diminishes. The initial current is the superposition of three effects; the charge of the cable acting as condenser, the current which corresponds to the phenomenon of electric absorption, and finally the leakage of the cable. The first rapidly ceases, the second lasts a shorter or longer time, the third alone is permanent; the resistance is deduced from the intensity of this latter.

The methods used in measuring resistance necessarily imply the condition of a permanent state. Owing to the phenomena of absorption and those of polarization, which are often met with in the case of solids, this state is often only attained after the lapse of some time.

When the resistances are very great the final current is sometimes too weak to act upon a galvanometer. An electrometer is then used.

984. A very simple method consists in measuring the quantity of electricity which the resistance in question lets pass, when a constant difference of potential is established between its two ends A and B.

* FOUSSEREAU. *Ann. de Chim. et de Phys.* [6], Vol. V., p. 260.

The extremity A is connected with a constant source at a very high potential, like the internal armature of a battery of great capacity, or one of the poles of a battery composed of a great number of couples, the other armature or the other pole being to earth. The end B might be connected with a discharging electroscope (824). The number of contacts in unit time will be inversely as the resistances.

The extremity B might also be connected with a condenser. If R is the resistance to be determined, C the capacity of the condenser, V the potential of the source, and V′ that of a condenser after the time t, we have the ratio

$$\frac{1}{R}=\frac{C}{t}l.\frac{V}{V-V'},$$

or, if V′ is very small in comparison with V, very nearly

$$R=\frac{t}{C}\frac{V}{V'}.$$

985. Without directly bringing in the potential of the source, we may still join the two armatures of a condenser by the resistance in question, and determine the time t required for the difference of potential to increase from V_0 to V. We have then

$$\frac{1}{R}=\frac{C}{t}l.\frac{V_0}{V}.$$

In this case the ratio of the potentials V_0 and V will be determined by an electrometer, or by the swings given to the needle of the ballistic galvanometer, when the condenser is discharged in its initial and in its final state.

The choice of units is immaterial for comparative experiments. For absolute measurements, the resistance R is expressed in electrostatic or electromagnetic units, according as the capacity C is evaluated in units of the former or of the latter system.

If the condenser leaks, then, whatever be the cause, it is equal to that which would be produced by a resistance R_0 interposed between the two armatures. R_0 is first determined by the same method. By connecting then the armatures of the resistance R, the condition of the condenser is the same as if it were closed by two parallel plates. Lastly, a fresh experiment gives the sum $\frac{1}{R}+\frac{1}{R_0}$ of the two conductivities, from which the value of the resistance R is deduced by difference.

This method may be applied directly to the determination of the insulation resistance of cables which themselves form conductors of great capacity. One end of the cable being insulated, and the outer coating to earth, the other end is connected with a source which produces the potential V_0 and an electrometer. Communication being then broken with the source, the time t is observed at the end of which the potential of the cable becomes V.

This method gives the product CR of the capacity of the cable by its resistance to insulation. The experiment is complicated in this case by the phenomenon of electrical absorption, and the results are very different, according to the length of time in which the end of the cable has been in contact with the source; it is therefore necessary to define the conditions. The rule of the French Telegraph Administration is to use as source a battery of 100 Daniell's, each with a resistance of 10 ohms. The cable is put in connection with the battery for 15 seconds, and V_0 *is* then measured by an instantaneous discharge. It is again charged for 15 seconds, and then the cable is discharged after being left to itself for one minute.

It is to be observed that the product CR is independent of the shape and dimensions of the cable, and that it is simply proportional to the product of the specific resistance of the insulator by its specific inductive capacity. If σ and μ are these two constants, the capacity of a cable, the core and armature of which may be compared with a condenser formed of concentric cylinders of length L and radii R_1 and R_2, has the value (80 and 123)

$$C = \frac{\mu L}{2 l . \frac{R_2}{R_1}}.$$

and its resistance (217)

$$R = \frac{\sigma}{2\pi L} l . \frac{R_2}{R_1};$$

consequently

$$CR = \frac{\mu\sigma}{4\pi}.$$

This ratio is a general one. It is a consequence of the correlation which exists between the flow of electricity and the flow of electrostatic induction (213).

986. Induction Balance.—Induced currents in a conductor, other things being equal, are inversely as its resistance; and this property has been already used (936) in comparing resistances by measuring induced charges. The use of induced currents is particularly valuable in a great number of special cases in which ordinary methods would be inapplicable.

The simplest arrangement consists in counterbalancing the effects of induction in two different circuits.

Suppose, for instance, that we interpose in a circuit containing a variable electromotive force E (Fig. 202) two equal coils A and A′, at such a distance from each other that their mutual induction may be disregarded.

Above them, and at the same distance, two other equal coils a and a' are placed in the same axis, so that the coefficients of mutual induction are the same for the two systems.

When the induced coils a and a' have the same coefficient of self-induction, and their circuits are closed respectively by wires

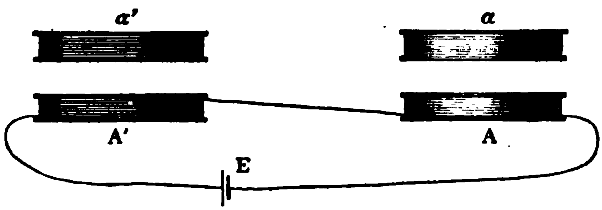


Fig. 202.

of equal resistances R and R′, the coefficients of self-induction are either equal or may be neglected. The currents induced in each case by any variation of the electromotive force are equal. The addition of two equal resistances r and r', without a coefficient of self-induction or having equal coefficients, does not affect the equality; so that if the induced currents are joined separately to the two coils of a differential galvanometer perfectly equilibrated, the needle should remain stationary.

As the galvanometer only measures total quantities of electricity, it is sufficient for equilibrium that the discharge or the integral value of the two induced currents is the same, and also approximately their duration, provided this duration is very short in comparison with that of the oscillation of the needle. This condition does not require complete symmetry of the two systems, but simply equality

of resistances and of the coefficients of mutual induction, whatever moreover be the coefficients of self-induction.

The galvanometer cannot be used if the inducing current, instead of the sudden changes in direction given by the opening or closing of the circuit, undergoes variations which succeed rapidly in contrary directions, like those produced by a vibrating break, or a microphone introduced into the circuit, or an electrometer with alternating currents. In this case recourse must be had to an instrument such as the telephone, which is sensitive merely to instantaneous variations of the current. A differential telephone, for instance, would be silent if the currents induced at each instant

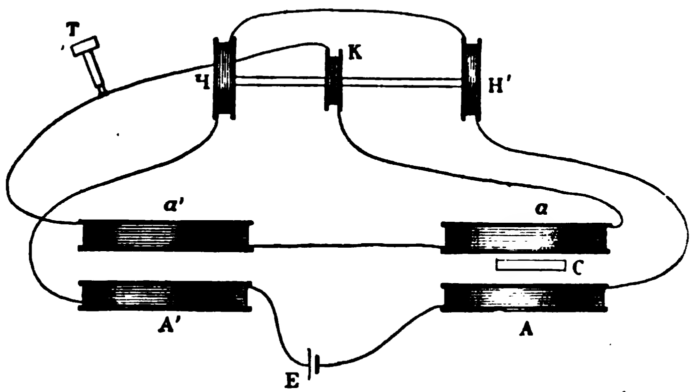


Fig. 203.

are equal, which implies that the coefficients of the two systems are equal (505).

It is, moreover, immaterial whether the differential telephone is worked by currents induced in the coils *a* and *a'*, or by two branches of the inducing circuit which respectively traverse the two coils A and A'.*

987. It is on this principle of the equilibrium of two circuits that is based the arrangement employed by Mr. Hughes,† and known as the *induction balance*, for comparing the electrical properties, and in particular the resistance of bodies which are met with in any given form.

* CHRYSTAL. *Phil. Trans. Roy. Soc. Edin.*, Vol. XXIX., p. 609. 1880.

† HUGHES. *Phil. Mag.* [5], Vol. II., p. 50. 1879.

A telephone (Fig. 203) is placed in the circuit of two induced coils a and a' connected so that their currents are in opposite directions. When the coefficients of mutual induction have the same value M_0, the balance is established, and the telephone is silent. The introduction of any conductor C between the coils A and a forms a partial screen (563) which destroys the equilibrium and makes the sound reappear.

Let L, R, I be the coefficient of self-induction, the resistance, and the intensity at the time t for the inducing circuit, l, r and i for the induced circuit, λ, ρ, γ for the conductor, or more exactly for a ring which should be equivalent to it; M and m are the coefficients of mutual induction of the conductor in respect of the coils A and a. Taking into account the original equilibrium, we have the equations

$$(44)\qquad \begin{aligned} &L\frac{dI}{dt}+M\frac{d\gamma}{dt}+RI=E,\\ &l\frac{di}{dt}+m\frac{d\gamma}{dt}+ri=0,\\ &\lambda\frac{d\gamma}{dt}+M\frac{dI}{dt}+m\frac{di}{dt}+\rho\gamma=0. \end{aligned}$$

If then the coefficient of mutual induction of the coils A and a is diminished by a quantity m_0, by separating them for example, the two former equations become

$$(45)\qquad \begin{aligned} &L\frac{dI}{dt}-m_0\frac{di}{dt}+M\frac{d\gamma}{dt}+RI=E,\\ &l\frac{di}{dt}-m_0\frac{dI}{dt}+m\frac{d\gamma}{dt}+ri=0. \end{aligned}$$

In order that there shall always be no current in the induced wire, we must have

$$m_0\,dI=m\,d\gamma,$$

or

$$\gamma=\frac{m_0}{m}\left(I_0+I\right);$$

the simultaneous variations being proportional, the two currents should have the same period, and the same phase, the constant current I_0 being further without action on the telephone. This

condition cannot generally be realised, and without extinguishing the sound in the telephone, we obtain thus only a minimum of sound.

Instead of altering the relative position of the principal coils, which would give rise to too sudden changes, Mr. Hughes found it more convenient to have recourse to an auxiliary apparatus which he calls a *Sonometer.*

This apparatus consists of two identical coils H and H′, fixed at the ends of a horizontal scale, between which a coil K may move parallel to itself; the former are inserted in the inducing circuit, and the movable coil in the induced circuit. The actions of H and H′ on K being of contrary signs, the resultant is null when the coil K is exactly in the middle of the scale; the induction increases on one system or the other according as the coil is displaced to the right or the left. That position of the coil, which in each case produces silence, furnishes an arbitrary measure of the effect produced by the conductor.

Another mode of measurement consists in merely modifying the balance by introducing a conductor between the coils A′ and a'. Mr. Hughes uses a zinc strip cut in the shape of a very acute wedge which he slides along; this is graduated, so that it is easy to determine the thickness of the plate which produces silence and balances the effect produced by the conductor C.

We may also place below the system $A'a'$ a small copper disc, or a ring of the same metal, movable about a horizontal axis. The action is null when the plane of the disc is perpendicular to that of the coils, and is a maximum for the parallel position.

It can be understood that this arrangement is better than the sonometer for compensating the action of a metal plate as it gives rise to a phenomenon of the same kind. If the letters with an accent denote quantities relative to the new conductor C′, the equations (44) should be replaced by the following:

$$(46)\qquad \begin{aligned} &L\frac{dI}{dt}+M\frac{d\gamma}{dt}+M'\frac{d\gamma'}{dt}+RI=E,\\ &l\frac{di}{dt}+m\frac{d\gamma}{dt}-m'\frac{d\gamma'}{dt}+ri=0,\\ &\lambda\frac{d\gamma}{dt}+M\frac{dI}{dt}+m\frac{di}{dt}+\rho\gamma=0,\\ &\lambda'\frac{d\gamma'}{dt}+M'\frac{dI}{dt}-m'\frac{di}{dt}+\rho'\gamma'=0. \end{aligned}$$

In the case of absolute silence, the second equation gives

$$m\frac{d\gamma}{dt}=m'\frac{d\gamma'}{dt}, \qquad \text{or} \qquad \gamma'=\frac{m}{m'}\gamma+C,$$

and the two latter

$$\left(\frac{\lambda}{M}-\frac{\lambda'}{M'}\frac{m}{m'}\right)\frac{d\gamma}{dt}+\left(\frac{\rho}{M}-\frac{\rho'}{M'}\frac{m}{m'}\right)\gamma-\frac{\rho'}{M'}C=0.$$

As this equation must continuously be satisfied, it follows that

$$C=0,$$

(47) $$\frac{\rho}{\rho'}=\frac{\lambda}{\lambda'}=\frac{Mm}{M'm'}.$$

The induction balance combined with a telephone is an apparatus of extreme sensitiveness. It renders evident the smallest differences of a weight, nature, degree of purity, or temperature of two conductors of the same dimensions, such as two coins placed in identical conditions in respect of the two systems of coils.

It enables us to detect very small masses of metal* in a badly conducting body, and may be employed with much advantage in verifying the insulation of the different windings of a coil the ends of which are open.† It is thus a very valuable instrument for qualitative researches, but lends itself less well to quantitative determinations; it is in general impossible to obtain complete extinction of sound, and, whatever be the mode of correction, it is difficult to interpret rigorously the results obtained.

988. Liquid Resistances.—The measurement of the resistance of an electrolysable liquid, presents special difficulties owing to the variable polarization of the electrodes. A given quantity of electricity cannot traverse the liquid without decomposing a proportional weight, and transferring the products of decomposition to the electrodes. Those which are identical at the origin when in contact with the liquid present falls of potential which are no longer equal and of opposite sign; if H and H′ are the values of these falls for each of them, I the intensity of the current, and R

* Graham Bell. *American Journ. of Sciences.* August, 1882.

† Lord Rayleigh and Mrs. Sidgwick. *Phil. Trans.*, p. 411. 1884.

the real resistance of the liquid between the electrodes, their difference of potential is equal to

$$H - H' + IR.$$

The difference $H - H' = e$ is what is called the electromotive force of polarization. This difference is null when the electrodes are identical, and goes on increasing to a maximum with the quantity of electricity which passes (**252**).

If the external electromotive force E is less than this maximum, the conditions tend towards equilibrium, and the current should completely cease; the current does not in fact disappear, but remains very feeble, without there being apparent any decomposition of the electrolyte; the current is just sufficient to keep the polarization constant, and to restore losses by diffusion.* If the electromotive force is greater than the maximum of polarization, gas bubbles are given off on the electrodes.

The immediate effect of polarization of the electrodes is then to diminish the intensity of the current, and produce an *apparent* increase of resistance. This effect was formerly attributed to a special resistance which electricity is supposed to experience in passing from a solid to a liquid, or conversely, and which was called *resistance of transition*. The presence on the electrodes of non-conducting deposits, gas bubbles for instance, may in certain cases introduce a new resistance which might be called resistance of transition, but in a very different sense to the preceding.

989. In order to get rid of polarization of the electrodes, Wheatstone† worked with liquid columns of different lengths, and kept the intensity constant by compensating variations of resistance by means of a rheostate. For if we suppose the electromotive force of the battery as well as the electromotive force e of polarization to be constant, and call $x, x' \ldots$ the resistances of the liquid, ρ, ρ' the corresponding resistances of the rheostate, we have

$$I = \frac{E - e}{R + x + \rho} = \frac{E - e}{R + x' + \rho'} = \ldots;$$

* HELMHOLTZ. *Pogg. Ann.*, Vol. CL., p. 483. 1873. *Wissenschaft. Abhandl.*, Vol. I., p. 823.

† WHEATSTONE. *Phil. Trans.*, Vol. CXXXIII., p. 303. 1843. *Scientific Papers*, p. 122.

consequently

$$x+\rho=x'+\rho',$$

or

$$x-x'=\rho'-\rho.$$

This method is not, however, free from objections; the deposits of gas bring about changes in resistance which it is impossible to allow for, and there is no certainty that the polarization of the electrodes is constant.

990. The difficulty disappears if we deal with solutions of metals, and use, as electrodes, plates of the metal of the solution. In this case polarization is always very slight, if not null. Pouillet* measured in this way the resistance of solutions of copper sulphate, of zinc sulphate, etc. He used a cylindrical tube divided into parts of equal length. This tube was closed at the bottom by a plate of the same metal, and a wire of the same metal fixed in a glass tube so that only the bottom was in connection with the liquid, could be adjusted at a variable distance from the bottom. Equal displacements of the liquid obviously represented columns of the same resistance. This arrangement forms a convenient rheostate.

M. Paalzow † has generalised the method. The liquid to be studied is contained in a U-shaped tube, the legs of which dip in porous vessels filled with the same liquid; the two porous vessels are placed in larger vessels containing a solution of zinc sulphate and two electrodes of amalgamated zinc. In order to compare two liquids, the porous vessel and the siphon are filled successively with the liquid. It is clear that in these conditions the polarization of the metal electrodes is got rid of, but we do not avoid, at all events completely, the variations of electromotive force at the surfaces of contact of the liquid with the zinc sulphate through the porous vessel.

991. Measurement of the fall of potential between two given points (**933**) gives a very simple method, and one not open to objection.‡ The liquid is contained in a cylindrical tube, closed at its ends by metal plates of the same section as the tube, and which serve as principal electrodes. The flow of electricity may be considered uniform, and the equipotential surfaces as perpendicular to the axis of the tube.

* POUILLET. *Comptes rendus*, Vol. IV., p. 786. 1837.

† PAALZOW. *Pog. Ann.*, Vol. CXXXVI., p. 419. 1869.

‡ BRANLY. *Ann de l'École Norm.* [2], Vol. II., p. 209. 1873.—LIPPMANN. *Comptes rendus*, Vol. LXXXIII., p. 192. 1876.

Two insulated platinum wires, called *parasite electrodes*, dip two points C and D of the liquid, and communicate with the electrodes of an electrometer. The deflection measures the difference of potential of the two points C and D. This difference is equal to the product IR, R being the resistance of the liquid column between the two planes C and D if the wires are not polarized. It is sufficient for this that they have not given passage to an appreciable current, which amounts to saying that the capacity of the electrometer is infinitely small compared with the capacity of polarization of the wire.

The capacity of a quadrant electrometer is always very small in comparison with that of parasite electrodes. With a capillary electrometer care should be taken that the immersed surface of the electrodes is very great compared with that of the mercury in the capillary tube of the electrometer. In this case it is advantageous to use platinised plates of platinum, the capacity of polarization of which is usually 20 to 25 times greater than ordinary platinum.

In order to compare directly the resistance of the liquid column with that of a metal conductor, the principal current is made to pass through a resistance box, and the two parasite electrodes of the liquid and those of the two points of this box are alternately connected with the electrometer. If the indication is the same in both cases, the resistances are equal. If there is an accidental difference between the electromotive forces of contact at C and D independent of the current and of the polarization, it is eliminated by reversing the direction of the current and taking the mean of the deflections observed.

992. USE OF SINUSOIDAL CURRENTS. — In measuring liquid resistances, Professor Kohlrausch* used a curious property of alternating currents sinusoidal in form.

Let us consider a circuit containing a voltameter and a source S. If R and L are the resistance and the coefficient of self-induction of the circuit, E the electromotive force of the source, e the electromotive force of polarization of the voltameter, we have for any given epoch

$$L\frac{dI}{dt}+RI+e=E. \tag{48}$$

* F. KOHLRAUSCH. *Pogg. Annalen*, Vol. CXXXVIII., pp. 280 and 370, 1869; Vol. CXLVIII., p. 143, 1873; *Jubelband*, p. 290, 1874.

If the polarization is always below its maximum value, it may be assumed that the electromotive force which corresponds to it is proportional to the quantity of electricity which has passed from the origin. If c is the capacity of the electrodes—that is to say, the inverse of the potential to which they would be raised by unit electricity—on the assumption of a strict proportionality, we may write then

$$e = \frac{1}{c}\int \mathrm{I}dt, \qquad \text{or} \qquad \frac{de}{dt} = \frac{\mathrm{I}}{c}.$$

Replacing this value in equation (48), we deduce from it

$$\text{(49)} \qquad \mathrm{L}\frac{d^2\mathrm{I}}{dt^2} + \mathrm{R}\frac{d\mathrm{I}}{dt} + \frac{\mathrm{I}}{c} = \frac{d\mathrm{E}}{dt}.$$

The electromotive force E being of the form

$$\mathrm{E} = \mathrm{E}_0 \sin 2\pi\frac{t}{\mathrm{T}},$$

the current is thus periodic; and when once the stable condition is established, it may be represented by the expression

$$\mathrm{I} = \mathrm{A} \sin 2\pi\left(\frac{t}{\mathrm{T}} - \phi\right).$$

As the equation (49) should be satisfied by this value of I for any given epoch, it gives

$$\text{(50)} \qquad \begin{aligned} \mathrm{A}^2 &= \frac{\mathrm{E}_0^2}{\mathrm{R}^2 + \left(\dfrac{2\pi\mathrm{L}}{\mathrm{T}} - \dfrac{\mathrm{T}}{2\pi c}\right)^2}, \\ \tan 2\pi\phi &= \frac{1}{\mathrm{R}}\left(\frac{2\pi\mathrm{L}}{\mathrm{T}} - \frac{\mathrm{T}}{2\pi c}\right). \end{aligned}$$

It will be seen that for given values of L and of c there is always one value of T—that is, a velocity of the sinusoidal inductor—for which the effects of self-induction and of polarization mutually destroy themselves. When this condition is realised, the change of phase is null, and the intensity is every instant equal to the quotient of the electromotive force by the true resistance.

993. We may point out in passing the analogy of this problem with that of a circuit which contained a variable electromotive force,

and a condenser of capacity placed as a shunt.* Let us suppose the two armatures of the condenser joined by a resistance R′. Let I′ be the current which traverses this resistance, I the current of the battery, and R the total resistance of the circuit. If at a given instant V is the difference of potential of the two armatures, we have

$$(51)\qquad \begin{aligned} V &= I'R', \\ I - I' &= C\frac{dV}{dt}, \\ L\frac{dI}{dt} + RI + V &= E. \end{aligned}$$

The elimination of V between these equations gives

$$(52)\qquad L\frac{d^2I}{dt^2} + \left(R + \frac{L}{R'C}\right)\frac{dI}{dt} + \frac{R+R'}{R'C}I = \frac{E}{R'C} + \frac{dE}{dt}.$$

If the electromotive force E is sinusoidal, this equation is also satisfied by a sinusoidal current when the permanent regime is set up. Moreover, equations (49) and (52) become identical if we suppose $R' = \infty$ and $C = c$—that is to say, the shunt R′ open and the capacity of polarization of the electrodes equal to that of the condenser. In both cases the condition necessary for the establishment of compensation is

$$(53)\qquad T^2 = 4\pi^2 cL;$$

it is independent of the resistance of the circuit and of the amplitude of the electromotive force.

This remarkable result may be put in a form which better brings out the mechanism of the phenomenon. Three electromotive forces are at work in the circuit—the principal electromotive force of the sources, $E = E_0 \sin 2\pi\frac{t}{T}$, and two inverse electromotive forces, consequences of the first, which have the values respectively

$$(54)\qquad \begin{aligned} E_1 &= -L\frac{dI}{dt}, \\ E_2 &= -\frac{1}{c}\int I\,dt. \end{aligned}$$

* MAXWELL. *Phil. Mag.* [4], Vol. XXXV., p. 360. 1868.

The resultant current being of the form

$$I = A \sin 2\pi \frac{t}{T},$$

we deduce from it

$$E_1 = -\frac{2\pi AL}{T} \cos 2\pi \frac{t}{T},$$

$$E_2 = \frac{AT}{2\pi c} \cos 2\pi \frac{t}{T}.$$

The constant of integration is null in the last equation, for, from symmetry, the charge of the condenser is null when the current is at its maximum value. The two electromotive forces have the same period, and a difference of phase equal to π. They are then represented by two sinusoids with the same nodes, and ordinates of opposite signs, and their sum is null if the amplitudes are equal in absolute values—that is to say, if equation (53) is satisfied.

994. Let T_0 denote the value of the period which corresponds to this condition. For any greater or less value of T the apparent resistance is greater than the true resistance, and the intensity of the current is less. The velocity T_0 is then that which corresponds to the greatest intensity for a given circuit.

This property may be utilised in measuring the resistance of liquids. If x is the resistance of the liquid, R that of the principal circuit, the maximum current would be observed; replacing then the liquid by a metal resistance r, the velocity is varied so as to obtain the same current. We shall have then

$$A^2 = \frac{E_0^2}{(R+x)^2} = \frac{E_0^2}{(R+r)^2 + \frac{4\pi^2 L^2}{T^2}},$$

from which is deduced

$$(R+x)^2 = (R+r)^2 + \frac{4\pi^2 L^2}{T^2}.$$

To get the maximum current it would be necessary to measure the period T, and ascertain the coefficient L, or eliminate it by a second experiment.

It is more advantageous to determine by experiment the metal resistance r, which, substituted for the liquid, and not itself introducing any coefficient of self-induction, gives the same intensity for the same velocity. We have in that case

$$A^2 = \frac{E_0^2}{(R+x)^2 + \left(\frac{2\pi L}{T} - \frac{T}{2\pi c}\right)^2} = \frac{E_0^2}{(R+r)^2 + \frac{4\pi^2 L^2}{T^2}}.$$

In order that the resistances x and r shall be equal, we must have

$$\left(\frac{2\pi L}{T} - \frac{T}{2\pi c}\right)^2 = \frac{4\pi^2 L^2}{T^2},$$

or

$$T^2 = 8\pi^2 cL = 2T_0^2,$$

that is to say, the period should be equal to $T_0\sqrt{2}$. For a greater or less value of the period than $T_0\sqrt{2}$, we shall have $r \gtrless x$.

In this case, again, it would be often difficult to determine and realise a convenient velocity. Professor Kohlrausch gets over this difficulty by using very weak currents with platinised electrodes of very large surface—that is to say, whose capacity is very large. If the capacity c is pretty great, and the period so small that the term $\frac{T}{2\pi c}$ may be neglected in comparison with $\frac{2\pi L}{T}$, the equation of condition is satisfied, and the two resistances are equal, whatever T may be. It can then be established that the equivalence of resistances between x and r is independent of the velocity. This verification will serve as control for the exactitude of the experiments.

995. Professor Kohlrausch also used Wheatstone's bridge with sinusoidal currents. The inductor and the coil, fixed to an electrodynamometer, are placed in the branch R of the battery. The movable coil occupies the ordinary place of the galvanometer. The two arms a and a' of the parallelogram are taken as equal. By means of a commutator the two other resistances b and b' may be interchanged, one of them being the resistance of the liquid. If the four branches have neither mutual induction nor self induction, the deflection of the movable coil does not alter, when the two resistances b and b' are interchanged, provided they are equal.

The diagonals being then conjugate, equations (23) of **949** show, in fact, that the current which at any given instant traverses the movable coil, in which there may be an electromotive force e of induction, has the value

$$i = \frac{e(a+b) - aE_b}{2ab + r(a+b)},$$

or

$$i = \frac{e(a+b) + aE_{b'}}{2ab + r(a+b)},$$

according as the liquid occupies the branch b or the branch b'. Now, these two values are identical if we allow for the direction in which the electromotive forces E_b and $E_{b'}$ have been measured.

The deflection itself depends not only on the amplitudes, but on the phase of the two currents I and i (**963**). It would be null if the electromotive force of polarization were itself null, for the intensity of the current would be null in the movable coil.

996. ATTRACTION BY INDUCTION.—Messrs. Guthrie and Boys* have endeavoured to deduce the conductivity of liquids from the intensity of the currents induced in their mass by the motion of a magnet. These currents, being closed and without electrodes, cannot produce any effect of polarization.

The liquid is contained in a glass vessel suspended by a metal wire about which it can rotate. A system of external magnets, producing a sensibly uniform horizontal field, can be rapidly rotated about the common axis of the wire and vessel. Screens interposed prevent any motion being transmitted through the air.

Other things being equal, the currents developed in the liquid mass are proportional to the intensity of the field, to the velocity of the relative displacement, and the conductivity of the liquid. They tend to oppose the motion. If there were no friction of the liquid against itself, or against the sides, the vessel would remain at rest, and the liquid would finish by acquiring the same velocity of rotation as the magnets. On the other hand, if the mass of liquid formed a rigid system with the vessel, this would be dragged in the direction until the torsion couple counterbalanced the moment of the electromagnetic actions. In fact, the various concentric layers acquire a very slow rotation, the angular velocity

* GUTHRIE and BOYS. *Phil. Mag.* [5], Vol. x., p. 398. 1880.

of which decreases from the centre to the circumference, and the mean value of which does not attain the $\frac{1}{20,000}$ that of the magnets. It may then be assumed, without appreciable error, that the torsion measures the electromagnetic action; and therefore, for the same velocity and the same field, the conductivity of the liquid.

It was found that the vessel alone produced no action. In order to allow for variations of the field and for the coefficient of torsion of the wire, each experiment is recommenced by suspending a brass disc in the vessel. The torsion should be the same for the same velocity if the field and the wire had not been modified.

The authors applied this method to mixtures of sulphuric acid and water. Their results agree well generally with those of Professor Kohlrausch, especially as regards the position of the maximum and the point of inflection of the curve which represents the conductivity of these liquids.

997. General Results.—As regards resistance, various bodies arrange themselves naturally in three categories: metals, pure metals or alloys, which do not alter in consequence of the passage of the current, and the resistance of which increases with the temperature; electrolytes, which are the seat of a chemical action correlated to the current; lastly, simple or compound non-metallic bodies, which have little or no conductivity, like the dielectrics, and the resistance of which diminishes as the temperature rises. The separation of these three classes is by no means absolute, and there are bodies to which it is difficult to assign an exact place.

998. Resistance of Metals.—Pure silver and copper are the best conductors, but a slight admixture of foreign substances greatly diminishes their conductivity.* The alloys of these metals with each other, or with gold, platinum, nickel, are far worse conductors than that which is the worst conductor. Alloys, on the other hand, which consist of only such metals as lead, tin, zinc, cadmium, etc., have a conductivity which is near the mean.

Between the temperatures of 0° and 100° the resistance r of a metal may be expressed as a function of the temperature t by a formula with two coefficients analogous to that for expansions, the coefficient β being very small compared with α.

$$r = r_0(1 + \alpha t + \beta t^2).$$

* Sir W. Thomson. *Proceedings of the Roy. Soc. Lond.*, Vol. VIII., p. 556. 1857.

It is very remarkable that for all pure metals the value of a is very near 0·0038, and is therefore sensibly equal to the coefficient of the expansion of gases.

The resistance of pure metals is then sensibly proportional to the absolute temperature. Nevertheless, the resistance of pure mercury in the liquid state offers far less variation (924); but MM. Cailletet and Bouty* have found that the conductivity of this body becomes 4·08 times as great as it solidifies, and that when solid, the coefficient of variation approaches that of other pure metals.

The coefficient a is still smaller in the case of alloys. With argentan, which is an alloy of copper and nickel, numbers are found for this coefficient varying from 0·00027 to 0·00044. The smaller values from 0·00022 to 0·00031 have been obtained with the alloy of platinum silver.

According to Sir W. Siemens,† parabolic formulæ no longer suffice when we get beyond the temperature of 100°. The resistances of platinum, iron, and copper are represented with sufficient accuracy by an expression of the form

$$r = r_0(1 + a\mathrm{T} + b\sqrt{\mathrm{T}}),$$

in which a and b are constants and T is the absolute temperature calculated from $-273°$.

The resistance of the metals is in general greater in the liquid than in the solid state. Bismuth forms an exception, as probably do all those metals which diminish in volume in changing to the liquid state.

999. Thermal and Electrical Conductivity.—Forbes‡ was the first to observe that the order of the metals as regards the conduction of heat is the same as for the electrical conductivity. Wiedemann and Franz,§ going still further, have ascertained that the two kinds of conductivity are sensibly proportional. It remained to be seen whether, as regards temperature, the thermal conductivity follows the same laws as the electrical conductivity, and, like it, diminishes as the temperature increases.

* Cailletet and Bouty. *Comptes rendus*, Vol. c., p. 1188. 1885.

† W. Siemens. *Journal of Soc. Tel. Engineers*, Vol. I., p. 123, 1872; Vol. III., p. 296, 1874.

‡ Forbes. *Phil. Mag.*, Vol. IV., p. 15. 1834.

§ Wiedemann and Franz. *Pogg. Annalen*, Vol. LXXXIX, p. 530. 1853.

This correlation seems to result from the experiments of Forbes,* of Angström,† of Neumann,‡ and of Lenz,§ who, working by the method of variable temperatures, endeavoured to determine, not merely the ratios of the conductivities of different bodies, but also their absolute conductivities. Nevertheless, the numbers obtained present some uncertainty. They are, moreover, insufficient, the calculations being made on the supposition that the two coefficients of internal and external thermal conductivity are independent of the temperature.

Professor Tait‖ considers the two coefficients as linear functions of the temperature. The method which he uses is that of Forbes. It requires two experiments—one statical, which consists in determining the distribution of stationary temperatures in a bar heated at one end; the other dynamical, in which the cooling of a very short piece of the same bar is investigated. This second experiment gives the quantity of heat lost by the outer surface at all temperatures. Tait's experiments were on iron, pure copper, and ordinary copper, lead, and argentan. It follows from the numbers obtained that the direction of the variations is the same in the two orders of phenomena, but that these two variations are far from being proportional.

1000. The physical properties of the metals vary so greatly from one specimen to another, that the question could only be strictly solved by studying the two phenomena in the same specimen. The experiments of F. Weber, of Kirchhoff and Hansemann, and of Lorenz, have been made under these conditions.

F. Weber¶ takes the metal to be studied in the shape of an anchor ring. In order to determine the thermal conductivity he heats one of the sections until the distribution of the temperatures is stationary; he then studies the cooling, observing the successive temperatures at two points, one at 45° and the other at 225°, from the section originally heated. Theory shows that if the dimensions of the ring are conveniently chosen, the temperature may, without appreciable error, be considered constant throughout an entire section, and the propagation as parallel to the mean circumference

* FORBES. *Phil. Trans. Roy. Soc. Edin.* Feb., 1860, and Feb., 1864.

† ANGSTRÖM. *Pogg. Ann.*, Vol. CXIV., p. 513, 1861; Vol. CXVIII., p. 423, 1863.

‡ NEUMANN. *Ann. de Chim. et de Phys.* [3], Vol. LXVI., p. 185. 1863.

§ LENZ. *Bull. de l'Acad. de St.-Pétersb.*, Vol. XV., p. 54. 1870.

‖ TAIT. *Phil. Trans. Roy. Soc. Edin.* Vol. XXVIII., p. 717. 1877.

¶ F. WEBER. *Bibl. Univ. de Genève* [3], Vol. IV., p. 107. 1880.

of the ring. It is assumed further that the two coefficients of conductivity, as well as the specific heat, are linear functions of the temperature.

It is sufficient to know the excess of temperature t_1 and t_2 at a given instant, at the two points in question, to get from it the two coefficients of conductivity.

The electrical conductivity was determined by the method of damping, the ring being placed vertically in the magnetic meridian, and a small magnetic needle being made to oscillate at the centre. The resistance R of the ring is deduced from formula (20)″ of **845**, in which the term in L can be suppressed, it being entirely negligable in the present case. If we suppose the induced currents distributed uniformly throughout the entire ring parallel to the mean circumference, which may, and even must, differ appreciably from the reality, and if a is the radius of the section, and b that of the mean circumference, the expression for the coefficient c of conductivity is

$$c = \frac{2b}{a^2 R}.$$

Weber's experiments do not favour the idea of a simple relation between the thermal and the electrical conductivity; but the author thinks he can infer that the ratio of the thermal conductivity to the electrical conductivity, is a linear function of the specific heat γ of unit volume of the metal, so that if a and b are two constants,

$$\frac{k}{c} = a + b\gamma.$$

As Weber remarks, this equation would enable us to explain why previous experiments seem to verify the proportionality of the two conductivities, as the metals employed, such as copper, iron, brass and argentan, have sensibly the same specific heat γ.

1001. Kirchhoff and Hansemann* have used a method of measuring thermal conductivity in which the coefficient of external conductivity only comes in as a term of correction of small importance. They consider an unlimited medium bounded by

* G. Kirchhoff and Hansemann. *Wied. Ann.*, Vol. ix., p. 1, 1880; and Vol. xiii., p. 406, 1881.—Kirchhoff. *Gesamm. Abh.*, p. 495.

a plane. The temperature being supposed uniform thoughout the whole extent of the medium, the entire surface of the plane is raised at a given moment to a different temperature, which is kept constant, and the course of the internal temperature is noted. If U is the excess of the temperature of the plane over the initial temperature, and if u is the excess at the time t at a point at distance r, we have

$$u = \mathrm{U}\frac{r}{2}\sqrt{\frac{k}{\gamma t}}. \tag{55}$$

The medium was represented by a cube 15 cm. in the side, the front face of which was fitted in a zinc plate; the temperature being uniform, and equal to that of the external medium, a jet of water of a fixed temperature, which was a few degrees higher or lower then the surrounding temperature, was directed against the face, and kept playing against it for the duration of the experiment. Thermoelectrical probes gave the variation of temperatures at different distances from the plane.

In order to obtain the electrical conductivity, the cube was then divided into prisms with a square base 5 mm. in the side, and length equal to the edge of the cube, and the method (**981**) was used. Unfortunately prisms from the same cube often showed differences amounting to 10 or sometimes 25 per cent., which greatly detracts from the definiteness of the conclusions which can be drawn from these experiments.

They tend to show that the ratio is virtually constant for the metals tried—that is to say, for lead, tin, zinc, and copper—except for iron. On the contrary, they do not confirm the law propounded by Weber.

1002. Lorenz* investigated the thermal and the electrical conductivity of bars 30 cm. in length, and 1·5 cm. in diameter.

Let us suppose the bar divided in n equal parts of length l, and let u_0, $u_1 \ldots u_n$ be the excess of temperature at the points of division 0, 1, 2 .... n. If q is the section, the quantity of heat which in unit time traverses the section A at an equal distance from the points 0 and 1, is

$$kq\frac{u_0 - u_1}{l},$$

* Lorenz. *Wiedemann's Annalen*, Vol. XIII., pp. 422 and 582. 1881.

and the section B, taken at equal distance from the points $n-1$ and n, is traversed by a quantity of heat

$$kq\frac{u_{n-1}-u_n}{l}.$$

The difference

$$\frac{kq}{l}\left[u_0-u_1-u_{n-1}+u_n\right]=\frac{kq}{l}\delta$$

represents a quantity of heat received in each unit of time by the portion AB of the bar; one part is used in heating it, and the other is lost by the external surface.

The quantity of heat employed in unit time to heat a portion l of the bar having its centre at the division p, is $\gamma lq\frac{du_p}{dt}$, γ being the specific heat of unit volume; the total quantity for the length AB is then

$$\gamma ql\left[\frac{du_1}{dt}+\frac{du_2}{dt}\cdots+\frac{du_{n-1}}{dt}\right]=(n-1)\gamma ql\frac{d\theta}{dt},$$

θ representing the mean temperature of this bar.

Lorenz assumes that the heat lost by the external surface is a function $f(\theta)$ of the mean temperature. We have then

$$\frac{kq}{l}\delta=(n-1)\gamma ql\frac{d\theta}{dt}+f(\theta).$$

If the heating be suddenly stopped, the difference δ sinks rapidly and acquires a value δ', θ taking the value θ'; we have then

$$\frac{kq}{l}\delta'=(n-1)\gamma ql\frac{d\theta'}{dt}+f(\theta');$$

if the temperatures θ' and θ are equal at the time in question, we get

$$(56)\qquad \frac{k}{l}(\delta-\delta')=(n-1)\gamma l\left(\frac{d\theta}{dt}-\frac{d\theta'}{dt}\right),$$

an equation from which the value of k is deduced. The formula holds so much the better the smaller is l, but Lorenz has shown that the error may be disregarded with bars 1·5 cm. in diameter when good conductors, whose length l is 2 cm.

This method was carried out in a very simple manner. The bar was perforated by 9 very fine holes 0·4 mm. in diameter, intended to receive the junctions of thermoelectric couples, and numbered 0, 1, 2, ... 8. Two couples, one with its junctions at 0 and 1, and the other at 7 and 8, are joined in opposite directions in the same circuit, and give δ directly. Seven other couples in series have their uneven junctions in the holes 1, 2, 3 ... 7, the even junctions being at the same temperature, and give θ directly. We have thus all the elements for calculating the value of k.

Electrical conductivities were determined on the same bars at temperatures of 0° and of 100°.

The results obtained by Lorenz confirm the law of Wiedemann and Franz, and bring out this other remarkable relation that the ratio $\frac{k_{100}}{c_{100}} : \frac{k_0}{c_0}$ is sensibly constant and equal to 1·367. Lorenz thinks that we might put generally

$$\frac{k}{c} = \text{CT},$$

T being the absolute temperature, and C a constant. In the case of very good conductors, the coefficient k varies very little with the temperature; the formula expresses then simply the known result that the resistance of pure metals is proportional to the absolute temperature. It is very remarkable that the relation pointed out by Lorenz holds for such alloys as brass and argentan; the coefficient of variation in electrical conductivity is then less, but that of thermal conductivity far greater, than for pure metals; there is exact compensation.

1003. Resistance of Electrolytes.—The most divergent numbers are obtained for the resistance of water, whatever pains be taken to purify it. The least traces of dissolved solids, or even gases, considerably increase the conductivity; the greatest resistances correspond to the purest water. Kohlrausch* obtained the value 7.10^6 ohms for the specific resistance—that is to say, 74 billion times

* F. Kohlrausch. *Bericht der Berl. Akad.* Oct., 1884.

the resistance of mercury. Foussereau* finds about 3.10^5 ohms for the resistance of ordinary distilled water. The resistance in the solid state is far greater than in the liquid state. The same specimen which in the liquid state had a resistance of $3 \cdot 231.10^5$ ohms, had a resistance of $3 \cdot 987.10^{10}$ at 0°, and of $4 \cdot 380.10^{10}$ at $-15°$.

M. Bouty† has shown that the resistance of any weak solutions follows a very simple law; the conductivity is the same for solutions which contain weights of salts proportional to the chemical equivalents; in other words, the molecular conductivity is the same for all the salts. The equivalents which satisfy this law are those which are determined by electrolysis — that is to say, electrochemical equivalents properly so called. M. Bouty's law should be considered as a limiting case, the phenomena becoming more complicated when the solution contains more than a few millionths of its weight of salts.

The resistance of all solutions, and especially of electrolytes, diminishes when the temperature increases; the conductivity c may be represented by an expression of the form

$$c = c_0 (1 + \alpha t + \beta t^2)$$

1004. Wiedemann,‡ considering the resistance of an electrolyte as due to the material transport of the elements of the salt through the mass of the solvent, investigated if there were no relation between the electrical resistance and the internal friction of a liquid.

Let us suppose that a liquid moves parallel to a fixed plane, the velocity of a point being a function $f(r)$ of its distance from the plane. Two infinitely near slices, parallel to the plane, and at distances r and $r+dr$, exert an action on each other parallel to the motion, proportional to their relative velocity, and equal to $\eta f'(r)$ for unit surface, η being a constant which depends on the nature of the liquid.

The constant η, which is called the coefficient of internal friction, may be deduced from experiments relative to the flow of liquids in capillary tubes. Poiseuille§ has shown that, if D is the diameter of

* FOUSSEREAU. *Ann. de Chim. et de Phys.* [6], Vol. V., p. 317. 1885.

† BOUTY. *Ann. de Chim. et de Phys.* [6], Vol. III., p. 433. 1884.

‡ WIEDEMANN. *Pogg. Ann.*, Vol. XCIX., p. 228. 1856.—*Die Lehre von der Elekt.*, Vol. II., p. 946.

§ POISEUILLE. *Mém. des Sav. Etrang.*, Vol. XI., p. 433. *Ann. de Chim. et de Phys.* [3], Vol. VII., p. 50. 1843.

the tube, L its length, p the difference of pressures at the two ends, and t the temperature, the quantity Q of liquid which flows in unit time is

$$Q = K\frac{pD^4}{L}.$$

The factor K, which depends on the nature of the liquid, is inversely as η; it only varies with the temperature t, and may be represented by

$$K = A(1 + at + bt^2);$$

we deduce from it, for the coefficient η, an expression of the form

$$\eta = \frac{\eta_0}{1 + at + bt^2}.$$

We may also, as Meyer* has shown, deduce the coefficient η from the method used by Coulomb in determining the *cohesion* of a fluid, and which consists in investigating the oscillations of a body immersed in a mass of liquid. If λ is the logarithmic decrement of the oscillations, R the radius of the oscillating disc, M the moment of inertia of the system, and T_0 the time of the oscillations in a vacuum, Mayer obtains the formula

$$\lambda\frac{M}{R^4} = \sqrt{\frac{5}{8}\pi^3 T_0 \eta}.$$

Grossmann,† applying this latter formula, suitably corrected, to the experiments of Kohlrausch and Grotrian, finds that the product of the electrical conductivity by the coefficient of internal friction is independent of the temperature for the same salt, and the same condition of solution; it follows from this that the two coefficients α and β, for the variation of conductivity, are respectively equal to the coefficients a and b relative to the variation of friction.

Bouty confirms this law for very weak solutions; he found that the molecular conductivity of any salt varies as the binomial

$$1 + 0{\cdot}033695\,t.$$

* *Pogg. Ann.*, Vol. CXIII., pp. 55, 193, 383. 1861.

† *Wiedemann's Annalen*, Vol. XVIII., p. 119. 1883.

According to Poiseuille, the internal friction of water is inversely as the expression

$$1 + 0.0336793\,t + 0.000209938\,t^2.$$

It follows accordingly that the variations with the temperature of the electrical conductivity of a very dilute saline solution depend simply on the internal friction of water.

The law of Grossmann applies also to fused salts, as follows from the experiments of Foussereau. The constancy in the value of the product of the conductivity into the coefficient of internal friction differs with different salts, and these different products do not seem to offer any simple ratio, either to each of the coefficients or to the chemical equivalent of the salt.

1005. Bad Conductors. Dielectrics.—Certain bodies, such as graphite, selenium, and various metallic sulphides, behave like metals, although their conductivity is much less; but their resistance diminishes as the temperature rises. The specific resistance of the carbons used for the electric light is about 0·004 ohm, or 42 times that of mercury; it varies about 0·0003 of a degree between the temperature of zero and 100°.

The resistance of selenium depends greatly on its structure; the metallic form conducts far better than the crystallised form. Mercadier* finds that the resistance of selenium decreases continuously as the temperature rises from 0° to 125°, and that then, towards 163°, it has a relative maximum corresponding to a change of state. Between zero and 36° the change is very rapid, and is sensibly proportional to the change of temperature. According to Shelford Bidwell, on the contrary, the resistance passes through a maximum between 20° and 30°.

Selenium has, moreover, the curious property, observed first by Willoughby Smith, of becoming a better conductor under the action of light. The increase of conductivity by illumination disappears with extreme rapidity when the selenium is kept dark.

Phosphorus, and particularly sulphur, present a great resistance when solid, but become relatively conductors when they are melted. Their conductivity increases rapidly with the temperature. Foussereau found that the resistance of these two bodies could be represented by an expression of the form

$$\log . R = a - bt + ct^2. \tag{57}$$

* *Comptes rendus*, Vol. XCII., p. 1407. 1881.

The experiments are very regular with phosphorus, but the numbers obtained for sulphur vary considerably, as do its other physical properties, with the previous conditions through which it has passed. The specific resistance of phosphorus at 50° is $1\cdot33.10^6$ ohms; from 25° to 100° it diminishes in the ratio of 6·6 to 1. The specific resistance of sulphur at 120° is about 10^{10} ohms.

Dry glass may be regarded as a perfect insulator at ordinary temperatures. Cavendish, however, had observed that at 300° it allows electricity to pass; the decrease of resistance is very rapid, and a difference of 6° to 9° is sufficient to double its conducting power. The resistance of various glasses may be represented, according to Foussereau, by an expression of the same form as that for sulphur and phosphorus.

It results from these experiments that plate glass has a much greater resistance than ordinary glass, and this, again, than Bohemian glass; for the specific resistances of these three bodies at 50° are respectively 3410.10^{12}, $2\cdot4.10^{12}$, and $0\cdot3.10^{12}$ ohms. It is to be presumed that the conductivity of heated glass is of the same order as that of electrolytes, and that the passage of a current is accompanied by chemical phenomena.

Experiments with gutta-percha are more difficult, owing to phenomena of electrical absorption, but the conductivity manifestly increases with the temperature. According to Bright and Clark, the resistance between the temperatures 0° and 24° may be represented by the formula

$$r = r_0\, 0\cdot8878^t.$$

1006. GASES AND VAPOURS.—It is known that gases and vapours, which are absolute insulators at low temperatures and under the atmospheric pressure, become conductors at high temperatures. Volta has observed that the flame of a candle or of a sulphur match acts like an infinitely fine point, and allows electricity to escape, at any rate for high potentials. Sir W. Thomson, utilising this property of heated gases, employs a slowly-burning strip of paper impregnated with nitrate of lead as an equaliser of potential, which is very useful in studying atmospheric electricity.

In like manner, the electric current in an arc light passes by the layer of hot gases and vapours which separates the two carbons. This is also the case with electrical sparks in air, and discharges in tubes of rarefied gas. The resistance of the gas diminishes with the pressure to a certain minimum, corresponding to a pressure of a few millimetres of mercury, which varies in different gases; it then

increases with very great rapidity, and all experiments tend to show that electricity cannot pass in a perfect vacuum.

Edlund* considers the resistance which gases present to the passage of electricity as due to two causes—the one a true resistance, which decreases indefinitely with the pressure; and to an inverse electromotive force developed by the contact of the gas and of the electrodes, which on the contrary increases with the rarefaction.

Another circumstance which complicates the phenomena, and to which it is difficult to assign its part, is the transport of electricity by the molecules of the gases themselves. And, notwithstanding numerous researches on this question, it seems difficult to form an exact idea on the true conductivity of gases.

* *Ann. de Chim. et de Phys.* [5], Vol. XXVII., p. 114. 1882.

CHAPTER IV.

MEASUREMENT OF ELECTROMOTIVE FORCES.

1007. STANDARDS OF ELECTROMOTIVE FORCE.—When a conductor is traversed by a permanent current, the electromotive force between corresponding elements of two equipotential surfaces, or the difference of potential of two points of these surfaces, is the product of the intensity of the current which traverses one of these elements by the resistance which separates them. The practical unit of electromotive force or the *volt*, is the electromotive force which maintains a current of one ampère in the resistance of a legal ohm. (920). The volt thus determined carries into the value, which results from its theoretical determination, the same relative error as that of the legal ohm.

It has not hitherto been possible to produce a standard of electromotive force so definite and invariable as for resistances, and for want of a better standard it is necessary to have recourse to couples which contain liquid elements. Daniell's element and that of Latimer Clark are those which are most trustworthy.

Various forms have been proposed for Daniell's element, either with a view of preventing the mixture of the two solutions, or to avoid the use of porous vessels, as for instance in Sir W. Thomson's standard, where the liquids are superposed in the order of their decreasing densities. In any case it is necessary to renew the liquids before proceeding to a new set of measurements.

The electromotive force depends on the density of the solutions. With saturated copper sulphate, and zinc sulphate of increasing density, Carhart* found that the electromotive force varies from 1·04 volt with pure water to a maximum of 1·137 for a 5 per cent. solution of zinc sulphate; it diminishes slowly to 1·111 for a solution of 10 per cent. sulphate, and then remains constant.

* H. S. CARHART. *Amer. Journ. of Science*, Vol. XXVIII. p. 374. 1884. *Journal de Physique* [2], Vol. IV., p. 98.

Unfortunately it is not sufficient to take liquids of the same degree of concentration to obtain a definite electromotive force; the differences, which no doubt arise from the unequal purity of the metals, may amount to as much as 2 or 3 per cent.*

Temperature moreover has great influence on the electromotive force of a Daniell's element. According to Von Helmholtz,† this influence is zero when, the solution of copper sulphate being saturated, that of the zinc sulphate is of density 1·04; the electromotive force decreases when the temperature increases for the most concentrated solutions of zinc, and increases on the contrary with the temperature for weaker solutions.

Light seems to have a still more distinct action; M. Pellat‡ found that the seat of this action of light is on the copper plate, and that it is due to the more refrangible rays; it may lower the electromotive force by one or two per cent.

In Sir W. Thomson's gravity battery, with concentrated solutions of sulphate,§ the electromotive force is very closely equal to 1·074 for a temperature of 15°.

Instead of using this couple in open circuit, Sir W. Thomson‖ prefers to take as standards the difference of potential of the two poles, when they are connected by a resistance of 250 ohms. This difference increases at first for some hours after closing the circuit, but it then becomes remarkably constant.

Latimer Clark's couple consists almost entirely of solid substances, which makes the transport and maintenance more easy, and it seems to give more constant results. The elements of which it is formed are—zinc, zinc sulphate, mercuric sulphate, and mercury.

A concentrated solution of zinc sulphate in boiling water is prepared, and mercur~~ic~~ous sulphate is added to the cold solution, so as to form a thick paste, and this mixture is then kept for some time at 100.° The paste is placed on the surface of pure mercury previously heated, and a zinc rod is fixed in it. The electrodes are platinum wires attached to the zinc and to the mercury. From its physical condition, the couple polarizes readily,

* Lord RAYLEIGH. *Phil. Trans. Roy. Soc.* for 1884, Part. II., p. 459.

† HELMHOLTZ. *Berichte der Akad. der Wiss., Berlin*, p. 26. 1882.—*Wiss. Abhand.*, Vol. II., p. 958.

‡ PELLAT. *Comptes rendus*, Vol. LXXXIX., p. 222. 1879.

§ Lord RAYLEIGH. *Loc. cit.*

‖ Sir W. THOMSON. *B. A. Report*, Southampton. 1882.

but soon resumes its original electromotive force; if the substance used are pure, the different specimens do not differ by more than a thousandth. In this form there is no reason to fear any diffusion, and the element once set up and sealed in the glass is always ready for use. From the experiments of Lord Rayleigh,* the electromotive force at 15° is 1·435 volts.

Rise of temperature diminishes the electromotive force of the Latimer Clark element. The coefficients of variation obtained by different physicists range from 0·00041 (Wright and Thomson), to 0·00082 (Lord Rayleigh, Von Helmholtz); these differences may arise from the mode of construction, but they may also be due to an inexact determination of temperature, which is difficult to ascertain from the form of the element itself.† By introducing a thermometer in the interior of the instrument, Pellat finds that the electromotive force of the Latimer Clark standard is exactly represented between 0° and 25° by the formula

$$E = E_0 (1 - 0{\cdot}000781\, t).$$

1008. In order to fractionate the electromotive force of a standard, the poles must be joined by a resistance which is very great in comparison with its own; from Ohm's law the difference of potential of two points which comprise between them any given fraction of this resistance is the same fraction of the total electromotive force.

Elliott's resistance boxes (**927**) enable us to make this experiment in a very simple manner. The ordinary plugs being placed at No. 9 of each dial, the standard is connected with the extreme terminals, which introduces a resistance of 9999 ohms. or of 10000 if a supplementary unit be added. If we place in the plates of the dials two auxiliary plugs separated by a resistance of n ohms, the difference of potential of these plugs is a fraction equalling n ten thousandths of that of the terminals, or virtually of the standard electromotive force.

When the boxes are not such as to enable partial resistances to be taken, two different boxes are joined end to end, the extreme terminals of which are connected with the standard. If n units of the first box are taken, and n' units of the second, the total

* Lord Rayleigh. *Phil. Trans. Roy. Soc.* for 1884. Part II. p. 459.

† Wright and Thomson. *Phil. Mag.*, [6], XVI. p. 33. 1883.—Helmholtz. *Berichte der Akad. zu Berlin*, p. 26. 1882.

resistance is $N = n + n'$, and the difference of potential of the terminals of the former is a fraction equal to $\frac{n}{n+n'} = \frac{n}{N}$ of that of the extreme terminals. It is advantageous in this case to vary the two numbers n and n', so that the sum $n + n' = N$ remains constant, and equal to 10,000 ohms, for instance.

1009. Electrometric Measurements.—Electromotive forces may be determined in absolute or relative value by the various methods pointed out in Chapter I. (976 *et seq.*) for determining differences of potential, as well as in numbers 868 and 869. With batteries and permanent currents, these methods are generally simpler than in experiments in statical electricity, for the apparatus themselves are sources of electricity, and losses of electricity due to the connections are soon repaired, so that in general no account need be taken of the capacity of the bodies concerned.

It must be observed that the electrostatic methods give results which are also measured in electostatic units, and that if we wish to pass from one system to another (610) we must know the ratio of the units (610). Sir W. Thomson* found, for instance, that the electromotive force of a Daniell's element is equal to 0·00374 electrostatic units (C.G.S.) If we assume for the ratio of the units the value $a = 3 \times 10^{10}$, this result corresponds to $0{\cdot}00374 \times 3.10^{10} = 1{\cdot}12.10^{8}$ in electromagnetic units, or 1·12 volts.

1010. Open Elements. Method of Opposition.—In the case of liquid elements, we should consider the electromotive force from two points of view, according as the circuit is open or closed, or again according as it has already been closed for sometime. In all cases the electromotive force is the sum of the differences of potential at the surfaces of contact of the successive elements, which form each of the couples. But if the battery has been traversed by a current, the chemical work has modified by polarization the differences of potential, and the electromotive force really put in play may have a totally different value; it is this latter which has more particularly a practical interest. It is then important to specify the condition in which the experiments have been made.

The current is zero in a circuit in which the algebraical sum of the electromotive force is zero—that is to say, when the electromotive forces divide into two opposed groups forming the same sum. In order to apply this method of direct opposition, we must

* Sir W. Thomson. *Reprint of Papers on Electricity and Magnetism*, p. 245.

be able to dispose of a series of couples the electromotive force of which is very small compared with that we wish to estimate. M. J. Regnauld* took as unit a bismuth-copper thermoelectric couple, the junctions of which were respectively at 0° and 100°.

In order to avoid the use of too great a number of thermoelectric couples, M. Regnauld used, as an auxiliary multiple, a Daniell's element, in which copper and copper sulphate were replaced by cadmium and cadmium sulphate, and the electromotive force of which was sensibly equal to 55 thermoelectric couples. The experiment consists in placing the couple under investigation in a circuit containing a delicate galvanometer, and in which the cadmium and the thermoelectric couples are interposed until the current is nearly zero. This condition is never exactly realised, and in fact we determine the two numbers n and $n+1$ of thermoelectric couples, which must be opposed to the electromotive force in question, to obtain deflections in opposite directions in the galvanometer. We thus obtain the electromotive force of the couple in question to within a unit.

The complementary fraction may be determined either by the ratio of the final deflections, or by seeking to what temperature the hot junction of the $n+1$ couple must be lowered to produce null current; but it would be superfluous to push the approximation so far. This method also, besides the practical difficulties which it involves, does not give the degree of accuracy which might be hoped. The cadmium couple is very constant; but, whatever precaution is taken in constructing thermoelectric couples, they present differences among each other, as Gaugain† found, arising from the crystallisation of the bismuth, and which may often amount to a tenth of the whole value. The gold-copper couple, the electromotive force of which is 62·5 times less than that of the bismuth-copper, would give a more satisfactory standard.

However this may be, Regnauld found that the electromotive force of the Daniell's element is between 175 and 176 units; the thermoelectric couple, bismuth-copper, between the temperatures of 0° and 100°, is accordingly about 0·0061 volts, and the Daniell's element, with zinc and cadmium, 0·34 volt.

1011. COMPENSATION METHODS.—By this are understood those methods in which the electromotive force to be determined is compensated by the difference of potential of two points of a circuit traversed by a permanent current.

* J. REGNAULD. *Ann. de Chim. et de Phys.* [3], Vol. XLIV., p. 453. 1855.

† GAUGAIN. *Ann. de Chim. et de Phys.* [3], Vol. LXV., p. 1. 1862.

The arrangement first used by Poggendorff is represented in Fig. 204.

The circuit of a constant battery E contains two rheostates R and r, the latter being between the points A and B. The couple e, to be measured, is connected with the points A and B through a galvanometer G, the currents produced by the electromotive forces E and e being all directed towards the point A. The two rheostates are connected, so that there is no current in the

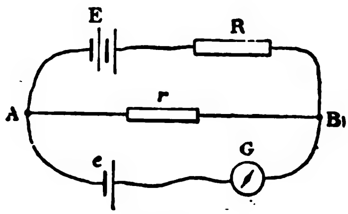


Fig. 204.

galvanometer. If, then, R is the resistance of the portion AEB of the circuit, and r that of the bridge AB,

$$\frac{e}{\mathrm{E}}=\frac{r}{\mathrm{R}+r}.$$

If we wish to dispense with a determination of the resistances R and r, they may be replaced by other values R′ and r', so as to obtain again no current. We have then

$$\frac{e}{\mathrm{E}}=\frac{r}{\mathrm{R}+r}=\frac{r'}{\mathrm{R}'+r'}=\frac{r'-r}{\mathrm{R}'-\mathrm{R}+r'-r}=\frac{1}{1+\dfrac{\mathrm{R}'-\mathrm{R}}{r'-r}};$$

and the ratio of the electromotive forces is given by the changes of resistance in each of the two rheostates.

1012. We may suppress the rheostate r, if we take for AB a homogeneous wire (Fig. 205), and seek the point C of the wire with which the second pole of the couple e must be connected, so that there is no deflection of the needle.* If l is the total resistance of the wire AB, x and x' the resistances of the lengths AC which satisfy

* DUBOIS-REYMOND. *Abh. der Berl. Akad.*, p. 787. 1862.—*Ges. Abh.*, I, p. 176.

the condition when the values of the total resistance of the circuit ABC are successively R and R′, we have

$$\frac{e}{E}=\frac{x}{R+l}=\frac{x'}{R'+l}=\frac{x'-x}{R'-R}.$$

Instead of comparing the electromotive force e with that which gives the principal current, and which, for that very reason, is badly defined, it is better to work by substitution, and to place successively in the circuit AGC the two electromotive forces e and e' to be compared.* As the two experiments give the ratios $\frac{e}{E}$ and $\frac{e'}{E}$, the ratio $\frac{e}{e'}$ is at once obtained.

In the second arrangement, for instance, if the value of R is constant, and x and x' are the values of the resistance AC, which

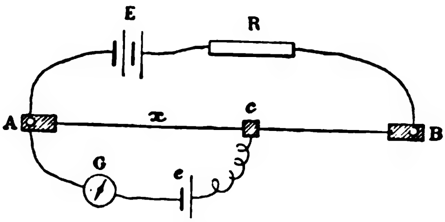


Fig. 205.

correspond to equilibrium for the two currents e and e', we have simply

$$\frac{e}{e'}=\frac{x}{x'}.$$

If the wire is homogeneous, the electromotive forces are proportional to the corresponding distances from the point of contact to the point A. The rheostat R serves to regulate the resistance so that the distances x and x', which satisfy the preceding equation, are comprised within the length of the wire AB.

The galvanometer may clearly be replaced by an electrometer, and particularly by Lippmann's capillary electrometer.

1013. The two experiments may be made simultaneously; the two couples e and e', connected at the point A by one pole, are

* PELLATT. *Ann. de Chim. et Phys.* [5], Vol. XXIV., p. 5, 1881.—*Journal de Phys.*, Vol. IX., p. 145.

joined by galvanometers to two different points e and e' of the wire AB, chosen so that simultaneously there is no current in the two galvanometers; this is the arrangement known as Clark's *potentiometer*.

Between two points A and B (Fig. 206), that one of the two couples which has the greatest electromotive force, e' for instance, is inserted along with a galvanometer G′, and the rheostate is regulated so that the needle is at zero. The second couple e is connected on the one hand to the point A, and on the other to the movable contact C, which is then displaced until there is no current in the corresponding galvanometer G. This galvanometer may be suppressed if necessary, for after the first adjustment the needle can only remain at zero provided no current passes in AeC; but the sensitiveness is then far less.

In all these methods, compensation is not effected at the first trial, and during the subsequent trials to get at the position of

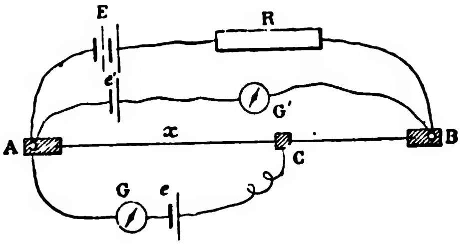


Fig. 206.

equilibrium, the couple whose electromotive force is to be measured produces currents, and is therefore more or less polarized. This objection is partially obviated by inserting in the circuit a key, which is pressed down for a very short time. Nevertheless, with elements which are easily polarized, it is better, before proceeding to the final reading, either to recharge the element or to leave it at rest so long that the polarization disappears.

1014. Absolute Electromagnetic Measurements.—The method of compensation gives, in absolute value, a measure of the electromotive force, if we know the resistance r between the points of the shunt and the intensity I of the current which traverses this resistance. This is the method used by Latimer Clark* in measuring the electromotive force of his standard element.

* *Phil. Trans. Roy. Soc., 1873.—Journal of the Society of Telegraph Engineers*, Vol. vii., p. 85.

Lord Rayleigh and Mrs. Sidgwick* have repeated this determination by the aid of a more complicated arrangement (Fig. 207).

C and C′ are two resistance boxes which close the circuit of an auxiliary battery E′, and the plugs are arranged so as always to have a fixed resistance of 10,000 ohms between the poles of the battery. The two keys K′ and K″ being left open, and the key K closed, the resistance r which compensates the electromotive force of the couple is first tried in the box C. If then the key K is opened and K′ and K″ closed, the resistance r' is sought

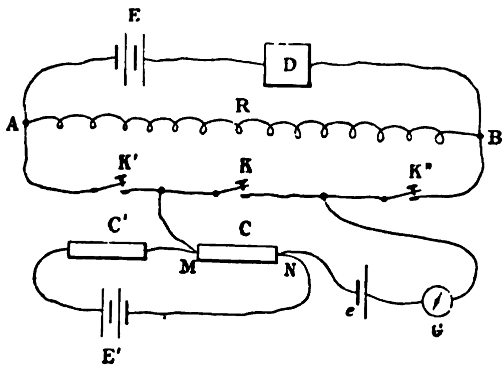


Fig. 207.

which balances the electromotive force $e - \mathrm{RI}$ of the circuit MABN. We have then

$$\frac{e}{e - \mathrm{RI}} = \frac{r}{r'}, \qquad \text{or} \qquad e = \mathrm{RI}\frac{r}{r - r'}.$$

The resistance R is known in absolute value, and the principal current I is measured by a balance electro-dynamometer D, analogous to that of Joule (792).

1015. Closed Battery.—When a polarizable battery of resistance R_0 and electromotive force E_0 in open circuit, is closed by a resistance r, it acquires a smaller electromotive force E, and its resistance has in general a different value R. The intensity of the current is defined by the equation

$$\mathrm{E} = \mathrm{I}(\mathrm{R} + r).$$

* *Transactions of the Royal Society, 1884*, p. 411.

The difference $E_0 - E$ represents the electromotive force of polarization. If the polarization is independent of the current, when this is beyond a certain limit, a series of equations will be obtained, by varying the interpolar resistance, such that

$$E = I(R+r) = I'(R+r') = I''(R+r'')\ldots.,$$

which give

$$E = \frac{II'}{I'-I}(r-r') = \frac{II''}{I''-I}(r-r'') = \ldots\ldots,$$

$$R = \frac{Ir - I'r'}{I'-I} = \frac{Ir - I''r''}{I''-I} = \ldots\ldots$$

The identity of the different values thus obtained will serve as a verification of the hypothesis of the constancy of the polarization. When only two electromotive forces E and E_1 are to be compared, we proceed in like manner for each of them, and we have

$$\frac{E}{E_1} = \frac{I_1 I_1'}{II'} \frac{I'-I}{I_1'-I_1} \frac{r_1 - r_1'}{r-r'}.$$

If the resistances are adjusted so that the intensities are the same in the two cases, the formula reduces to

$$\frac{E_1}{E} = \frac{r_1 - r_1'}{r-r'};$$

the ratio of the electromotive forces only depends on the ratio of the variations of resistances, and the galvanometer need not even be graduated. This latter method was used by Wheatstone.

1016. The following method, indicated by Poggendorff,* leads to the same results in a more direct way. The two couples E and E′ (Fig. 208) are placed in the same circuit in such a manner that their electromotive forces add themselves, and a shunt AB is introduced between them containing a galvanometer. The point A being arbitrary, the point B is displaced or the resistances modified so that there is no current in the shunt. The resistances R and R′, of two portions AEB, AE′B of the total circuit, are then as the electromotive forces (982).

* See BOSSCHA. *Pogg. Ann.*, Vol. XCVII., p. 172. 1851.

Without changing the position of the point B, resistances r and r' are added to the segments R and R′ respectively, so that the current is null; and then, without its being necessary to determine the resistances R and R′ of the circuits containing the electromotive forces,

$$\frac{E}{E'}=\frac{R}{R'}=\frac{R+r}{R'+r'}=\frac{r}{r'}.$$

1017. USE OF GALVANOMETERS OF HIGH RESISTANCE.—If

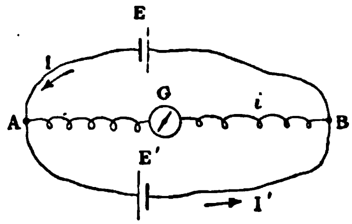


Fig. 165.

the resistance r, which closes the circuit of a battery, is very great in comparison with the internal resistance R, the expression

$$E=I(R+r)=Ir\left(1+\frac{R}{r}\right)$$

reduces sensibly to

$$E=Ir.$$

In like manner, when we wish to compare two electromotive forces, we have simply

$$\frac{E_1}{E}=\frac{I_1}{I}\frac{r_1}{r};$$

and if we arrange the experiment so that the resistances or the intensities are the same in both cases, the ratio of the electromotive forces is equal to the ratio of the intensities observed, or of the resistances introduced into the circuit.

The sensitiveness of galvanometers of high resistance being, so to say, unlimited, this method is the most exact, the most expeditious, and almost the only one in use at present. The couples

to be compared are closed by a constant resistance of 30,000 to 40,000 ohms, inserting a shunt if necessary. With mirror instruments the electromotive forces are proportional to the deflections. The method is equivalent to the use of an electrometer (868), and the electromotive forces thus determined are those which correspond to open circuits, or, at any rate, to currents so weak, and of such short duration, that their polarization may be entirely neglected.

1018. The method may also be applied to measuring any given difference of potential between two points on the path of a current, when the resistance s which separates them, is not of the same order of magnitude as that of the galvanometric circuit. If, as above (867), g is the resistance of the wire containing the galvanometer, R that of the circuit outside the points touched, I_1 and I the intensities of the principal current before and after the introduction of the galvanometer, the difference of potential V observed by the galvanometer is $V = Is = ig$, and the original difference $V_1 = I_1 s$. We have further

$$I_1(R+s) = I\left(R+\frac{gs}{g+s}\right) = I\left(R+\frac{s}{1+\frac{s}{g}}\right).$$

The two currents I and I_1 are sensibly equal, as well as the differences of potential V and V_1, if the ratio $\frac{s}{g}$ is very small in comparison with unity.

Let us suppose even, in a more general manner, that the resistance s contains an electromotive force e, in the same direction for instance as the current, and let E be the total electromotive force. The intensity of the primitive current is

$$I_1 = \frac{E}{R+s},$$

and the difference of potential at the two points in question is

$$V_1 = I_1 s - e.$$

When the galvanometer is introduced, we have

$$V = ig = Is - e,$$
$$E = (I+i)R + Is,$$

from which is deduced

$$I=\frac{E+e\dfrac{R}{g}}{R\left(1+\dfrac{s}{g}\right)+s}.$$

If the ratio $\frac{R}{g}$ is itself very small, the two currents I and I_1 are sensibly equal; and unless the differences V and V_1 are extremely small, which would represent a very special case, they are virtually equal.

1019. Measurement of Polarization.—When a battery of electromotive force e_0 and resistance r_0 is closed by a circuit of resistance ρ, too weak for the polarization to attain its maximum value, the new values e and r give a current of strength

$$i=\frac{e}{r+\rho},$$

and the difference of potential V of the two poles is

$$V=i\rho=e\frac{\rho}{r+\rho}.$$

If the method of compensation (1012) be applied in the two cases by joining each time one pole of the battery to the point A (Fig. 205) and determining the position of the point C which corresponds to no current in the galvanometer, we get the ratios m_0 and m of the differences of potential e_0 and V to the electromotive force E of the principal battery. We shall thus have

$$\frac{V}{e_0}=\frac{e}{e_0}\frac{\rho}{r+\rho}=\frac{m}{m_0},$$

an equation which gives the ratio of the electromotive force of the polarized battery to its initial electromotive force.*

* Paalzow. *Pogg. Ann.*, Vol. cxxxv., p. 326. 1868.

In order to eliminate the resistance r, it is sufficient to replace ρ by another value ρ', and to determine the corresponding value m'. We have then

$$\frac{e}{e_0}=\frac{r+\rho}{m_0\frac{\rho}{m}}=\frac{r+\rho'}{m_0\frac{\rho'}{m'}}=\frac{\rho-\rho'}{m_0\left(\frac{\rho}{m}-\frac{\rho'}{m'}\right)}.$$

We might, moreover, determine the ratio of the differences of potential Δ and e_0 by an electrometer or by a galvanometer of high resistance.

1020. Investigation of a Battery at Work.—When a battery is at work it might be useful to investigate its properties, taking as much care as possible not to open it.

Let us consider the case of an effective electromotive force E and resistance R closed by a circuit of resistance ρ, and let I be the strength of the current. The difference of potential at the two terminals is

$$V=E-IR=I\rho=E\frac{\rho}{R+\rho}.$$

If the current is opened, the difference of potential V_0 between the same points, before the polarization has time to disappear, is

$$V_0=E.$$

The values of V and V_0 being determined by comparison with a standard, by means of an electrometer or a galvanometer of high resistance, we have

$$R=\rho\frac{V_0-V}{V}.$$

If the battery consists of n couples arranged in series, the electromotive forces of which are $e_1, e_2\ldots, e_n$ and the resistances $r_1, r_2\ldots, r_n$, the preceding experiment will give first

$$E=\Sigma e=V_0,$$
$$R=\Sigma r=\rho\frac{V_0-V}{V}.$$

The same test is then made for each of the elements. The differences of potential v and v_0, before and after opening the battery, will also give

$$v = e - \mathrm{I}r = v_0 - \mathrm{I}r,$$

whence

$$r = \frac{v_0 - v}{\mathrm{I}} = \rho \frac{v_0 - v}{\mathrm{V}}.$$

As an experimental verification, it is clear that the total electromotive force and resistance of the battery should be respectively equal to the sum of the values obtained for the different couples, which requires that we have

$$\mathrm{V}_0 = \Sigma v_0,$$
$$\mathrm{V}_0 - \mathrm{V} = \Sigma (v_0 - v),$$

or, more simply,

$$\mathrm{V}_0 = \Sigma v_0,$$
$$\mathrm{V} = \Sigma v.$$

A couple has no useful effect in the series unless its electromotive force is greater than the fall of potential produced by the resistance it introduces—that is to say, if the difference of potential v is positive, and we have

$$e > \mathrm{E} \frac{r}{\mathrm{R} + \rho}.$$

The mere sign of v would thus enable us to discriminate the couples which are in bad order, without its being necessary to open the circuit.

If all the elements are of the same kind, they have the same electromotive force, and only differ in resistance. We may then put $\mathrm{E} = ne$, and the preceding condition becomes

$$r < \frac{\mathrm{R} + \rho}{n}.$$

From this it follows, that if the internal resistance of an element is not less than the nth part of the total resistance, which is simply

indicated by the sign of v, it is advantageous to withdraw this couple from the battery.

1021. Method of Discharges.—When two points in an electrical system which has attained a permanent state are connected with the coatings of a condenser, the charge of the condenser is proportional to the difference of potential V of the two points in question. Further, if the permanent state is kept up by constant electromotive forces, and the losses of the condenser may be neglected, this difference of potential is the same as if the condenser had not been brought in.

The capacity of the condenser being C, the charge m is equal to CV. A comparison of the discharges measured by a ballistic galvanometer (883) will give the ratio of the electromotive forces. The method is equivalent to employing an electrometer or galvanometer of high resistance.

In order to avoid complicated corrections, it is advantageous that the swings to be compared be of the same order of magnitude, even when the electromotive forces are very different. Instead of using shunts, the inconveniences of which have already been noted (884), it is better to use standard capacities, such a value being used in each experiment that the quantities are of the same order.

1022. The experiment may be arranged so as to determine the resistance r of a battery. The poles of the battery are connected up with a condenser through a ballistic galvanometer. The swing δ of the needle is proportional to the electromotive force e of the battery in open circuit. Leaving this connection, the poles are joined by a shunt of resistance ρ. The swing δ of the needle is in the opposite direction, and it is proportional to the excess $e-e'$ of the electromotive force over the difference of potential of the poles in the new state. We deduce

$$r=\rho\frac{e-e'}{e}=\rho\frac{\delta'}{\delta}.$$

By means of two keys the changes are rapidly made, and polarization has not time to establish itself.

We may work in the opposite direction. The battery being first closed by a resistance ρ, the poles are connected with the condenser by the galvanometer. The shunt is then cut. Swings are thus obtained in the same direction, which are respectively

proportional to e' and to $e-e'$; but in this case the resistance and the electromotive force correspond to the closed battery.*

1023. Seat of the Electromotive Force.—In any battery, open or closed, the electromotive force should be regarded as the algebraical sum of the differences of potential existing in the circuit. With the exception of those which are due to the Thomson effects (276), these differences of potential should be sought for at the various bounding surfaces of solids and liquids.

Volta ascribed the seat of the electromotive force solely to the contact of the two metals, the function of the liquid being merely to reduce the metals of two successive pairs to the same tension, or, in other words, to the same potential. An ingenious experiment of Sir W. Thomson seems to confirm Volta's theory.†

Two insulated half-circles, or two half-rings, the one of zinc and the other of copper, are placed below a light needle suspended horizontally. They are brought almost in contact, and so that the surface of separation is exactly in the plane of equilibrium of the needle. When two points of the ring are joined by any conductor, and the needle is positively electrified, it turns towards the copper, and if it is negative towards the zinc. When the apparatus is well adjusted, these deflections on either side are equal for equal charges and of opposite sign to the needle. Experiment demonstrates then the general fact of the difference of potential of two metals in contact (186). But if, instead of connecting the two rings by means of a solid conductor, they are connected by a drop of water or of alcohol, the needle remains in the plane of equilibrium whatever be its charge, proving that the two metals are at the same potential.

In order to determine the difference of potential of the two metals, Sir W. Thomson connects them respectively with the two points of a conductor traversed by a current (1007), and seeks that position of the movable contact which reduces the needle to zero. He thus found that the difference of potential changes with the state of the metals. For instance, it increases considerably when the copper is oxidised by heating it and when the zinc plate is carefully scraped.

1024. Measurement of the Electromotive Force of Contact.—Case of Two Metals.—A general method of deter-

* Kempe. *Handbook of Electrical Testing*, p. 195.

† Sir W. Thomson. *Reprint of Papers on Electricity and Magnetism*, p. 317.—Jenkin. *Electricity and Magnetism*, p. 48.

mining the electromotive force of contact of two metals consists in using them as armatures of a condenser which is joined by a wire, and the charge of which is measured after they have been insulated from each other and put at a great distance apart (187). The charge is proportional to the product of the capacity by the difference of potential, or merely by the latter if the capacity is constant.

Kohlrausch* measures the charge by means of a Dellmann's electrometer; and, in order to reduce the measurements to a given standard, he makes three experiments by joining the two plates A and B (Fig. 209); first by a wire only; then by a wire in which is interposed a Daniell's element; and, thirdly, by the same wire

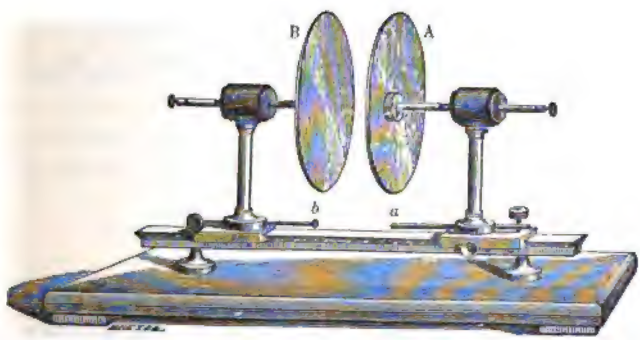


Fig. 209.

and the same element, but turned in opposite directions. He thus obtains three deflections corresponding to the differences of potential V_1, V_2, and V_3. If δV is the electromotive force of contact of the two metals, D that of the Daniell's element, we have

$$V_1 = \delta V,$$
$$V_2 = D + \delta V,$$
$$V_3 = D - \delta V;$$

from which is deduced

$$\delta V = 2D \frac{V_1}{V_2 + V_3},$$

* R. KOHLRAUSCH. *Pogg. Ann.*, Vol. LXXXII., pp. 1, 40, 1851; Vol. LXXXVIII., p. 465. 1853.

with the equation of condition

$$V_2 - V_3 = 2V_1.$$

The method requires that the plates are exactly at the same distance in the three experiments, and that the electrometer is graduated. By means of the stops *a* and *b* this first condition is satisfied.

This method has been employed in analogous conditions by different physicists. Pellat* greatly improved it by connecting the two plates by a conductor which contains, instead of a fixed electromotive force, a variable one, obtained by means of a contact moving along a wire traversed by a permanent current. The contact is regulated so that the charge of the two plates is zero. The electromotive force of contact is then equal and opposite in sign to that which the wire comprises. A single experiment is sufficient, and both the

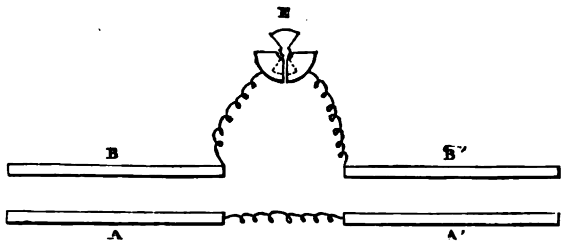


Fig. 210.

distance of the plates and the graduation of the electrometer need not be regarded.

1025. We may mention, further, an ingenious arrangement used by Professors Ayrton and Perry,† although it is more complex and less certain.

Four plates, A, A′, B, B′ (Fig. 210), form two systems of condensers. The lower plates, A and A′, are made of the two metals of which we desire to know the electromotive force of contact. The upper plates B and B′, both of brass, are insulated, and communicate respectively with the quadrants of an electrometer, the needle of which is kept at a very high potential.

* PELLAT. *Ann. de Phys. et Chimie* [5], Vol. XXIV., p. 5. 1881.—*Journal de Physique*, Vol. IX., p. 145. 1880.

† AYRTON and PERRY. *Transactions of the Royal Society, 1880*, p. 15.

A and A′ being in permanent connection, B and B′ are connected. The needle of the electrometer comes to zero. The connection between B and B′ being broken, the system of the two plates A and A′ is turned through 180°, so that the plate A′ comes under the plate B and A under B′. The deflection of the needle is proportional to the difference of potential between A and A′.

For let c and c' be the capacities of the plates B and B′, including the corresponding quadrants, q and q' the quantities of electricity taken by each of them in the first experiment, V the potential of A, $V+\delta V$ that of A′, and V′ the common potential of B and of B′; we have

$$q=c(V'-V), \qquad q'=c'(V'-V-\delta V).$$

After being turned through 180°, the four plates A, A′, B, and B′ have potentials represented respectively by U, $U+\delta V$, U′, and U″, which, since the charges and capacities of the two systems B and B′ have not altered, gives

$$q=c(U'-U-\delta V), \qquad q'=c'(U''-U).$$

From this follows

$$1=\frac{V'-V}{U'-U-\delta V}=\frac{V'-V-\delta V}{U''-U}=\frac{\delta V}{U'-U''-\delta V},$$

or

$$\delta V=\frac{1}{2}(U'-U'');$$

the difference $U'-U''$ is given by the electrometer.

This method presupposes that the capacity of the electrometer is independent of the deflection (813). Moreover, the necessity of bringing the plates exactly to the same distance requires a perfection in the mechanism which is realised with difficulty.

1026. Metal and Liquid.—In order to obtain the electromotive force of contact between a solid and a liquid, Hankel*

* Hankel. *Abhhandlung. der König. Sachs. Gesell.; Math.-Phys. Klasse.* 1861 and 1865.—*Pogg. Ann.*, Vol. cxv., p. 57, 1862; Vol. cxxvi., pp. 286, 440, 1865; Vol. cxxxi., p. 607, 1867.

places the liquid in a siphon (Fig. 211) which on one side is widened out as a funnel, the liquid being level with the edge of the funnel. The surface of the liquid forms one plate of the condenser, the other being the copper plate B, which, by means of the platinum wire p, is in connection with the metal M to be investigated. This dips in the second limb of the siphon, and is also put to earth by a platinum wire p''.

If the connections p and p'' are broken, and, raising the plate B, it is connected with an electrometer by the wire p', the charge is proportional to the difference of potential V_1 of the copper and of the liquid—that is to say, to the sum of the electromotive

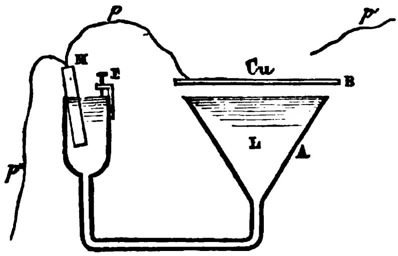


Fig. 211.

forces of contact of copper with the metal M, and of this again with the liquid L,

$$Cu|M + M|L = V_1.$$

The funnel is emptied, and instead of the surface of the liquid, a plate of the metal M is used, which is connected by a platinum wire with B and with the earth; for the same distance the charge observed corresponds to the difference of potential

$$Cu|M = V_2,$$

whence

$$M|L = V_1 - V_2.$$

If as lower plate A (Fig. 210) we take the surface of the liquid in question, connected by a siphon with the metal of the plate A′, Ayrton and Perry obtain the electromotive force of contact of the metal or of the liquid.

The vapour from the liquid, and which condenses on the upper plate, introduces a source of error which it is difficult to get rid of, and which cannot be allowed for.

1027. Two Liquids.—The method of Ayrton and Perry may be applied to the case of two liquids. The two lower plates A and A′ are replaced by the surfaces of the two liquids, which are connected by a siphon. In order to allow the vessels to be reversed, while only leaving a very small distance between the surfaces of the liquid and the upper plates, these are suspended by a frame in the shape of a parallelogram, so that they can be raised parallel to themselves. This complication adds to the difficulty of bringing the surfaces to exactly the same distance. Condensation of vapour on the upper plates is also to be feared.

MM. Bichat and Blondlot* used a method which is not open to these objections.

Let L and L′ be the two liquids to be compared. The two vessels X and Y, both containing the liquid L, communicate respectively by a platinum wire with the quadrants of an electrometer. The liquid L′, contained in the vessel Z, communicates with the vessel X by a siphon closed by a porous diaphragm, and filled with liquid L. The liquids of the two vessels Y and Z being brought to the same potential by a special device, the difference of potentials between the vessels X and Y is equal to the electromotive force of contact L|L′. The device in question consists in allowing the liquid L′ to flow out in a fine stream, and divide into droplets in the centre of a large tube, on the inside of which the liquid L flows continuously. The two vessels from which the liquids flow, communicate respectively with the vessels Y and Z, by means of siphons filled with the same liquid, and which therefore introduce no difference of potential.

1028. Maxwell was the first to draw attention to the nature of the phenomenon, which is measured in all methods in which bodies are separated by a dielectric layer. In the experiment on the electromotive force of contact of two metals M and M′, for instance, there are in reality three contacts to be considered—that of the two metals with each other, and that of each of the metals with the medium A in which it is immersed. We really measure then the sum

$$A|M+M|M'+M'|A=\Delta,$$

* Bichat and Blondlot. *Comptes rendus*, Vol. xc., pp. 1202 and 1293. 1883.—*Journal de Physique* [2], Vol. II., p. 533. 1883.

† Maxwell. *Electrician*. April 26th, 1879.

and this sum would only be equal to the true electromotive force $M|M'$ of the two metals so long as the two electromotive forces $A|M$ and $A|M'$ are zero, which is not at all demonstrated; or, in the opposite case, are equal, which is not probable. We may call Δ the apparent electromotive force, and this applies to all the preceding methods. Thus, in the method of Bichat and Blondlot, it is clear that the difference observed represents the sum

$$L|L' + A|L' - A|L.$$

Maxwell assumes that Peltier's phenomenon gives the true electromotive force between two metals. As the results obtained by calorimetric and by electrometric methods are very different (**248**)—the former being incomparably smaller than the latter, and sometimes of contrary signs—it would follow from this hypothesis that the electromotive force of contact between the bodies in question and air should form great part of the phenomenon observed.

This point of view is not in contradiction with the experiment of Sir W. Thomson (**1022**). The liquid L, interposed between the two halves of the ring, comes then into play, and it is seen that we have

$$A|M + M|L + L|M' + M'|A = 0,$$

or

$$M|L + L|M' = M|A + A|M',$$

which merely proves that the contact electromotive force of metal with air, and with an oxygenated liquid such as water and alcohol, are sensibly equal.

It would thus be very interesting to measure the electromotive forces in other conditions, and particularly without the intervention of a dielectric.

1029. The simplest plan would be to repeat the measurements in a vacuum; but the experiments would only be conclusive provided the vacuum were perfect. Pellat, by reducing the pressure to 1 cm. or 2 cm. of mercury, or by replacing the air by a gas which does not act on the metals, such as nitrogen or hydrogen, only found very slight variations of electromotive force. In the case of copper or of zinc the difference of potential rises when the pressure diminishes, and the variation is greater for oxygen than for hydrogen.

Brown,* on the contrary, found very great differences by using gases which attacked the metals, such as hydrochloric acid and sulphuretted hydrogen. The arrangement used is that of Sir W. Thomson's divided ring (**1025**). By placing the apparatus in a bell jar, in which air and sulphuretted hydrogen are alternately introduced, the needle deflects alternately to the left or right, until the copper is covered with a blue layer of sulphide of copper. But this permanent alteration of the surface of one of the metals raises doubts as to the conclusions which can be drawn from these experiments.

After having discussed the results furnished by electrostatic methods, Maxwell thus concludes :—

> "These experiments seem to show that the agreement between the results obtained by the ordinary methods for the electromotive forces of contact, and those obtained by immersing the metals in water, or in any other oxygenated liquid, is due less to the extreme smallness of the force between a metal and a gas, or between a metal and an electrolyte, than to the fact that the properties of air agree to a certain point with those of oxygenised electrolytes. And, in fact, if the active component of the electrolyte is sulphur, the results change altogether; and the same is the case if the air is replaced by sulphuretted hydrogen."

1030. M. Garbe† has deduced from the properties of surface tension, demonstrated by M. Lippmann, an ingenious method of measuring the electromotive forces of contact. This method, as Bichat and Blondlot ‡ observe, may furnish absolute values independent of the external medium.

We have seen that the surface tension of mercury in contact with a liquid, is a function solely of the difference of potential between the liquid and mercury, and this difference is null when the surface tension A is a maximum, for the capacity X of unit surface is then zero. The electromotive force E, which must be introduced between the two liquids is then equal, and of opposite sign to the pre-existing difference of potential.

In a capillary electrometer in the ordinary condition, the external electromotive force, which produces the maximum tension, is equal to the difference of potential of contact Hg|L of mercury with acidulated water. If the acidulated water is replaced by a liquid L′,

* J. BROWN. *Phil. Mag.* [5], Vol. VI., p. 142, 1878; [5], Vol. VII., p. 109. 1879.

† *Comptes rendus*, Vol. XCIX., p. 123. 1884.

‡ *Comptes rendus*, Vol. C., p. 791. 1885.

we shall have, in like manner, the value of $Hg|L'$ by the electromotive force E', which produces the maximum tension.

Let us now put the two liquids L and L' in two vessels, above a layer of mercury, and let them be connected by a siphon filled with either of the two liquids, and provided with a diaphragm. By measuring the electromotive force E_1 of the couple thus formed, and of which the two layers of mercury are the electrodes, we have

$$E_1 = Hg|L + L|L' + L'|Hg = E + L|L' - E',$$

from which is deduced the value of $L|L'$ as a function of the three electromotive forces determined directly.

This method gives results which differ completely, not only in magnitude, but also in sign, from those which the ordinary methods furnish. Such a divergence can only be explained by the fact that there is an electric difference between a liquid and air, as had been pointed out by Maxwell. This point of view is confirmed by results deduced from the consideration of the Peltier effects.

1031. The explanation of thermoelectric currents on the principle of Volta alone (**295**) leads to this conclusion—that all couples should have a uniform course; and in this case (**286**) the electromotive force of contact of the two metals should be proportional to the absolute temperature. M. Potier* arrives directly at this result.

Suppose two plates of zinc and copper, for instance, forming a condenser, are joined by a conducting wire, and kept at a constant temperature. The capacity of the system being C, and the difference of potential of contact H, the plates are brought nearer each other by an indefinitely small quantity, and the capacity then increases by dC; if the value of H is constant, a quantity of electricity passes from one to the other

$$dM = H dC,$$

and the thermal energy absorbed at the junction is

$$dQ = H dM = H^2 dC.$$

On the other hand, the electrical work of the system, which is at constant potential (**97**), is equal to the change of energy—that is to say,

$$d\left(\frac{1}{2}CH^2\right) = \frac{H^2}{2} dC,$$

* Potier. *Journal de Physique* [2], Vol. IV., p. 220. 1885.

so that the work of the external forces is

$$dW = -\frac{H^2}{2} dC.$$

As the current is infinitely weak, the heat disengaged, according to Joule's law, may be disregarded; the phenomenon is reversible, and we may apply Carnot's principle. Referring to (**645**), we have here $a = H^2$, $b = -\frac{H^2}{2}$, and equation (5) of (**646**) gives

$$\frac{H^2}{T} = \frac{\partial}{\partial T}\left(\frac{H^2}{2}\right) = H\frac{\partial H}{\partial T},$$

or

$$H = AT.$$

As experiment does not favour this conclusion, it follows that, in using a condenser, the difference of potential of the plates cannot in general be considered as constant; it depends on the quantity of electricity with which they are charged, and it must be assumed that, no doubt by the action of the surrounding medium, a phenomenon is produced analogous to the polarisation of electrodes.

1032. MEASUREMENT OF THE PELTIER EFFECT.—SOLIDS.—The Peltier phenomenon (**247**) may be considered as furnishing a general method for measuring local variations of potential. Between two points, whose difference of potential H is independent of the existence of a current, the energy furnished or absorbed by a current I in unit time is $W = IH$. If this energy is simply transformed into a quantity of heat Q, we have

$$JQ = IH,$$

and the determination of H reduces to a calorimetrical measurement. The only difficulty consists in eliminating the heat disengaged between the same points, in accordance with Joule's law.

If all the measurements have been made in C. G. S. units, the mechanical equivalent J of heat, expressed in grammes and centigrade degrees, is $4{\cdot}17 \times 10^7$ (**917**). Without changing the unit of heat, if the current is in ampères and the difference of potential H in volts, we have simply

$$IH = 4{\cdot}17\,Q, \qquad \text{or} \qquad H = 4{\cdot}17\,\frac{Q}{I}.$$

The ratio $\frac{Q}{I}$ represents the quantity of heat which corresponds to the passage of unit electricity, or a coulomb.

We are indebted to Le Roux* for very careful determinations of the absolute value of H for a great number of metals at the mean temperature 25°. For the couple formed by copper and an alloy containing 10 of bismuth and 1 of antimony, the experiment was made at 25° and 100°; the quantities of heat disengaged were in the ratio of 308 to 398.

The ratio 1·29 of the quantities of heat disengaged at the temperatures of 100° and of 25° does not much differ from the ratio 1·25 of the absolute temperatures; the electromotive force of contact H is then proportional to the absolute temperature, and the bismuth-copper couple should have a uniform course, which agrees with experiment (280).

The electromotive forces of contact are very slight. For the preceding couple, which gives the highest value, Le Roux obtained 0·0219 volt. Bellati† found that a coulomb liberates 0·0006065 thermal units in the iron-zinc couple at 13·8°, which agrees with 0·00253 volt.

1033. Solids and Liquids.—A current cannot traverse the bounding surface of a solid and an electrolysable liquid without producing a chemical action defined by Faraday's law (255).

If it be assumed that a coulomb decomposes 0·09316 mgr. of water (918), or 1035×10^{-4} grammes of hydrogen, the action of a coulomb on any body will be represented by the same fraction $0{\cdot}1035 \times 10^{-4}$ of its electrochemical equivalent expressed in grammes.

Let us again observe that, if the heat of combination of the body in question is q thermal units, the heat relative to a coulomb will be

$$q \times 0{\cdot}1035 \times 10^{-4}\text{c.},$$

and the corresponding electromotive force, expressed in volts,

$$4{\cdot}17 \times q \times 0{\cdot}1035 \times 10^{-4} = \frac{q}{10{,}000}\, 0{\cdot}432.$$

As nothing authorises us to suppose that the Peltier effect can then be neglected, we must in general assume that, concurrently with the chemical operation, there is a disengagement or an absorption of

* Le Roux. *Ann. de Chim. et de Phys.* [4], Vol. x., p. 201. 1867.

† Bellati. *Atti del R. Inst. Veneto* [5], Vol. v. 1879.

heat, and that the energy of the current IH is equal to the algebraical sum of these works.

The measurement of the heat disengaged on the electrode is very difficult, owing to its dissemination by conductivity and convection in the surrounding liquid. Bouty got over this difficulty very happily by taking as electrode the thermometer itself. In order to graduate the thermometer in thermal units, he winds round the bulb a carefully insulated wire, and, the apparatus being immersed in the liquid, he passes a current of known strength through the wire; the ascent of the column, corresponding to a known quantity of heat disengaged in each second at the surface of the bulb, is thus measured. The spiral being removed, the surface of the bulb is silvered, and a thin layer of copper, for instance, is deposited on it by electrolysis. The thermometer may then be used with another similar thermometer as electrodes in a copper bath.

At the contact of a copper electrode, for example, with a solution of copper sulphate, a disengagement of heat is observed if the electrode is positive, and a cooling if it is negative. As these quantities of heat are proportional to the intensity of the current, the phenomenon has all the characteristics of a Peltier effect. Bouty* has observed that, with such metals as copper, zinc, and cadmium, the effect is independent of the nature of the acid, and of the degree of concentration of the solution, provided they are not too dilute. For copper, the quantity of heat per coulomb is 0·05078, which would represent a difference of potential of 0·212 volt; zinc gives 0·241 volt. These numbers greatly exceed those which are found by the same method for the contact of metals with each other.

The positive electrode is the seat of the positive chemical work. The formation of an equivalent of hydrated cupric oxide disengages 19,000 thermal units, and the combination of this oxide with dilute sulphuric acid 9,200, or 28,200 in all. The quantity of heat produced at the electrode by chemical action is thus 0·292 for a coulomb, which corresponds to a difference of potential of 1·217 volt. The heat disengaged is the excess of the chemical energy over the heat absorbed by the rise of the electrical level. The excess of potential of the solution over that of the metal, or the electromotive force of contact, is thus

$$H = 1{\cdot}217 - 0{\cdot}212 = 1{\cdot}005 \text{ volt.}$$

* BOUTY. *Journal de Physique* [1], Vol. IX., p. 229. 1880.

An analogous calculation would be made for zinc sulphate, starting from the numbers 53,500 for the heat of formation of zinc sulphate; the electromotive force of contact is then

$$H' = 2{\cdot}309 - 0{\cdot}241 = 2{\cdot}068 \text{ volt.}$$

1034. Measurement of the Thomson Effect.—The Thomson effect is analogous to the Peltier effect, and, like it, is reversible with the direction of the current; it only differs in the fact that the fall of potential, instead of being localised at the surface of contact of two different substances at the same temperature, is produced between two portions of the same substance at different temperatures. We have explained the manner (285) in which Sir W. Thomson has verified the electric transport of heat, which follows as a necessary consequence.

Le Roux has measured this effect by an ingenious arrangement. Two bars of the same metal, AB, A'B' (Fig. 212), are arranged

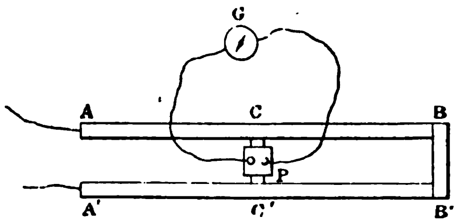


Fig. 212.

parallel to each other. The ends A and A' are kept in a bath at zero, and the ends B and B', joined by a plate of copper, in a bath at 100°. When the ends A and A' are connected with a strong battery, the temperature of the intermediate points is modified in accordance with Joule's law, and with the electric transport of heat. The difference in temperature of two similarly situated points C and C', which would correspond to twice the Thomson effect, is measured by a thermoelectric pile between the two bars. If the symmetry of the apparatus were complete, the galvanometer G of the thermoelectric circuit should be at rest if the current in the rods is suppressed; but this condition is not necessary, for, if the direction of the current is reversed, the difference of the deflections observed in the galvanometer corresponds in all cases to four times the Thomson effect. The experiment may be recommenced by replacing one by the other bath at constant temperature, and the mean of the results eliminates all defects of symmetry.

If it be assumed that the variation of temperature from one bath to the other is uniform, the effect observed in the centre of the bar is proportional to the specific heat of the metal for the mean temperature. Le Roux found first that this effect is proportional to the intensity of the current. It follows also from his experiments that the specific heat of electricity is zero for lead; positive for brass, copper, silver, zinc, cadmium, antimony, and an alloy of bismuth with a tenth of antimony; negative for tin, aluminum, platinum, argentan, and pure bismuth. In the two lists the metals have been arranged in the order of increasing values of the specific heat of electricity.

1035. THERMOELECTRIC ELECTROMOTIVE FORCES.—In a thermoelectric couple, the solderings of which are at different temperatures, the electromotive force E is the sum of the Peltier and the Thomson effects, of which it is the seat. We have seen how Sir W. Thomson* established, between the electromotive force, the Peltier effects, and the specific heat of electricity, the equation

$$\frac{dE}{dT}=\frac{dH}{dT}+\sigma'-\sigma.$$

If we compare the couple to a reversible thermal machine, we deduce

$$(1) \qquad H=T\frac{dE}{dT},$$

so that the change of potential at the soldering of the two metals may be deduced from the variations of the electromotive force of the couple which they form.

We have nothing to add to what has been said above on the measurement of thermoelectric electromotive force. Gaugain's experiments have shown that the electromotive force of a couple at work, as a function of the difference of temperatures, may be represented by a parabola (**373**). We shall only mention here two experiments, made with a view of verifying the formula (1), by M. Bellati for the iron-zinc couple, and by M. Bouty for the couple copper-copper sulphate.

* THOMSON. *Transactions of the Roy. Soc.*, Vol. XXI., p. 751. 1854.—*Math. and Phys. Papers*, Vol. I., p. 232.

For the couple iron-zinc, one of the junctions of which is at zero, the electromotive force, expressed as a function of the temperature t of the hot junction, is

$$E = 917{\cdot}77t - 1{\cdot}9488t^2.$$

From this is deduced, for the value of the Peltier effect at 13·8°, the number 0·00235 volt, instead of the number 0·00253 volt, which had been obtained directly (1032).

For thermoelectric couples metal-liquid, working between the temperatures t and t', Bouty found that the electromotive force is represented by an expression of the first degree

$$E = a + m(t - t'),$$

in which the parameter only depends on the nature of the metal; it follows that the Peltier effect is proportional to the absolute temperature. With copper, taking the EMF of a Daniell's element at 1·1 volt, experiment gives

$$m = 0{\cdot}00078 \text{ volt}.$$

From this is deduced 0·218 volt for the Peltier effect at 12°; direct measurement gave 0·212 volt.

1036. THERMOELECTRIC DIAGRAM.—The quantity $\left(\frac{dE}{dT}\right)_T$ has been called by Sir W. Thomson the thermoelectric power of the two metals at the temperature T.

Clausius gave the name *entropy* to a function such that the quantity of heat relative to an infinitely small transformation of a body is equal to the product of the corresponding absolute temperature by the variations of entropy. The thermoelectric power of the two metals presents a property analogous to the function of Clausius, and Maxwell* has proposed to call it *electric entropy.*

We have seen that, taking lead as standard of comparison, its specific heat of electricity being null (284), the curves which represent the thermoelectric power as a function of the temperature are, for most metals, straight lines, the angular coefficient of which represents the specific heat of electricity. The only exceptions

* MAXWELL. *Element. Treatise on Elect.*, p. 137.

seem to be iron and nickel, the curves for which have numerous sinuosities.

At the end of the volume is the diagram for the principal metals, compared with lead as neutral metal. If the temperatures are counted from the absolute zero, those curves give a very simple representation of the quantities of heat disengaged in the circuit, in various forms, during the passage of the current.

Let us suppose that the curve A (Fig. 213) represents the thermo-

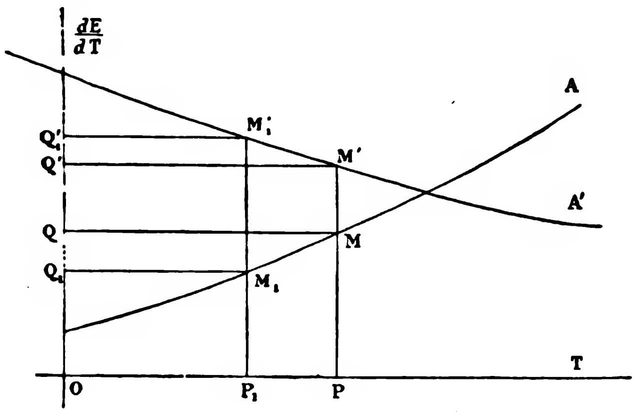


Fig. 213.

electric power $\frac{dE}{dT}$ of a metal A. Equation

$$H = T\frac{dE}{dT}$$

shows first that, at the temperature T, the difference of potential of contact with the neutral metal is represented by the area of the rectangle PQ. In like manner, the value of H, relative to the temperature T_1, is represented by the rectangle P_1Q_1. Lastly, as we have already seen (**279**), the electromotive force E, between the temperatures T_1 and T, is represented by the area of the curvilinear trapezium M_1P. The specific heat of electricity of the standard metal being zero, we have (**282**)

$$\int_{T_1}^{T} \sigma dT = H - H_1 - E,$$

that is to say, that this integral is given by the area of the curvilinear trapezium MQQ_1M_1.

Considering a circuit of any two metals A′ and A, the electromotive forces of contact H and H_1 are represented by the rectangles MQ' and $M_1Q'_1$, the electromotive force by the curvilinear trapezium $MM'M'_1M_1$, and the total differences of potential corresponding to the Thomson effect by the curvilinear trapeziums MQQ_1M_1 and $M'Q'Q'_1M'_1$.

These different surfaces also represent the calorific energies absorbed or liberated at the corresponding points of the circuit during the passage of unit current.

1037. Relation of the Electromotive Force to the Chemical Energy in Closed Couples.—In any closed hydro-electrical battery, the electrical work is evidently taken from the chemical energy; and we cannot conceive any battery in which the reactions, considered as a whole, are not exothermic. The expression for the sum of the chemical works for a current of intensity I is $I\Sigma Jap$ (**259**). On the other hand, if the current does no external work, the electrical work is represented by EI. If these two orders of work are equivalent, we have

$$E = \Sigma Jap.$$

In the Daniell couple with sulphates the chemical work reduces to the replacement of copper by zinc, equivalent for equivalent. This replacement corresponds to the disengagement of 25,300 for the chemical equivalent, which, from the preceding relations, gives for the electromotive force of Daniell's element

$$2{\cdot}53 \times 0{\cdot}432 = 1{\cdot}09 \text{ volt.}$$

a number which agrees perfectly with direct determinations.

It may be observed that if we assume that the difference of potential is null at the contact of two solutions of copper sulphate and zinc sulphate, the excess $H' - H$ of the differences of potential of the solution over the corresponding metal, leads (**1033**) to a value very near 1·06 volt for the difference of potential of copper and of zinc.

The agreement is also satisfactory for the couple zinc-cadmium and sulphates of Regnauld. So remarkable an agreement between theory and experiment had led Sir W. Thomson to propound the

following law:—*The absolute value of the electromotive force of a hydroelectric couple is equal to the mechanical equivalent of the chemical action which is developed in it by unit electricity.*

It must, however, be observed that the elements in question present exceptional conditions: they give rise to no disengagement of gas, to no secondary action; they do not heat during the passage of the current, or at any rate the calorific effects of the two electrodes are virtually equal and of opposite signs; they are, lastly, completely reversible. The agreement does not seem to be so complete with other couples.

1038. The application of the law requires an exact knowledge of the reactions which take place in the element when at work, which in the present state of our knowledge is at times impossible. Voltaic couples in which amalgamated zinc is combined with metals supposed to be unalterable, such as copper, silver, platinum, etc., have not the same electromotive force, although the same quantity of zinc is dissolved for unit electricity. The unattacked metal does therefore exert an influence which it seems difficult to define with precision.

A distinction must further be made between the direct effect of the current and the secondary effects. The direct effect is the separation of the elements of the electrolysis, the secondary effects depend on the nature of these elements and of that of the electrodes. They are due to the properties of the bodies concerned, and are, apparently at least, independent of the current. To this must be added changes of state, disengagement of gas arising from the decomposition of water, the crystallisation of a salt formed by electrolysis in a saturated solution, etc. Among all these actions, which are we to allow for and which must be disregarded in calculating the electromotive force?

Let us consider with Berthelot* the electrolysis of potassium sulphate. For each unit of electricity, an equivalent of oxygen is liberated at the positive and an equivalent of hydrogen at the negative electrode. Sulphuric acid, further, is liberated about the former and potass about the latter. This result is usually explained by assuming that the decomposition of potassic sulphate takes place in the same way as that of copper sulphate, with this difference, that the potassium set free at the negative electrode decomposes water by a secondary action, and in turn liberates hydrogen. The sulphuric

* Berthelot. *Comptes rendus*, Vol. XCIII., p. 661. 1881.—*Ann. de Chim. et de Phys.* [5], Vol. XXVII., p. 89. 1882.

acid on the one hand, and the reformed potass on the other, diffuse through the liquid and reproduce the original potassium sulphate.

Are we to attribute solely to the effect of the current the liberation of an equivalent of potassium, the other actions being considered as purely chemical and independent; or is there a simultaneous decomposition of an equivalent of potassic sulphate into acid and base and an equivalent of water into hydrogen and oxygen; or again, given that the acid and base are constantly recombining, and that the final result reduces to the disengagement of oxygen and hydrogen, the mere decomposition of an equivalent of water? The work corresponding to these different cases would be 98,000, 50,200, or 34,500 thermal units. Berthelot has shown that it is the second which is realised. He associates constant elements, and of graduated electromotive forces, such as Daniell's or Regnauld's element, and investigates the least electromotive force which produces decomposition. The electrolysis of potassium sulphate requires an electromotive force corresponding to at least 50,000 thermal units. This electromotive force is insufficient when mercury is used as negative electrode, as the potassium amalgamates instead of decomposing the water. The electromotive force necessary in that case approaches 98,000 thermal units, though still inferior; for we must allow for the heat of amalgamation of potassium.

An interesting case is that in which electrolysis may take place in several different ways. Experiment shows that if the electromotive force is progressively increased, the reaction which absorbs least heat shows itself first. Each kind of decomposition appears in turn when the electromotive force has attained the desired value, although the preceding ones continue to act.

The same takes place for a mixture of salts, and a method of analytical separation of certain metals has been based on the action of gradually increasing electromotive forces.

Berthelot concludes from his experiments that the sum of the energies necessary to produce electrolysis must comprise all reactions which take place in the passage of the current, without distinguishing between what are called primitive and what are considered as secondary reactions; at any rate, as regards those which take place in the immediate vicinity of the electrode, the experiment of potassium sulphate showing that we need not take into account the recomposition of the potassium sulphate which is due to the diffusion of acid and base across the electrolyte.

1039. If we compare electromotive forces of different couples, calculated according to these rules, with the electromotive forces

measured directly, differences are observed which it is difficult to explain, owing to insufficient knowledge either of the reaction or of the data necessary for calculating them. Chemical energy is generally greater than electrical energy, but it is sometimes less. In the former case the couple becomes heated during the course; it becomes cooled in the second, and borrows heat from the external medium.

Braun* has investigated from this point of view a great number of elements on the Daniell type, where the sulphates are replaced by chlorides, bromides, and iodides, and the copper by mercury and silver. He finds that Thomson's law often fails, and particularly in the case of couples in which the salt which surrounds the positive pole is insoluble.

Braun assumes that only a portion of the chemical heat can be converted into electrical work, and that for each compound there is a constant ratio between these two quantities, which he calls the "useful effect." The heat which is not converted into electrical work heats the couple and produces the rise of temperature which is ordinarily attributed to secondary actions. From this point of view a couple is analogous to a gas machine in which, from Carnot's principle, only a fraction of the disposable chemical energy is converted into mechanical work. Pushing the analogy still further, Chaperon† assumes that Braun's coefficient is that which would be defined by Carnot's theorem for a thermal machine working between the actual temperature and the temperature of dissociation of the compound; but the experimental data which are necessary for the verification of this hypothesis are at present wanting.

1040. Von Helmholtz‡ has endeavoured to bring these phenomena within a strict theory only taking into consideration reversible couples; and he arrived at this conclusion, that chemical energy is not in general transformable into electrical energy. We shall give this theory in the form in which it has been stated by M. Lippmann.§

* BRAUN. *Wiedemann's Annalen*, Vol. v., p. 182, 1878; Vol. XVI., p. 561, 1882; Vol. XVII., p. 593, 1882.

† CHAPERON. *Comptes rendus*, Vol. XCII., p. 786. 1881.

‡ HELMHOLTZ. *Wiedemann's Annalen*, Vol. III., p. 201. 1877.—*Wissenschaft. Abhandl.*, Vol. I., p. 840.

§ LIPPMANN. *Comptes rendus*, Vol. XLIX., p. 845. 1884.

We may consider the state of an element as defined by three independent variables—the temperature T, the quantity x of electricity which has traversed it starting from a certain initial condition, and the degree of concentration of the solutions. In order to allow for this latter variable, suppose the system contained in the barrel of a pump in which is water in a state of saturation. By raising or lowering the piston, the concentration can be varied at pleasure, and the volume v of the system taken as a third independent variable. If the couple can be regenerated by the current, we may make it traverse a closed cycle. In this case the work produced $-W$ is equivalent to the thermal energy absorbed JQ. Let p be the maximum elastic force for the temperature T, E the electromotive force of the couple, U the internal energy of the system; generalising the reasoning used in **645** and **646**, we see that the expression

$$dU = JdQ + dW$$

should be an exact differential. We have further

$$-dW = pdv + Edx,$$
$$dQ = cdT + l_1 dx + l_2 dv,$$

c being the thermal capacity of the couple, l_1 and l_2 the coefficients defined by the equation itself.

If the external resistance is so great that the heat disengaged in virtue of Joule's law may be neglected, we may consider the cycle as reversible, and apply Carnot's principle.

The influence of the concentration of the liquid may be studied by the properties of the coefficient l_2. Consider only the independent variables T and x; observing that in the present case

$$a = Jl_1, \qquad l = Jc, \qquad b = E,$$

equation (5) of **646** gives

$$Jl_1 = T\frac{\partial E}{\partial T}.$$

From this follows the important fact that to keep an element of a given concentration at a constant temperature, a thermal energy must be imparted to it of $T\dfrac{\partial E}{\partial T}$ for each unit of electricity

which traverses it. If q is the sum of the chemical energies for unit electricity, we shall have

$$q = \mathrm{E} - \mathrm{T}\frac{\partial \mathrm{E}}{\partial \mathrm{T}}.$$

We have seen that the electromotive force of an ordinary Daniell's element varies very little with the temperature, and we shall therefore virtually have $q = \mathrm{E}$; but this is not the case for all couples of the same kind. Thus, when silver is substituted for copper, we have for the ordinary temperature $\frac{d\mathrm{E}}{d\mathrm{T}} = -0{\cdot}0012$ volt. It follows that E is less than q by about 0·36 volt, which is moreover confirmed by experiment. For Latimer Clark's element we have about $\frac{d\mathrm{E}}{d\mathrm{T}} = -0.008$ volt.

When the electromotive force increases with the temperature, it exceeds the sum of the chemical energies. The couple tends then to cool when the current passes. This is the case with Von Helmholtz's calomel couple, in which the electromotive force slightly increases with the temperature, or an analogous couple in which chlorine is replaced by bromine. The electromotive force measured is about 1·7 times that deduced from the chemical heat.

1041. In order that a couple shall strictly satisfy Thomson's law, we must have $\frac{\partial \mathrm{E}}{\partial \mathrm{T}} = 0$, or $l_1 = 0$. From this condition it follows from equation (4) of **646** that

$$\frac{\partial c}{\partial x} = 0.$$

The partial differential represents the variation of the thermal capacity of the couple in consequence of the passage of unit electricity. The condition amounts to saying that the capacity of the system is the same whether the chemical elements which enter into the reactions are free or combined—in other words, that for the set of reactions produced by the current, the law of Wœstyn and of Kopp is verified. Hence only those couples which satisfy this law satisfy also the law of Thomson.

But Berthelot has shown* that Wœstyn's law only holds throughout a series of transformations as long as they do not bring about changes of state. In the contrary case, the heat of combination is moreover changeable with the temperature. It is just those elements with a solid depolarizer, such as mercuric sulphate, mercurous chloride, and argentic chloride, which are most sensitive to changes of temperature, and which diverge most widely from Thomson's law. The two orders of phenomena are therefore connected: if the law of thermal capacities is verified, the chemical heat and the electromotive force are equal to each other, and are independent of the temperature; in the opposite case, they are unequal, and vary with the temperature.

Numerous experiments have been made by Crapski† and Gockel‡ to verify the conclusions from Von Helmholtz's theory. The values of E and of $\frac{\partial E}{\partial T}$ have been measured for a great number of elements. If the values obtained for the expression $T\frac{\partial E}{\partial T}$ be compared with the difference between the chemical energy and the electromotive force, it is found that the verifications always succeed as regards the direction of the phenomenon, but not when we come to compare the numbers. The quantity of heat represented by $T\frac{\partial E}{\partial T}$ is never more than a fraction of the positive or negative difference between the chemical energy and the electromotive force. It is scarcely possible, in most cases, to ascribe these divergences to experimental errors; and these are some obscure points which the theory must clear up.

* BERTHELOT. *Essai de Mécanique Chimique*, Vol. I., p. 110. 1879.
† CRAPSKI. *Wiedemann's Annalen*, Vol. XXI., p. 209. 1884.
‡ GOCKEL. *Wiedemann's Annalen*, Vol. XXIV., p. 612. 1885.

CHAPTER V.

MEASUREMENT OF CAPACITY. DIELECTRICS.

1042. Characteristics of Condensers.—The capacities to be measured are almost always those of condensers (78), the type of which is the Leyden jar. They consist of two armatures, separated by a dielectric, and raised to different potentials. Submarine or subterranean cables, which consist of a system of conducting wires immersed in a layer of gutta-percha, satisfy this condition, and frequently possess great capacities. The conducting wire or the core of the cable forms the internal armature; the outer coating is formed either of a metal sheath which protects the cable, or by the earth or water in which it is immersed.

Of all forms, the Leyden jar is that best suited for keeping a charge of electricity. Certain qualities of glass appear, at ordinary temperatures, to be absolutely impermeable to electricity. Franklin kept electricity for several months in a Leyden jar, the neck of which had been sealed. Sir W. Thomson repeated this experiment with a sealed flask, the internal armature of which was a layer of sulphuric acid; the loss was inappreciable even after several years.

The use of glass, however, makes the apparatus too fragile, and it is not possible to obtain plates so thin as to have great capacity in a small space. Condensers in sheets are then much better. Alternate layers of tinfoil and of thin sheets of mica, or sheets of paraffined paper, are superposed on each other, all the tinfoil sheets of even order projecting on one side, and the odd ones on the other; they are connected separately, and thus furnish in a small volume very large surfaces close together.

All solid or liquid dielectrics present, in a greater or less degree, the property (which is still imperfectly understood) of *absorbing* electricity. If we take a charged condenser, the dielectric of which is not a gas, and after having discharged it by connecting the armatures for a few moments, again insulate the armatures, we observe that it then spontaneously acquires what is called a *residual* charge, which

depends not merely on the duration and strength of the primitive charge, but also on the previous charges it has received.

If a Leyden jar, for instance, has had first a positive charge for several weeks, then a negative charge for twenty-four hours, and a new positive charge for five minutes, the residue may give oscillations of potential which are alternately positive and negative.* Everything happens as if electricity gradually penetrated into the dielectric, to become dissipated by a displacement in the contrary direction, when the armatures are brought to the same potential.

These residual charges have been observed since the invention of the Leyden jar. The observations of M. Gaugain,† in particular, have shown that—

1st. The charge of a condenser for a given difference of potential increases with the duration of the connection with the source. This charge is only definite for a pretty short contact; it is then said to be *instantaneous.*

2nd. The instantaneous discharge—that is to say, that which corresponds to a connection of the armatures for less than two seconds—is virtually equal to the instantaneous charge.

Apart from losses by conduction, the residual charge is virtually equal to the excess of the total charge over the instantaneous discharge.

Clausius‡ and Maxwell§ have endeavoured to ascribe the phenomena of electric absorption to the heterogeneity of the dielectric. Their theories agree in showing that there would be no absorption in a perfectly homogeneous medium, and the fact has been verified by Professor Rowland for Iceland spar, the natural substance which suggests itself as purest and most homogeneous. Other natural crystals gave greater or less residues; the absorption of quartz was found to be the ninth of that of glass.‖

When quantities of electricity are to be exactly measured, air-condensers are the only ones which offer complete certainty, provided that no dust gets between the two plates. Unfortunately, they have only a small capacity.

1043. Standards of Capacity.—In the electrostatic system (607) the capacity of a conductor is a length, and it might be

* Sir W. Thompson. *Congrès Intern. des Électriciens.* Paris, 1881, p. 217.

† Gaugain. *Ann. de Chim. et de Phys.* [4], Vol. II., p. 264. 1864.

‡ Clausius. *Pogg. Ann.*, Vol. LXXXVI., p. 337. 1882.

§ Maxwell. *Electricity and Magnetism*, Vol. I., p. 376.

‖ Rowland and Nichols. *Phil. Mag.* [5], Vol. XI., p. 414. 1881.

determined from the geometrical dimensions of the bodies employed.

The problem is easily solved in certain cases—for example, a sphere at an infinite distance from any conductor; the capacity is then equal to the radius (73). But these conditions are impossible to realise, and external objects—for instance, the sides of the room—greatly increase the real capacity.

This difficulty is avoided with a condenser formed of two concentric spheres* (77). If R and R_1 are the internal and external radii, the value of the capacity is

$$C = \frac{RR_1}{R_1 - R} = \frac{R}{1 - \frac{R}{R_1}}.$$

Sir W. Thomson has used a standard of this kind. The radii were deduced from the weight of water contained in the external sphere alone, and in the interval between the two spheres when they were placed concentrically. Allowance must also be made for the insulating wedges which support the inner sphere, and which do not act like the air which they displace; allowance must further be made for the orifice which must be made in the outer armature, to allow passage to the rod which communicates with this internal sphere, as well as the influence of this rod. These corrections can only be made approximately. In the apparatus of Sir W. Thomson, for which $R_1 = 5{\cdot}857$ cm., $R = 4{\cdot}511$ cm., and therefore $C = 63{\cdot}264$ cm., the corrections amounted to $0{\cdot}255$ cm., and raised the capacity to $63{\cdot}519$ cm.

Concentric cylinders (80) might also be used, but in this case a correction for the ends must be introduced into the formula.

Condensers formed of two parallel planes are to be preferred; it is comparatively easy to ascertain if the two surfaces are truly plane, and to measure accurately the distance between them. It is to be observed, however, that the density of the electric layer is greater at the edge than in the centre, and that therefore the true capacity is greater than that deduced from the dimensions. Further, we do not consider the electricity which exists on the external surface of the disc, and which, in fact, would not exist if the system were infinitely

* This formula holds particularly for an insulated sphere in the centre of a hall of mean radius R.

distant from all other conductors. We have seen how Sir W. Thomson gets over this difficulty by using a *guard ring*. The influence of the edges is not altogether eliminated in consequence of the small interval which must be left between the plate and the ring; this is allowed for by replacing the area a of the movable plate by the mean between this area and the aperture a' of the ring. The capacity C formed by the plate a, with a parallel disc at a distance e, and of such dimensions in reference to the plate as to be considered infinitely small, is expressed by

$$C = \frac{a + a'}{8\pi e}.$$

1044. It is desirable in practice to have the capacity of a condenser in electromagnetic units; as the measurement deduced from the dimensions is expressed in electrostatic units, it is necessary to know exactly the ratio of the units in the two systems (610). It is best to determine the capacity of standards directly, by electromagnetic methods.

The capacity of a condenser being the ratio of a charge of electricity to a difference of potential, the measurement of capacity in absolute value will be determined by that of a charge (883) and of an electromotive force (1000).

We observe, further, that the capacity may be regarded (609) as the quotient of a time by a resistance, or of the square of a time by a coefficient of induction. The measurement of a capacity may thus be referred to that of a resistance, or of a coefficient of induction.

The practical unit of capacity is the microfarad equal to 10^{-15} C.G.S. units (613).

Sheet condensers are made which have only very slight absorption, and which, conveniently subdivided, form boxes analogous to resistance boxes.

The association of capacities in cascade (85) gives a result comparable with that of conductivity boxes (928). The reciprocal of the capacity of a battery of condensers arranged in cascade is equal to the sum of the reciprocals of the capacities which form the cascade.

It is interesting to remember that Cavendish had used an arrangement analogous to the actual boxes.* He used Franklin's

* CAVENDISH. *Electrical Researches*, published by Maxwell, p. 157. Cambridge, 1879

panes—that is to say, condensers made of a glass plate with tinfoil on each side, and having definite ratios to each other. He had observed that the influence of the edges prevented the capacities from being proportional to the surface; and he found that, to allow for this effect, it was sufficient to add to the real surface of the tinfoil a circular band 1·5 mm. in breadth for glass 5 mm. thick, and 2·25 mm. for glass 1·7 mm. thick.

1045. Sliding Condensers.—It is often useful to have capacities which can continuously vary. This is what is obtained with a plate condenser provided with a guard ring, and in which the disc, which moves parallel to itself, may be displaced in the direction of the perpendicular by a micrometric screw. The same result is arrived at more conveniently with condensers in which one of the armatures moves parallel to itself, so as to vary the extent of the two surfaces without altering the distance which separates them.

Consider in particular the system of conductors discussed in **98**, consisting of two cylindrical envelopes A and B (Fig. 214) with a

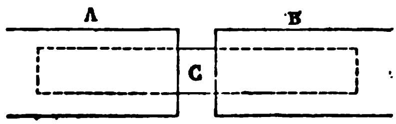


Fig. 214.

common axis and a cylinder C having the same axis as A and B and moving along this axis.* The envelope B being put to earth, the cylinders A and C are connected with each other and insulated. Let C_0 be the capacity of the system formed by the union of A and C when the system is in a given position which serves as mark, α the capacity of unit length of the condenser CB for the mean region where the distribution of the densities is uniform. If the cylinder C be made to slide towards the right by a length x, the capacity of the system AC becomes $C_0 + \alpha x$; it varies then proportionally to the quantity x. This would also be the case for the envelope B, if it were alone insulated, and the system AC communicated with the ground.

* The idea of this condenser is due to Sir W. Thomson. See also Gibson and Barclay, *Transactions of the Roy. Soc.*, *1871*, p. 573.

If R and R_1 are the radii of the internal and external cylinders, the value of the coefficient a in electrostatic units is*

$$a = \frac{1}{2l.\frac{R_1}{R}}.$$

The capacity C_0, which corresponds to the standard or zero of the scale, is determined by comparison.

The capacity of the sliding condenser is somewhat small; but a suitable capacity may always be added, and the first only used to effect the adjustment.

In order, finally, to avoid any perturbation arising from adjacent conductors, the insulated part AC is enclosed in a conducting cylinder in connection with the earth.

Fig. 215 represents a sliding condenser used by Sir W. Thomson,

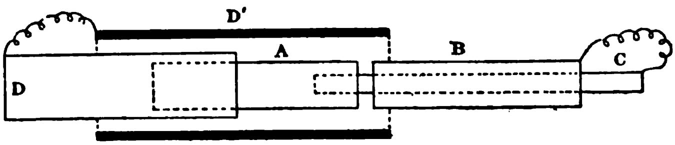


Fig. 215.

which, by a double adjustment, enables us to obtain a given capacity with great approximation. The cylinder A is insulated, the cylinders B and C communicate with each other and with the earth, as well as with the cylinders D and D'; the cylinders D and C are movable. The capacity of A increases when the cylinder D or the cylinder C is pushed in, but much more in the first case than in the second for the same displacement. The first motion will enable us to obtain an approximate equilibrium in a comparative experiment, and the adjustment is effected by moving the cylinder C in either direction.

1046. Comparison of Two Capacities. Methods of Opposition.—A method of opposition employed so long ago as last century by Volta and by Cavendish enables us to verify the equality of two condensers. They are charged to equal and opposite

* In the condenser employed by Gibson and Barclay the central cylinder was moved by a screw sliding in a groove of the envelope A, in front of a divided scale. Here $R_1 = 2\cdot4807$ cm., $R = 1\cdot2515$, which gives $a = 0\cdot6514$. The envelopes A and C were 30 cm. in length, and the cylinder 36 cm.

potentials, and then discharged on each other. If their capacities are equal, they will both be brought to the neutral state.

Let us consider, for instance, two Leyden jars the internal armatures of which are A and A′ and the external armatures B and B′. A is connected with B′ and B with A′; and one of these systems being put to earth, the other is electrified by a source at high potential. The connections with the source and with the earth are then broken, and the internal and external armatures are connected respectively with each other. The two capacities are equal provided the jars are entirely discharged. In this form the method is only exact provided the capacity of the condensers is independent of the choice of the armatures—that is, if they could be considered as quite closed. In condensers with the guard ring the acting surfaces alone should thus be compared—that is, the surface of the plate comprised within the guard ring.

A simple method of charging the two surfaces to equal and opposite potentials consists in putting them in communication re-

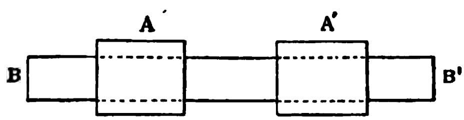


Fig. 216.

spectively with the two poles of a battery the centre of which is to earth. After having broken connection with the battery, the two capacities are connected. The charge is reduced to zero if they are equal.

1047. PLATYMETER.—Sir W. Thomson* has given this name to a double condenser formed of two cylindrical rings A and A′ (Fig. 216) of the same length and radius, perfectly insulated, and placed about a cylinder BB′ of the same axis which is connected with an electrometer or an electroscope. Keeping at first the cylinder B in connection with the earth, one of the rings A is charged to a potential V; connection with the source and with the earth is then broken. If the two rings are then joined, the common potential becomes $\frac{V}{2}$ by the division of the charges,

* Sir W. THOMSON. *British Association Report, 1855.* Glasgow.—See also GIBSON and BARCLAY, *Transactions of the Royal Society, 1871*, p. 570.

and the electroscope shows that the potential of the cylinder B is zero. Excepting for the unequal influence of the edges, the same would take place if the rings A and A′, while always of the same radius, were of different lengths, their capacities being proportional to their lengths. It must be assumed, however, that these rings are so far from each other, that their reciprocal inductive action may be neglected in comparison with that which they exert on the internal cylinder.

In order to compare any two capacities C and C′, they may be connected separately with the two rings A and A′ of the platymeter. If the experiment, repeated in the same conditions, shows that the potential of the cylinder BB′ is zero after the two rings have been connected, it follows that the total capacity of the two systems are in the ratio of the capacities of the rings, which gives

$$\frac{a}{a'}=\frac{C+a}{C'+a'}=\frac{C}{C'}.$$

We may then verify the equality between a and a'. It is sufficient to permute the capacities C and C′ in respect of the rings. If equilibrium still exists, we have

$$\frac{C'}{C}=\frac{a}{a'},$$

and therefore $a=a'$.

When this condition is not realised, and equilibrium has first been established with the capacities C and C′, and then in a second experiment with the capacities C″ and C, equations

$$\frac{C}{C'}=\frac{a}{a'}, \qquad \frac{C''}{C}=\frac{a}{a'},$$

give

$$C=\sqrt{C'C''}.$$

The experiment is analogous to that which consists in determining the weight of a body, the arms of which are unequally long.

1048. Balance of Capacities.—This name may be given to several experimental arrangements which suggest those of Wheatstone's bridge.

In the method used by De Sauty,* two condensers C and C′ (Fig. 217) take the place of the two branches *b* and *b′* of a Wheatstone's bridge, their external armatures being in connection with the earth. The two resistances *a* and *a′* are adjusted so that, by opening or closing the key K which sets up connection with the battery, there is no current in the galvanometer.

The condition of equilibrium requires evidently that the extremities B and B′ of the bridge are at the same potential—that is to say, that at a given moment the charges Q and Q′ are proportional

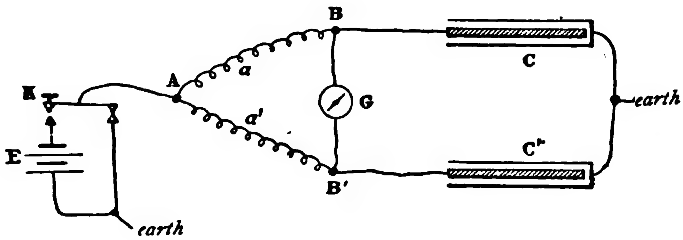


Fig. 217.

to the corresponding capacities C and C′. As the charges are proportional to the currents which produce them, and these are inversely as the corresponding resistances *a* and *a′*, it follows that

$$\frac{C}{C'}=\frac{a'}{a}.$$

It is clear that the galvanometer may be replaced by an electrometer.

This method is good for capacities of mean value, and in which electrical absorption has but little influence. It requires, in fact, that the duration of the charge is the same for the two condensers. Thus it cannot be applied to large capacities, such as submarine cables. This is apparent from the fact that the adjustment of resistances which is suitable for the charge is not so for the discharge.

1049. Sir W. Thomson † has pointed out two methods which do not depend on the time of the charge, but simply on the final state

* L. Clark and R. Sabine. *Electrical Tables and Formulæ*, p. 62. 1871.

† Sir W. Thomson. *Journal of the Society of Telegraph Engineers.* Vol. I., 5, p. 394. 1873.

of equilibrium, and which consequently are applicable to capacities of all kinds.

The first requires the use of three condensers of comparison, one of which at least has a variable capacity.

Let C, C', C_1, C_1' be the four capacities to be compared (Fig. 218). Two of them C and C_1 are charged to the same potential V_0, and then C is connected with C', and C_1 with C_1', all the outer coatings being to earth. Let V and V_1 be the potentials on either side; we have

$$(C+C')\,V=CV_0,$$
$$(C_1+C_1')\,V_1=C_1V_0.$$

If the two potentials V and V_1 are equal, a galvanometer inserted between the two systems shows no current; we have then

$$\frac{C}{C_1}=\frac{C+C'}{C_1+C_1'}=\frac{C'}{C_1'}.$$

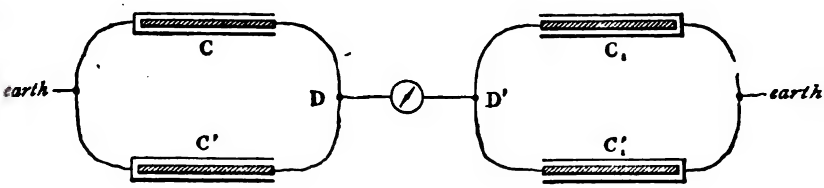


Fig. 218.

It is advantageous to replace the galvanometer by an electrometer; for, as each system retains its charge, there is every opportunity of adjusting the capacity of the variable condenser so that the condition is satisfied.

In the case in which the galvanometer is used, we must wait until the charge is in equilibrium before closing the ~~current~~ of the galvanometer between D and D'. This time may amount to several seconds when we are dealing with large capacities and high resistances.

The second method only requires one standard of comparison, which, however, must be unchanged. The outer coatings of the two capacities C and C' to be compared (Fig. 219) are insulated and connected with each other by a wire D, and the internal armatures are in relation with the ends of a considerable resistance AB

traversed by a permanent current. The point P of the resistance AB is sought at which, when the wire of the galvanometer G is attached connecting P and D, the needle is stationary. The position of the point P does not change when the current of the battery is reversed by the commutator M.

As the charge of the two condensers is the same, in consequence of their being connected (85), and if V and V′ are the potentials of the points A and B, and V_1 that of the external armatures,

$$C(V-V_1)=C'(V_1-V'),$$

or

$$\frac{V-V_1}{C'}=\frac{V_1-V'}{C}.$$

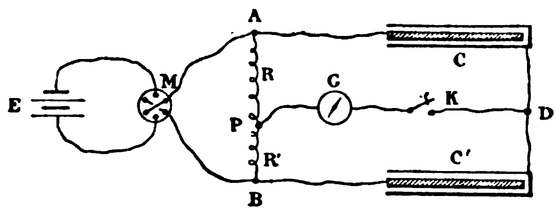


Fig. 219.

On the other hand, if x be the potential at the point P, R and R′ the resistance AP and PB, we have also

$$\frac{V-x}{R}=\frac{x-V'}{R'}.$$

In order that there be no current in the galvanometer, the potentials x and V_1 must be equal, which requires the condition

$$\frac{C}{C'}=\frac{R'}{R}.$$

It is again advantageous to replace the galvanometer by an electrometer, which makes it easier to find by trial the position of equilibrium.

This method is susceptible of great accuracy, and lends itself very well to a verification of the standards of capacity.

1050. Measurement of Potentials.—In order to compare two capacities C and C′, we may compare by an electrometer the potential V which the first acquires for a definite charge M of electricity, with the potential V′ of the system when the two capacities C and C′ are connected; we have, in fact (83),

$$M = CV = (C + C')V',$$

from which follows

$$\frac{C'}{C} = \frac{V - V'}{V'} = \frac{V}{V'} - 1.$$

This, for instance, is the method which Faraday* used to determine the ratio of two spherical condensers of the same dimensions, one containing air and the other a solid dielectric. By means of the torsion balance (801) Faraday measured the quantities of electricity taken by an insulated ball from the same point of the first condenser before and after contact with the second. The ratio of the charges was equal to the ratio of the potentials V and V′.

If the ratio $\frac{V}{V'}$ be determined directly by a graduated electrometer, and the capacity γ of the instrument cannot be neglected in comparison with that which is to be estimated, the former must be eliminated. For this the experiment must be repeated by first charging the electrometer alone to a potential U_0 and then connecting it with a capacity C, which lowers the potential to a value U; we have then

$$\frac{C}{\gamma} = \frac{U_0}{U} - 1.$$

As the comparison of the capacities C and C′ gives

$$\frac{C'}{C + \gamma} = \frac{V}{V'} - 1,$$

it follows that

$$\frac{C'}{C} = \left(\frac{V}{V'} - 1\right)\left(1 + \frac{1}{\frac{U_0}{U} - 1}\right).$$

* Faraday. *Experimental Researches*, Vol. I., p. 371.

We have again assumed that the capacity of the electrometer is independent of the deflection, which is not strictly correct. Care must finally be taken that the capacities to be compared are without inductive action on one another, and that the capacity of the connecting wires may itself be neglected.

The method becomes very exact, especially for capacities which are very near each other, when we work by opposition. The charges of the two capacities being M and M′ when they are joined crosswise, and when one of the common armatures is raised to potential V_0, we have

$$M = CV_0, \qquad M' = C'V_0.$$

Joining the armatures of contrary signs, the final potential is defined by the equation

$$M - M' = (C + C')V;$$

it follows that

$$\frac{C'}{C} = \frac{V_0 - V}{V_0 + V}.$$

If the capacities are very near, we may write

$$\frac{C'}{C} = 1 - 2\frac{V}{V_0}.$$

1051. Measurement of Charges.—We may also deduce the ratio of two capacities from the ratio of the charges which they acquire for the same potential, or, more generally, in the case of condensers, from the ratio of the charges for the same difference of potential between their armatures. This is the method employed by Gaugain* in an important research on the relations between the distribution of statical electricity in a system of conductors and the permanent currents in a correlated system (213).

The quantities of electricity were measured by a discharge electrometer (824) with which the electrified bodies were connected by a cotton thread. As the discharge is not complete, we may allow for the residue which corresponds to the final state of the

* Gaugain. *Ann. de Chim. et de Phys.* [3], Vol. LXIV., p. 174. 1862.—See Mascart. *Traité d'Electricité Statique*, Vol. I., p. 467.

electrometer. This residue does not, however, occur in the present case; for the discharge observed corresponds to the same fall of potential from the primitive value V_0 to that which can no longer produce contact with the gold leaf.

As the experiment has a certain duration, the total charge is thus measured. If we wish to know the instantaneous discharge, it will be sufficient, after having determined the total charge, to electrify the condenser afresh to the same potential, then to discharge it for a very short time, and finally to determine the residual discharge. The difference of the two values thus obtained represents the instantaneous discharge.

In place of the total charges, we may solely determine the rate of leakage, or the lengths of time t which correspond to the same flow $V_0 - V$ of potential. For if R is the resistance of the connecting wire, and C the capacity in question, we have (985)

$$t = RCl.\frac{V_0}{V};$$

the time t is thus proportional to the capacity. This method only requires a delicate electroscope, without any attention being needed as to its graduation.

1052. If the discharge of a condenser is determined by a ballistic galvanometer (883), the swing α of the needle, corrected for damping and graduation, gives the equation

$$m = CV = \frac{H}{G}\frac{T}{\pi}\alpha.$$

As the potential is the same in two successive experiments, the ratio of the capacities is equal to the ratio of the swings.

In order to compare very unequal capacities, which would give very different swings, it would be useful to take electromotive forces V and V′, which are in a known relation—for instance, with the numbers n and n' of couples of the same kind. We shall then have

$$\frac{C}{C'} = \frac{\alpha}{\alpha'}\frac{V'}{V} = \frac{n'\alpha}{n\alpha'}.$$

The experiment is very exact when the capacities are very near and of the same kind. For very different capacities, the duration

of the discharge introduces a source of error which it is difficult to eliminate. We know further (**884**) that the use of a shunt may give rise to serious errors.

1053. This method gives a means of determining a capacity C in absolute units in the electromagnetic system. For if the deflection δ of the galvanometer for the current produced by the electromotive force V in a resistance R be directly or indirectly produced, we shall have

$$V = R\frac{H}{G}\delta,$$

and therefore

$$C = \frac{T}{R}\frac{a}{\pi\delta}.$$

The capacity is thus determined by the resistance R and the time of oscillation T of the needle.

This is the method employed by the late Fleeming Jenkin,* for instance, in the name of the British Association, to determine the absolute value of a condenser which was to serve as standard, the capacity being near 10 microfarads (10^{-14} C.G.S. units).

The experiment presents great difficulties with condensers having a solid dielectric. Unless we make the oscillations of the needle extremely slow, which is inconvenient in practice, and which reduces the swings, we can never be sure that the duration of the discharge is only a very small fraction of the time of oscillation (**889**). Moreover, in consequence of the absorption of electricity by the dielectric, the capacity appears as a function of the time of charge and of discharge.

Thus in Jenkin's experiments with a Thomson's astatic galvanometer (**849**), in which the moment of inertia of the movable system had been so increased as to raise the time of oscillation to about 20 seconds, the charge was obtained by a battery of 20 Daniell's elements. The condenser was discharged after being charged a minute. According as the discharge contact was 1·7, 3·4, or 5 seconds, the deflection was 156, 161, 164, or 166 divisions, this latter being the same as for a permanent contact. Here again it is necessary to specify the duration of the discharge.

1054. Differential Galvanometer.—When two capacities are equal, and after having charged them by the same battery they are discharged through a differential galvanometer, the needle

* Jenkin. *British Association Report*, Dundee, 1867.—*Reprint*, p. 146.

remains at rest. If they are unequal, equilibrium may be established either by a variable capacity, or by a convenient shunt added to one of the coils of the galvanometer.

If G and G′ are the constants of the two coils, m the multiplying power of the shunt in the first, C and C′ the corresponding capacities, we have

$$G\frac{C}{m} = G'C', \qquad \text{or} \qquad \frac{C'}{C} = \frac{G}{mG'}.$$

The use of shunts is then allowable, provided the discharges are of the same duration. This is the weak point of the method devised by Varley.*

1055. Intermittent Currents.—A capacity may be evaluated by a series of discharges in the same direction, succeeding at

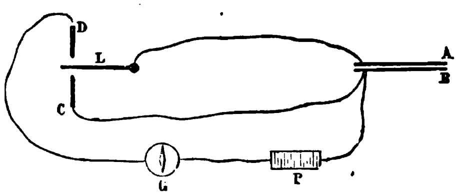


Fig. 220.

very short intervals in reference to the time of oscillation of the needle of a galvanometer, on which they produce the same effect as a continuous current.

Figs. 220, 221, give two arrangements used by Werner Siemens.† A and B are the two armatures of a condenser, L a vibrating plate

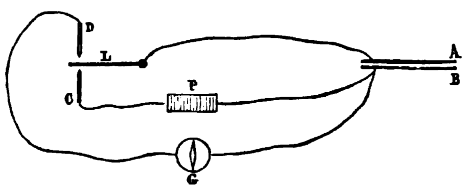


Fig. 221.

between two contacts C and D, and P the charging battery. When the plate vibrates, the galvanometer is traversed in the first place by

* L. Clark and R. Sabine. *Electrical Tables and Formulæ*, p. 63, 1871.
† *Pogg. Annalen*, Vol. cii., p. 66, 1857.

charging currents, and in the second by discharging currents. If the plate makes n double vibrations in a second with a battery of electromotive force E, the current is nEC. Measuring the deflection α of the needle, and the deflection δ which the current of the same battery would give through a resistance R, the deflections being reduced by the graduation, we have again

$$nEC = \frac{H}{G}\alpha, \qquad E = R\frac{H}{G}\delta;$$

consequently,

$$C = \frac{\alpha}{nR\delta}.$$

If the resistance R is chosen so that the current is the same in both cases, we have simply

$$C = \frac{1}{nR}.$$

We thus obtain the value of the capacity relative to a time of discharge determined by the duration of the contacts.

1056. By means of a rocking commutator we may arrange the experiment in such a way that the charging and discharging currents

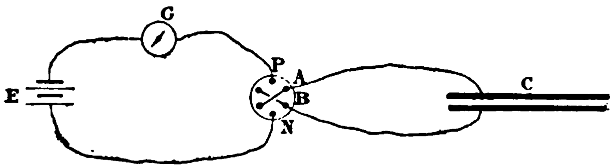


Fig. 222.

successively traverse the galvanometer in the same direction. The galvanometer G (Fig. 222) being placed in the circuit of the battery which terminates at the binding screws P and N of the commutator, the armatures of the condenser C are connected with the terminals A and B. When the commutator is worked backwards and forwards, the charge of the condenser is alternately $\pm$ C, and at each operation the quantity of electricity which traverses the galvanometer is equal to 2EC. If there are n reversals per second, the mean intensity of the current is $2n$EC. If ρ is the resistance of the circuit which would give the same permanent current, we have

$$2nC = \frac{1}{\rho}.$$

A mechanical arrangement is necessary to make the oscillations of the rocking commutator regular; the result is obtained more easily with a rotating commutator.

1057. In this method it has been supposed that at each operation the contact is long enough for the discharge to be complete. Suppose this is not the case. Let R be the resistance of the circuit comprising the connections with the condenser, and m the charge of the condenser at the time t. If the effects of induction may be neglected, the current is defined by the equation

$$\frac{m}{C}+E+R\frac{dm}{dt}=0,$$

or, if M is the initial charge, before the reversal,

$$m=(M+EC)e^{-\frac{t}{RC}}-EC.$$

During the time of contact θ, the charge passes from $+M$ to $-M$, which gives

$$M\left(1+e^{-\frac{\theta}{RC}}\right)=EC\left(1-e^{-\frac{\theta}{RC}}\right).$$

The mean current I is equal to nM or $2n$M, according as the discharges are simple, or with the reversal of the condenser. In the second case, for instance, we have

$$I=2nEC\frac{1-e^{-\frac{\theta}{RC}}}{1+e^{-\frac{\theta}{RC}}}.$$

1058. Direct Comparison of Capacities with Resistances.—Let us consider the problem more generally. Between two points N and P of a network of conductors containing constant electromotive forces a capacity C is inserted, and by one of the preceding arrangements it is discharged n times per second. If E is the difference of potential of the two points in the permanent regime, and if we suppose that after each break the current has time to re-establish itself, the quantity of electricity which in each second traverses the connecting wire is, with the second method of breaking,

$$I=2nCE.$$

The intensity is modified in all the branches of the network; but as the condition is the same after each break, the sum of the induced currents by mutual or *self-induction* is zero. Hence in each branch the mean intensity is independent of the manner in which the discharges succeed between N and P; consequently it is the same as if these two points were joined by a conductor of resistance R_1, which allowed the same quantity of electricity I to pass as a continuous current, and conversely. But in this case, if R_2 is the resistance which originally separated the two points, we have, from the theorem of M. Thevenin (946),

$$I = \frac{E}{R_1 + R_2}.$$

If the resistance R_1 is adjusted so that in any given branch the intensity is the same as with the discharges of the condenser, it follows that

$$2nC = \frac{1}{R_1 + R_2};$$

the capacity is thus found to be determined as a function of two known resistances, and of the number of breaks.

We may account more completely for the manner in which the currents of discharge are distributed in the network, by referring to the general equations. Let i be the intensity at a given moment in any branch r which contains an electromotive force e, L the coefficient of self-induction of this branch, and Q the flow of external forces, magnets or currents. The difference of potential at the two ends of the branch is

$$ri + \frac{d}{dt}(Q + Li) - e.$$

For a closed circuit of which it forms a part, we have

$$\Sigma ri + \Sigma\frac{d}{dt}(Q + Li) - \Sigma e = 0,$$

or calling $m' = \int_0^\theta idt$ the quantity of electricity which traverses the branch during the discharge,

$$\Sigma rm' + \left[\Sigma(Q + Li)\right]_0^\theta - \Sigma e\theta = 0.$$

If the intensities are the same at the beginning and at the end of each of the discharges—that is to say, if the permanent regime has time to re-establish itself, and that no magnet or external current has been displaced or modified—the quantity comprised within brackets is null, and we have

$$\Sigma rm' - \Sigma e\theta = 0 .$$

Let i_0 be the intensity of the current for the permanent regime, and $m = m' - i_0\theta$ the excess of the discharge over the quantity of electricity which would correspond to this regime. The preceding equation may be written

$$\Sigma rm + \theta\left[\Sigma ri_0 - \Sigma e\right] = 0 ,$$

and, as the second term is null, we have simply

$$\Sigma rm = 0 .$$

Thus, in the conditions mentioned, we may say that the discharge m adds itself to the permanent currents, and divides itself between the different branches according to their respective resistances, and independently of the electromotive forces which they contain.

1059. This general theorem leads to several special methods.

If the network consists of a single wire of resistance r, and if the same current is established in the two experiments, either by the discharges of the condenser or by an auxiliary resistance R_1, we have

$$\rho = \frac{1}{2nC} = R_1 + r ;$$

this is the method of **1056**.

1060. If the points P and N are united by two resistances r and r' (Fig. 223) the first of which contains an electromotive force E_0, the resistance of the network is

$$R_2 = \frac{1}{\frac{1}{r} + \frac{1}{r'}} = \frac{rr'}{r + r'} .$$

In order that the current shall be null in the branch r' in the two experiments, the compensating resistance R_1 must be null; in this case we have simply

$$\rho = \frac{1}{2nC} = \frac{rr'}{r+r'}.$$

Instead of bringing the currents to the same value, we may determine the intensity I_0 of the primitive current and the intensities

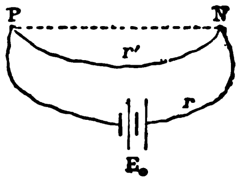


Fig. 223.

i and i' of the current in the two branches when the commutator is at work; we have in that case

$$E_0 = I_0(r+r'),$$
$$E = I_0 r' = E_0 \frac{r'}{r+r'}.$$

The mean intensity of the discharge current is

$$I = 2nCE = \frac{E}{\rho} = I_0 \frac{r'}{\rho}.$$

If I' and $I - I'$ are the discharge currents in the wires r' and r, we have

$$I'r' = (I - I')\, r,$$

or

$$I'(r+r') = Ir = I_0 \frac{rr'}{\rho}.$$

The mean currents during the action of the commutator are

$$i' = I_0 - I' = I_0 \left[1 - \frac{rr'}{\rho(r+r')}\right] = I_0\left(1 - \frac{R_2}{\rho}\right),$$
$$i = I_0 + (I - I') = I_0\left(1 + \frac{r'}{r}\frac{R_2}{\rho}\right).$$

The ratio $\frac{R_2}{\rho}$ may then be determined, either by the ratio of two successive currents I_0 and i or I_0 and i', or by the ratio of two simultaneous currents i and i'.

A galvanometer with two rectangular coils (854) interposed in the two branches r and r' would give directly the ratio of the currents i and i', and therefore

$$\frac{\rho}{R_2}=\frac{1+\frac{i'}{i}\frac{r'}{r}}{1-\frac{i'}{i}}$$

The value of ρ would still be determined by the comparison of the discharge current I, either with the principal current I_0, or with

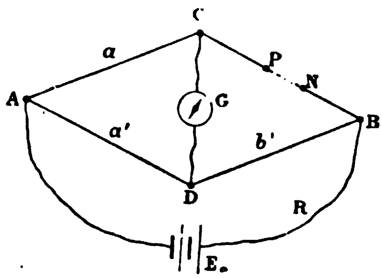


Fig. 224.

one of the currents i and i'. If, for instance, we introduce a differential galvanometer in the branch r', and in the wire connecting one of the points P or N with the condenser, we may choose the resistance r', so that the needle is at zero. The galvanometer being adjusted, the condition $I=i'$ gives

$$\rho=r'+R_2.$$

1061. Let us assume that the capacity which can be discharged is inserted between two points P and N on the branch b of a Wheatstone's bridge (Fig. 224).

The resistances a, a' and b', being adjusted so that there was no current in the bridge during the alternating discharges of the

condenser, the capacity is then replaced by a resistance R_1, which re-establishes equilibrium, and satisfies the equation

$$R_1 = b'\frac{a}{a'}.$$

The value of the resistance R_2, of the network between the points B and C is

$$R_2 = \frac{a'(a+R)(r+b') + aR(r+b') + rb'(a+R)}{a'(a+R+r+b') + (R+b')(a+r)},$$

and the capacity is given by the condition that

$$\rho = \frac{1}{2nC} = R_1 + R_2.$$

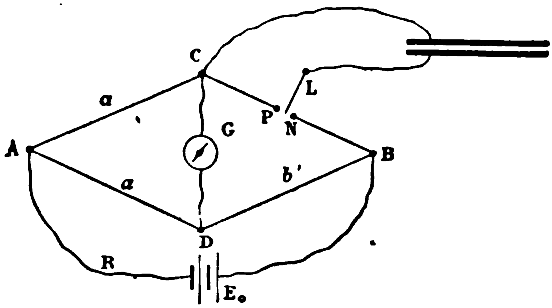


Fig. 225.

1062. Professor J. J. Thomson* has applied this method, which was first suggested by Maxwell,† by introducing into the experiments a modification which permits the use of a vibrating plate.

One of the armatures of the condenser is connected with the summit C of the bridge, and the other with a plate L which oscillates between the contacts P and N (Fig. 225). The condenser discharges itself, and its charge alone interposes in the general current. For n double oscillations per second, the mean intensity

* J. J. Thomson. *Transactions Roy. Soc. for 1883.*

† Maxwell. *Electricity and Magnetism*, Vol. II., p. 375.

of the charging current is nCE, and if equilibrium of the bridge is produced in the two experiments, we have

$$\rho = \frac{1}{nC} = R_1 + R_2.$$

The plate L is put in motion by an electrical trembler, and the contacts take place on platinised surfaces; the number of vibrations is given by the height of the note. The values obtained for the capacity are very concordant, with a less divergence than 0·002 when the number of oscillations is varied from 16 to 128 per second, and when the relative duration of the contacts is modified within wide limits. The phenomenon, therefore, is well defined, and the condenser had time to take at each contact what is called an instantaneous discharge.

The expression for R_2 may be simplified if we suitably choose the different resistances, as Mr. Glazebrook* has done. Writing this expression in the form

$$R_2 = \frac{ar(a'+R+b') + Rr(a'+b') + b'[a(a'+R)+a'R]}{(a+r)(a'+R+b') + a'(R+b')},$$

we see that, if the resistances R and a' are small in comparison with b' and r, we have sensibly

$$R_2 = (a+R)\frac{r}{a+r},$$

$$\rho = \frac{1}{2nC} = a\frac{b'}{a'} + (a+R)\frac{r}{a+r}.$$

1063. Comparison of a Capacity with a Coefficient of Mutual Induction.—The discharge of a condenser presents the same characteristics as induced discharges.

If a capacity C, communicates with two points of a circuit traversed by a permanent current I, and separated by a resistance R, the charge it takes,

$$m = CE = CRI,$$

* R. T. Glazebrook. *Phil. Mag.* [5], Vol. xviii., p. 98. 1884. In Mr. Glazebrook's experiments, $a' = 10$, $R = 5$ to 6, $a = 240$ to 1800, $b' = 1000$, and $s = 11{,}000$ ohms.

sent into a ballistic galvanometer, will give

$$m = \mathrm{CRI} = \frac{\mathrm{H}}{\mathrm{G}} \frac{\mathrm{T}}{\pi} a .$$

Suppose that the circuit of the primary current contains a coil, in front of which is another coil forming part of a closed circuit, the resistance of which is r, and the coefficient of mutual induction of the two coils M. When the principal current is made or broken, the induced discharge is

$$m' = \frac{\mathrm{MI}}{r},$$

and if we measure it by the swing a', which it produces in the same galvanometer, we have also

$$\frac{\mathrm{MI}}{r} = \frac{\mathrm{H}}{\mathrm{G}} \frac{\mathrm{T}}{\pi} a' .$$

It follows that

$$\frac{\mathrm{CR}r}{\mathrm{M}} = \frac{a}{a'}, \qquad \text{or} \qquad \mathrm{C} = \frac{\mathrm{M}}{\mathrm{R}r} \frac{a}{a'} .$$

The two systems of discharge may still be transformed into continuous currents.* That of the condenser being repeated n times per second, and the induced discharge n' times per second, if i and i' are the mean currents in the same galvanometer,

$$\frac{n\mathrm{CR}r}{n'\mathrm{M}} = \frac{i}{i'}, \qquad \text{or} \qquad \mathrm{C} = \frac{\mathrm{M}}{\mathrm{R}r} \frac{n'}{n} \frac{i}{i'} .$$

With a suitable mechanical arrangement, which establishes suitable connections, the two kinds of discharges may be sent alternately in the galvanometer, and the resistances, or the values of M and C, regulated so that the needle is at zero; we have $n = n'$, $i = i'$, and therefore

$$\mathrm{C} = \frac{\mathrm{M}}{\mathrm{R}r} .$$

1064. Specific Inductive Capacity. — By comparison with solid or liquid dielectrics, the specific inductive capacity of air may

* Roïti. *Atti del R. I. Veneto* [6], Vol. ii. 1884.

be considered as sensibly equal to unity. The specific inductive capacity of a dielectric will then be determined by the ratio of the capacities of a condenser, when the medium interposed is alternately the dielectric in question and air (108).

This is the method first used by Cavendish.* The ratio found by experiment between the capacity of a Franklin's pane of surface S and thickness e, and that of a system of two concentric spheres separated by air, has always been greater than that which would result from the theoretical formula $\frac{S}{4\pi e}$. These experiments lead to very high specific inductive capacities, 3 to 10, owing to the influence of time and of phenomena of absorption.

In a first series of researches on this subject, Faraday† compared, by a division of charges, the capacities of two spherical condensers of the same dimensions, one containing air, and the other a solid dielectric. He thus found the following values:—

Spermaceti	1·45	Shellac	2·0
Glass	1·76	Sulphur	2·24

1065. Comparing the charges taken for the same difference of potential (1034) for plate condensers of the same dimensions, Gaugain brought out the influence of the duration of the charge on the apparent value of the specific inductive power. With commercial stearic acid, this power was found to be 1·3, 1·85, 2·17 and 7, according as the charge was a small fraction of a second, or two seconds, or a minute, or lastly, several hours. These variations are very unequal for different bodies, and even the order of the inducing powers depends on the duration of the charge. He obtained, for instance:—

Charge of a Fraction of a Second.		Charge of Two Seconds.	
Stearic acid	1·30	Sulphur	1·71
Wax	1·50	Stearic acid	1·92
Sulphur	1·57	Wax	2·21

Traces of foreign substances, such as a layer of powder or of moisture, render the surface of bodies better conductors, and have the effect of increasing the apparent value of the specific inductive

* CAVENDISH. *Electrical Researches* (published by Maxwell), p. 183.

† FARADAY. *Experimental Researches*, Series XI., § 1187. 1837.

power. Spontaneous alteration of the surface produces the same result.

1066. Gibson and Barclay,* using concentric spheres, determined the specific inductive power of paraffine by comparison of the capacities with a platymeter (**1047**) and a sliding condenser (**1045**).

In his experiments on the inductive capacity of different kinds of optical glass, Hopkinson† used the guard-ring condenser, and determined the thickness of the equivalent layer of air. The sliding condenser was adjusted so as to give the same capacity as the apparatus with the glass plate; the glass plate being removed, the disc was approached until equilibrium was restored; the sliding condenser was then simply used as a tare. Only the acting surfaces

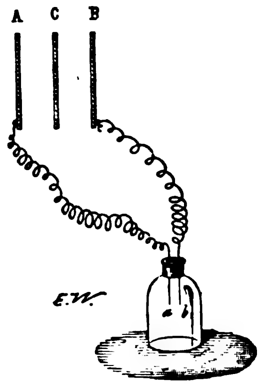


Fig. 226.

are compared—those of the plate, that is to say—all other condensers comprising the guard ring being connected with the earth.

1067. Electrostatic Induction Balance.—The part which the dielectric plays has been made evident by Faraday,‡ by means of an ingenious arrangement which he called the induction balance, or ***differential inductometer***. Between two plates A and B (Fig. 226), which are respectively connected with two gold-leaves *a* and *b*, an insulated plate of the same dimensions is placed at equal distances. The plates A and B are connected with the earth, C is positively electrified, and connection broken; the leaves *a* and *b* are then at

* Gibson and Barclay. *Phil. Trans. Roy. Soc. Lond. for 1871*, p. 573.

† Hopkinson. *Phil. Trans. Roy. Soc. Lond. for 1878*, p. 17.

‡ Faraday. *Exper. Researches*, Series xi., § 1307, Vol. i., p. 413. 1838.

the same potential, and have no action on each other. It is clear that, if the plate B is brought nearer, the induction will be increased on this side, the leaf *b* will become positive, the leaf *a* negative, and attraction will be observed.

Putting the plates again in position, a dielectric plate of thickness e is interposed between C and B. If the distance of the plates is small in comparison with their dimensions, the plate introduced is equivalent (**122**) to a layer of metal of thickness $e\left(1-\frac{1}{\mu}\right)$, and produces the same effect as if the plate had been brought nearer B by the same amount. An attraction of the gold leaves is observed, and, measuring the quantity d by which the armature B must be removed so as to restore equilibrium, we have

$$d=e\left(1-\frac{1}{\mu}\right) \quad \text{or} \quad \mu=\frac{e}{e-d}.$$

This arrangement has the drawback that the want of equilibrium, on whatever side it takes place, always produces attraction of the gold leaves. If the plates A and B are connected with the quadrants of an electrometer, the needle of which is connected with the plate C, it is no longer necessary first to connect these plates with the ground. The needle stays at zero if the induction is the same on each side; and, whatever be the sign of the electrification, the deflection always shows on which side is the greatest induction.

1068. Gordon* has used a more complete induction balance, the general idea of which is due to Sir W. Thomson and Maxwell.

In order to eliminate the influence of external bodies and to realise as much as possible the case of unlimited parallel surfaces, the plates A and B of Faraday which communicate with the quadrants of an electrometer (Fig. 227) are placed in the intervals of the three plates C, A_1 and B_1 of greater dimensions; the external plates A_1 and B_1 are connected with each other, while the central plate is connected with the needle of the electrometer.

When once equilibrium is established the needle should be stationary, whatever be the difference of potential V which is established between the conductors C and A_1B_1. This is what would be the case if, for instance, the system is quite symmetrical in respect of the intermediate plate C. One of these extreme plates A_1 is

* J. E. H. GORDON. *Trans. Roy. Soc.*, *1879*, p. 417.—*Physical Treatise on Electricity*, Vol. I., p. 110.

provided with a screw, by which it can be moved parallel in each case until the condition is realised.

A dielectric plate D of thickness e is introduced between A and A_1, and, by means of a micrometric screw M, the displacement d of the plate A_1 is measured which is necessary to re-establish equilibrium; this gives the specific inductive capacity.

As this inductive capacity is a function of the time of charge, the conductors C and A_1B_1 are connected with the poles of an induction coil, the inducing current of which is broken by a vibrating plate, or by a rotating electromagnetic apparatus, which can give as many as 12,000 breaks in a second.

Mr. Gordon investigated a great number of substances. He found that for glass, in particular, the manner in which the surfaces

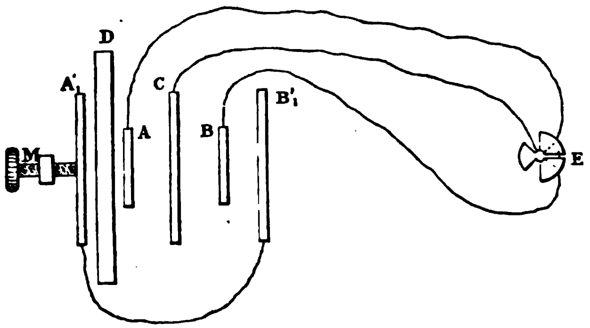


Fig. 227.

are cleaned plays an important part, and that the spontaneous alteration which they undergo in the course of time manifests itself by a material increase of apparent inductive power.

1069. Electrical Oscillations.—The theory of electrical oscillations (538) furnished Schiller* with a method which has the advantage that the results correspond to a smaller time of charge than any other.

The general nature of the experiment has been mentioned above (911). The time T of a single oscillation is given by the formula

$$T^2 = \frac{\pi^2}{\frac{1}{CL} - \frac{R^2}{4L^2}};$$

* Schiller. *Pogg. Ann.*, Vol. CLII., p. 535. 1874.

we may write

$$T^2 = \pi^2 CL\left(1 + \frac{R^2}{4L^2}\frac{T^2}{\pi^2}\right),$$

and if, as is easily done, we choose the experimental conditions so that the second term of the bracket may be neglected in comparison with unity, it reduces to

$$T^2 = \pi^2 CL.$$

Schiller made three consecutive experiments: with the wire of the induced coil simply open; with this wire connected with a plate condenser and comprising an insulating plate; and, finally, with this same condenser when the insulating plate is removed and replaced by a plate of air. If T_0, T_1, and T_2 are the times of oscillations in the three cases γ, $\gamma+\mu C$, $\gamma+C$ the corresponding capacities, we deduce

$$\mu = \frac{T_1^2 - T_0^2}{T_2^2 - T_0^2}.$$

The mean time of charge was 0·00005 of a second.

1070. Action on a Dielectric Sphere.—The specific inductive power may also be deduced from the action which a constant electrical field γ produces on bodies of small dimensions. For a homogeneous field of volume u, the component of this action in any direction (179) is expressed by

$$X = \frac{uK}{2}\frac{\partial\phi^2}{\partial x} = \frac{u}{2}\frac{3}{4\pi}\frac{\mu-1}{\mu+2}\frac{\partial\phi^2}{\partial x}.$$

The action X_0 which would be exerted on a conducting sphere of the same volume is

$$X_0 = \frac{u}{2}\frac{3}{4\pi}\frac{\partial\phi^2}{\partial x};$$

from this follows

$$\frac{X}{X_0} = \frac{\mu-1}{\mu+2},$$

$$\mu = \frac{X_0 + 2X}{X_0 - X} = \frac{2 + 2\dfrac{X}{X_0}}{1 - \dfrac{X}{X_0}}.$$

This method was used by Boltzmann.*

In the first series of experiments the ratio of the forces X and X_0 was determined by deflections from the vertical. Two small spheres of sulphur 0·7 cm. in diameter, one of which is coated with gold leaf to make it a conductor, are suspended at a distance of 9 cm. from two wires 2 metres in length placed in front of a divided scale. When a metal ball 2·6 cm. in diameter, originally electrified, is inserted between them, the two balls are attracted unequally. If the experiment be so arranged that the active ball is in the centre of the interval of the movable balls, the ratio of the attractions is equal to the ratio of the deflections, which is observed by the microscope, or, more generally, by the product of the ratio

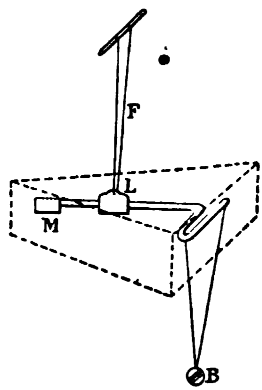


Fig. 228.

of the deflections by the ratio of the weight of the balls. The method is not very accurate, because it is difficult to insure equality of distance and to avoid the action of currents of air.

1071. Another arrangement employed by Professor Boltzmann is more delicate. The movable ball is attached by two wires to the end of a metal rod (Fig. 228), itself supported by a bifilar suspension F, so as to form a small torsion balance. A mirror M serves to measure the deflections. The needle of this balance is protected by a screen against air currents. The active ball communicates with a Leyden jar, to which a definite charge has been imparted by a spark electrometer (**824**). By means of a conducting torsion balance connected with the earth, and analogous to the first,

* BOLTZMANN. *Wiener Sitz. Berichte*, Vol. LXVIII., Pt. ii., p. 81. 1873.

it is ascertained whether the potential of the Leyden jar has the same value in successive experiments. If the movable ball is formed alternately of a dielectric and of a conductor, the ratio of the deflections is equal to the ratio of the corresponding attractions. There are, however, small corrections to be made which arise either from the difference of volume of the balls or from the change of distance produced by unequal attractions.

As the actions observed are always attractive, whatever be the electrical condition of the Leyden jar, the experiment may be repeated several times by successively charging and discharging by hand, or even by altering the signs by the action of a plate which vibrates between the armatures of two batteries electrified in contrary directions. In this way we determine the influence of the time of charge on the specific inductive capacity.

In experiments of attraction the phenomenon depends on the entire mass of the body electrified by induction, and not merely on the surface, as with conductors. This has been directly verified by Romich and Fajdiga* with balls of sulphur covered with paraffine, or of shellac.

1072. The method has the particular advantage of only requiring a very small quantity of the body under experiment, and of being applicable to crystallised bodies.

Electrification by induction follows the same laws as magnetisation by induction, the intensity of magnetisation being equivalent to intensity of electrification—that is to say, to the electrical moment $K\phi$ (**178**) of unit volume. In like manner for anisötropic bodies (**391**), the electrical condition is the superposition of three conditions which would be produced separately by the three components of the electrical field parallel to the principal axis of elasticity.

By measuring the attraction exerted between spheres of crystallized sulphur, parallel to the three principal axes, Professor Boltzmann† obtained three very different values for the specific inductive capacity

$$\mu_1 = 3{\cdot}811\,,$$
$$\mu_2 = 3{\cdot}970\,,$$
$$\mu_3 = 4{\cdot}773\,.$$

MM. Romich and Nowak‡ have applied the same method to several

* Romich and Fajdiga. *Wiener Sitz. Berichte*, Vol. lxx., Pt. ii., p. 367. 1874.
† Boltzmann. *Wiener Sitz. Berichte*, Vol. lxx., Pt. ii., p. 342. 1874.
‡ Romich and Nowak. *Wiener Sitz. Berichte*, Vol. lxx., Pt. ii., p. 380. 1874.

isotropic or crystallized bodies, by varying the duration of the discharge between pretty wide limits. Bad insulators soon show themselves as conductors; even for very short contacts it is difficult to prove that specific inductive capacities are unequal in various directions.

1073. Liquids.—The determination of the specific inductive capacity of liquids presents special difficulties, owing to the fact that the molecules of the liquid carry electricity from one armature to the other. We know that it is impossible to give a somewhat strong charge to a liquid dielectric, and particularly to keep it charged.

With this reserve, we may employ the methods described above. Professor Silow* used a modification of the quadrant electrometer. The quadrants are replaced by four cylindrical sectors, consisting of tinfoil fixed on the inside of a glass cylinder, and the needle of two cylindrical sectors concentric with the first. The needle is in connection with the earth, and the two systems of opposed quadrants with the poles of an insulated battery, the centre of which is to earth. The deflection is measured when the apparatus is full of liquid—oil of turpentine, for instance. For the same difference of potential, the deflection is proportional to the capacity, and this, again, to the specific inductive capacity; by reversing the connections of the poles with the quadrants, and measuring the double deflection, a zero reading is avoided.

1074. Gases.—The objection relative to the transport of electricity by molecules does not seem to have the same importance for gases, as experiment shows that the charge of a gas condenser can be maintained for a long time. Faraday,† notwithstanding numerous attempts, was unable to show any difference in the specific inductive capacity of various gases; these differences are too small to be shown by the methods which he employed. Boltzmann,‡ and afterwards Professors Ayrton and Perry,§ succeeded in the following manner.

Boltzmann's condenser consists of two parallel metal plates, A and B, placed under a bell jar, and protected by larger plates against any external influence. The plate A is permanently connected to the positive pole of a battery of 300 elements, the other pole of which is to earth, and to one pair of quadrants of an electrometer, the

* Silow. *Pogg. Ann.*, Vol. CLVI., p. 389. 1875.

† Faraday. *Exper. Researches*, § 1290, Vol. I., p. 407. 1837.

‡ Boltzmann. *Wiener Sitz. Berichte*, Vol. LXIX., p. 795. 1874.

§ Ayrton and Perry. *Mem. lu à la Soc. As. du Japon.* Yokohama, 1877.

needle of which is electrified; the plate B is connected with the other pair of quadrants.

The needle having been brought to zero, by putting the plate B to earth for an instant, the deflection α is observed, either if a vacuum is made, or if air is replaced by another gas.

If V is the electromotive force of the battery, μ and μ' the coefficients for the two gases, the difference of potential between the two plates, which was first V, soon becomes $V\frac{\mu'}{\mu}$ (123), for the charge of the plate is virtually the same. The change δV, indicated by the electrometer, is equal to $V\left(1-\frac{\mu}{\mu'}\right)$, which gives

$$\frac{\mu}{\mu'}=1-\frac{\delta V}{V}.$$

As the electrometer must be very sensitive for this kind of observation, the potential V is determined by the deflection which a single couple produces.

If we take as unit the specific inductive power of vacuum, we find a number greater than unity for all gases, but less for hydrogen than for air. It follows that the variation δV is negative when any gas is exhausted, or when air is replaced by hydrogen.

Professors Ayrton and Perry worked by Thomson's method (1049), using two condensers, a standard air one, and a lamellar one of twelve plates, between which any given gas could be introduced, or a vacuum made.

The determination of the specific inductive capacity of a dielectric has great theoretical interest, owing to the relation established by Maxwell's theory between this constant and the refractive index (634).

1075. Pyroelectricity.—Pyroelectricity is scarcely manifested in any other than bad conductors, and is naturally connected with the question of specific inductive powers. Older observations showed that a rod of tourmaline, when heated, acquires the property of attracting light bodies at its two ends. These phenomena, which have been called pyroelectrical, have given rise to a great number of researches, and the experimental laws have been established by Gaugain.*

* Gaugain. *Ann. de Chim. et de Phys.* [3], Vol. lvii., p. 5. 1859.—See Mascart. *Traité d'Électr. Stat.*, Vol. ii., p. 494.

A tourmaline, which is kept at a constant temperature for a long time, seems absolutely neutral; the electrical properties of which it may be the seat are compensated by a superficial layer, which is produced spontaneously by conductivity (619), or which has been produced by connecting the surface with the earth—for instance, by passing the crystal through the flame.

When the crystal is heated, one end becomes negative and the other positive; the reverse is the case if the crystal is cooled. We may, with Riess and Rose,* call that end of the crystal, the electrical sign of which is the same as that of the change of temperature, the *analogous pole;* and that end whose electrical sign is contrary to the change of temperature, the *antilogous pole.* The *axis of polarization* is the direction from the antilogous to the analogous pole.

The ends of the tourmaline being covered with a conductor—tinfoil or wire being rolled round it, for instance—Gaugain connects one of them with the earth, and the other with a discharging electrometer; he has thus found that—

1st. The quantities of electricity disengaged at the two poles are equal and opposite for the same change of temperature, and are independent of the velocity of heating or of cooling.

2nd. The quantities of electricity disengaged at one end are equal and of opposite sign for two changes of temperature which are the inverse of each other.

3rd. These quantities of electricity are proportional to the section of the crystal, perpendicular to the optical axis, and independent of its length.

In order to explain these curious properties, Sir W. Thomson† assumes that the crystal is naturally in a state of electrical polarization.

A change of temperature, starting from what is apparently the neutral state, produces a polarization equal to the difference of those which correspond to two determinate temperatures t_0 and t_1, and the crystal is comparable with a body magnetised uniformly. Canton observed, in fact, that when an electrified tourmaline is broken, the pieces have each two poles, and the faces corresponding to the fracture are oppositely electrified.

If we consider a cylindrical rod of tourmaline, terminated by faces perpendicular to the axis, and insulated during the change of temperature, the potential is positive on one half and negative

* RIESS and ROSE. *Archives de l'Électricité*, Vol. III., p. 585.

† THOMSON. *Phil. Mag.* [5], Vol. V., p. 24. 1878.

on the other, and for a given point is proportional to the distance from this point to the median section. The quantity of electricity produced on each terminal is proportional to the extent of this surface, and is sensibly proportional to the difference of the temperatures.

We may state, again, that the apparent polarization is proportional to the expansion for unit length between the temperatures in question.

1076. Pyroelectric polarization is connected with hemihedry of oblique faces, as Haüy* long ago remarked.

In tourmaline which crystallizes in rhombohedra (Fig. 229), the faces which terminate the cylindrical rods are not distributed in the same way at the two poles. On the analogous pole, the faces P of

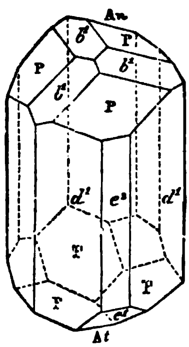


Fig. 229.

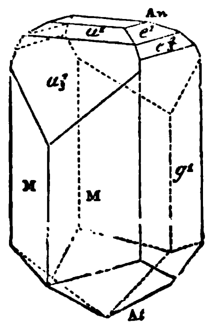


Fig. 230.

the primitive rhombohedron rest on the three lateral faces of the triangular prism e^2, and the hemihedral elements are represented by the facets b^1 and e^1 of two distinct rhombohedra.

Hydrated silicate of zinc, or *calamine*, which crystallizes as a right rhombohedral prism, is also pyroelectric, and the axis of polarization is parallel to that axis of the crystal which has the characteristic of hemihedry. In Fig. 230, when this axis is vertical, the analogous pole is terminated by a horizontal truncature P, and the antilogous pole by the summit of a octahedron.

Pyroelectrical phenomena have been observed in a great number of mineral substances belonging to various crystalline systems—*boracite*, *scolezite*, *prehnite*, *topaz*, etc.; and, among artificial crystals, *struvite*, the *tartaric acids*, *tartrates*, etc.

* HAÜY. *Ann. de Chim. et de Phys.* [1], Vol. IX., p. 59. 1800.

Kundt* has described a very simple method of showing the electrification of various plates, which consists in powdering the crystal with the mixture of sulphur and red lead which is used for producing Lichtenberg's figures.

1077. Various observers have noticed, in certain crystals, such as *topaz*, *prehnite*, *axinite*, and *boracite*, the existence of two, three, or even four axes of pyroeletricity; but Freidel has observed that two different directions are never found to which the regular characteristics of tourmaline can be assigned. This multiplication of electrical axes is, in fact, contrary to the idea of polarization. Two or three polarizations in different directions would be equivalent to a single polarization in a definite direction, as with the magnetisation of anisotropic bodies (**391**); in a homogeneous crystal, at a uniform temperature, experiment can only show one resultant axis.

Most of the experiments in which it has been supposed that there are different axes have been on macled crystals. This is particularly the case with *topaz* and *prehnite*, which belong to the right rhombic prism. The lozenge-shaped tablets present three poles, one analogous in the centre of the lozenge and two antilogous at the ends of the shortest diagonal, so that there are two opposite electrical axes; but these plates are formed of two halves, the orientation of which differs by 180°, and the crystal has really only one pyroelectrical axis. Nevertheless, the experimental verification is often difficult because the crystals are usually too small and too confused.

Scolezite presents the same characters and lends itself better to experiments.

In other circumstances the effects observed arose in great part from the shape of the specimens employed.

1078. In order to eliminate the source of error due to the shape of the crystal, Friedel† uses a face with parallel faces cut perpendicular to the direction of the hemihedral axes, which are always axes of pyroelectricity. By a change of temperature one of the faces becomes positive and the other negative. The plate being placed on a conductor connected with the earth, if the upper face is covered with tinfoil connected with the needle of an electrometer, a disengagement of electricity is observed which, for a given change of temperature, is proportional to the extent of the surface. If the plate is not perpendicular to the pyroelectrical axis, the

* KUNDT. *Sitzb. der Akad. der Wiss. zu Berlin*, p. 421. 1883.

† C. FRIEDEL. *Bulletin de la Société de Minéralogie*, Vol. II., p. 31. 1879.

quantity of electricity is proportional to the projection of this axis on the perpendicular to the plate.

Another method is still simpler. The plate being kept at the ordinary temperature, and connected with the earth by its lower face, a conducting test body is placed on the other face at a higher temperature—for instance, the base of a hemisphere; the test body is connected with the needle of the electrometer, or rather with one pair of quadrants, the other pair of which is to earth while the needle is kept at a constant potential. A deflection is at once observed. The electricity changes its sign when the opposite face is observed, or when the test body is at a lower temperature than that of the plate. The results thus obtained with tourmaline are in complete agreement with those which would be given by regular heating.* No effect is observed with homohedral substances, or at most a slight deflection in the same sense for two opposite faces.

1079. The method of the hot or cold test body enables us to demonstrate a special kind of pyroelectricity which seems to have multiple axes, and which consequently would seem to contradict what has been said above. With plates of *quartz* parallel to the axis, or even simply with an uncut quartz prism, the existence of three pyroelectrical axes is observed perpendicular to the axis of the crystal and making angles of 120° with them. These axes are in planes which pass by the edges of the hexagonal prism, and directed from one edge on which are the rhombic facets, towards the opposite edge which is not modified; they are then alternately in opposite directions. A plate perpendicular to the optical axis gives no regular indication.

In the same conditions *tourmaline* presents three axes of pyroelectricity perpendicular to the axes of the crystal, and forming with each other angles of 120°.

As the lateral axes of electrical polarization in the rhombohedric system have from symmetry a resultant null, the crystal should not manifest pyroelectrical phenomena by a homogeneous variation in temperature; and experiment then shows only traces of electricity distributed in a wholly irregular manner. For the same reason, if the surface of the test body is greater than that of the plate, the heating is more regular and the electrification less.

This apparent pyroelectricity appears, then, to be attributed to the compression produced in the crystal by an unequal expansion

* C. Friedel and J. Curie. *Comptes rendus*, Vol. xcvi., pp. 1262, 1389; Vol. xcvii., p. 61. 1883.

of the various regions which are not at the same temperature (1081).

Cubical crystals with a tetrahedric hemihedry have the same characteristics. In this case the axes of electrical polarization due to unequal temperatures are parallel to the longer diagonals of the cube. Their resultant is still null, and all trace of regular electrification disappears when a homogeneous variation of temperature is produced. Pyroelectricity, on the contrary, is observed when a plate perpendicular to one of the axes is touched with a heated test body. This is the case with blende and sodic chlorate.

Boracite is particularly remarkable. This body is powerfully pyroelectrical, and has long been supposed to crystallise in cubes, which would be contradictory of the preceding remarks; but we know from the researches of M. Mallard that a crystal of boracite at the ordinary temperature is made up of twelve orthorhombic hemihedral pyramids, and that it really becomes cubical at a temperature of 265°, retaining this condition to the fusing point.

Now Friedel and Curie have shown that boracite has no pyroelectrical properties at temperatures higher than 265°—that is to say, when it is cubical. But at the moment when, in cooling, the crystal resumes its crystalline complex, there is suddenly a considerable disengagement of electricity with a well-marked polarization; for the effect changes its sign according as one or the other face is observed of a plate parallel to one of the faces of the tetrahedron.

1080. Pyroelectrical Currents. — When two poles of a pyroelectrical crystal are joined while its temperature is changing, an extremely feeble current is produced in the wire, which may, however, be shown by a galvanometer of very high resistance. The intensity of this current is proportional to the quantity of electricity disengaged on the crystal in unit time, and therefore is sensibly proportional to the velocity of cooling.

This method of observation enables us to recognise the existence of electrical properties in certain crystals which are better conductors, and with which electrometers would give no appreciable effect.

The crystal is cut in the form of a plate with parallel sides, and pressed between two metal plates connected with a galvanometer. Friedel has thus found* that, for regular heating or cooling, *fahlerz*, or *grey copper ore*, gives a very appreciable current changing its sign with the variations of temperature, and that the axes of maximum

* C. D. Friedel. *Ann. de Chim. et de Phys.* [4], Vol. XVII., p. 92. 1869.

effect are parallel to the diagonals of the cube. *Chalcopyrite* shows the same properties, without its being possible to determine the direction of the axes. These two crystals also show hemihedry with inclined faces. The multiplicity of axes in grey copper does not seem to show that we are dealing with regular polarization or with a true pyroelectricity.

1081. PIEZOELECTRICITY.—It has long been known that pressure may electrify certain bodies. Surfaces are thus obtained which are charged with opposite electricities; but the electrification of the compressed bodies remains after the pressure has ceased, and presents no character of polarity. The phenomenon appears to be a particular case of electrification by contact.

MM. J. and P. Curie* have discovered that mechanical changes in form also produce in hemihedral crystals with inclined faces a special polarization which they call *piezoelectrical.*

Take, for instance, the case of tourmaline terminated by two bases perpendicular to the axis. Tinfoil is applied on each face, and the crystal is compressed in the direction of the axis. The two tinfoils are charged with opposite electricities, and in the same direction as if the crystal had been cooled or undergone a thermal contraction (**654**). When the pressure is removed, the tinfoils, passing through the neutral state, become electrified in the contrary direction—that is to say, in the same manner as by heating. By first stretching, and then letting go, effects are observed resembling those produced by removing and putting on pressure. In all cases electricity is disengaged on the bases only.

The pressure may be applied laterally and the electricity collected on the bases. The direction of electrification along the axis is the same in both cases.

Consider, in like manner, a rectangular quartz prism with two faces parallel to the axis, and two others perpendicular to a lateral axis of pyroelectricity, the tinfoils being on the faces perpendicular to the electrical axis. A pressure parallel to the electrical axis produces polarization, and the negative tinfoil corresponds to the edge which supports the face of the ditrihedron. Polarization also takes place, but in the opposite direction, with pressure perpendicular to the plane passing through the electrical axis and the optical axis. Pressure parallel to the optical axis produces no electricity.

* JACQUES and PIERRE CURIE. *Comptes rendus*, Vol. XCI., p. 294, 1880, *passim.—Journal de Physique* [2], Vol. I., p. 245.

The lateral axes of tourmaline produce effects of the same kind, but far feebler than for the principal axes, and the two polarizations are superposed.

Homohedral crystals present nothing analogous, while all those which possess dissymmetrical hemihedry manifest electrical polarization under the influence of mechanical pressure or tension.

In all cases the experimental laws are the same as for pyroelectricity:—

1st. The quantities of electricity disengaged at the two ends of an axis, for the same change of form, are equal and of opposite signs.

2nd. The quantities of electricity disengaged at one end are equal and opposite for two changes of shape which are the converse of each other.

3rd. These quantities of electricity are proportional to the change of pressure or of traction.

4th. For the same variations of pressure the electricity liberated on the electrode is independent of the dimensions of the crystal if the pressure is parallel to the polarization observed.

In other words, the electrical density or the intensity I of the piezoelectrical polarization is proportional to the pressure for unit surface—that is to say, to the contraction of the crystal for unit length and to the extent of the surface. For a pressure P on a surface S we may then write

$$I = k\frac{P}{S}.$$

5th. If the pressure is on a surface S′ perpendicular to the optical axis and to a lateral axis, the polarization may still be represented by $k'\frac{P}{S'}$, and the quantity of electricity liberated on the surface S perpendicular to the axis of polarization is equal to $k'P\frac{S}{S'}$; it is proportional to the quotient of the length of the crystal parallel to the pressure by the thickness parallel to the electrical axis.

For reasons arising out of symmetry, it appears that k' should equal $-k$, which is in agreement with experiment.

1082. In order to determine in absolute value the quantity of electricity liberated, MM. Curie worked by a method of opposition.

The needle of an electrometer being electrified, one of the pairs of quadrants is charged from a Daniell's element, the other pair of

quadrants communicates with one of the tinfoils, and with a standard condenser of capacity C formed of concentric cylinders. The second tinfoil is connected with the earth. The pressure P is then regulated by variable weights, until the needle of the electrometer which is at first deflected by the Daniell's element, reverts to zero. If D is the electromotive force of a Daniell, and c the capacity of the system formed by the tinfoil, the conductors, and the corresponding pair of quadrants, the value of the quantity of electricity is $m = (C + c)\ D$.

If then the standard of capacity C is removed, the pressure P′ necessary to produce equilibrium is again determined; the quantity of electricity is then $m' = cD$. The difference P – P′ of the pressure is then capable of producing a quantity of electricity

$$m - m' = CD.$$

It is thus found that a pressure of 1 kilogramme exerted along the axis of the crystal of tourmaline liberates a quantity of electricity equal to 0·053 C.G.S. units; the same pressure applied along the electrical axis of quartz gives 0·063 units.

1083. The electrical properties which crystals acquire by compression lead to a remarkable consequence relative to the changes of dimensions which they should experience under the action of electricity. For a parallelopipedon of quartz having two faces perpendicular to the optical axis, and two other faces perpendicular to an electrical axis, the establishment of a difference of potential between the latter should produce either expansion along the electrical axes and contraction in the direction perpendicular to the optic and electrical axes, or the reverse effects, according to the sign of the difference of potential, the length parallel to the optical axis being unchanged. This curious experiment has been made by MM. Curie by using the difference of potential of a battery charged by a Holtz machine. The expansions observed were found to be very nearly equal to that shown by calculation.

1084. There is a close analogy between the charge of a condenser and the polarization of electrodes (252). Any quantity m of electricity which passes through a voltameter, without giving rise to any apparent disengagement of gas, produces a difference of potentials e between the electrodes; if we put

$$m = ce,$$

the factor c may be defined as the capacity of polarization of the

electrode. The simplest method of measuring this constant will be to connect it like a condenser with a battery of known electromotive force less than the maximum electromotive force of polarization, and then to measure the charge obtained, when the voltameter is closed by a galvanometer. This method was used by Varley,* but it is rendered difficult by this circumstance, that, as equilibrium is not attained instantaneously either by charge or discharge, there is a considerable loss owing to depolarization by diffusion.

M. Blondlot† endeavoured to avoid this cause of error by first investigating how polarization is set up as a function of the time of charge. By means of a pendulum break (910), he connects the voltameter with a known electromotive force for a very short time, then short-circuits the voltameter by a very small resistance to depolarize it. The quantity of electricity corresponding to the charge

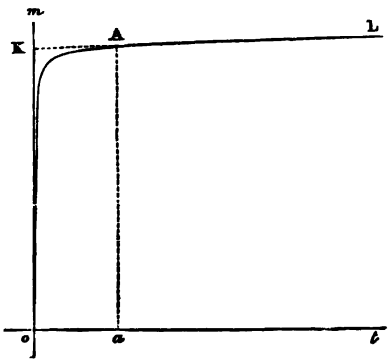


Fig. 231.

current is measured by the swing of the needle of a galvanometer. By varying the time of contact, the curve of the changes as a function of the time may be constructed. This curve rises rapidly, and without leakage would soon become horizontal; it tends in short to merge into the line AL (Fig. 231) the inclination of which with the axis of the abscissæ represents the leakage. The ordinate with the origin OK measures the true charge which can give the electrodes a difference of potential equal to that of the battery.

The experiment being repeated with variable electromotive forces, the curve representing the charges as a function of the electromotive

* VARLEY. *Transactions of Royal Society*, Vol. CVXI. p. 129. 1871.

† BLONDLOT. *Journal de Physique* [1], Vol. X., pp. 277, 333, 434. 1881.

force may be constructed. If the capacity were constant this curve would reduce to a straight line passing through the origin. It does really turn its convexity towards the axis of abscissæ—in other words, the charges increase far more rapidly than the electromotive forces. The angular coefficient of the tangent represents the true capacity at each point. We may consider as the *initial capacity* the limiting value of this coefficient when the electromotive force tends towards zero.

1085. The difference of potential of the two electrodes represents the difference which polarization establishes between the electromotive forces of contact of the two electrodes with the liquid, which were originally equal. If the surface of one electrode is very great, and as it were infinite in reference to the other, it may be assumed that the surface of the first, and therefore its electromotive

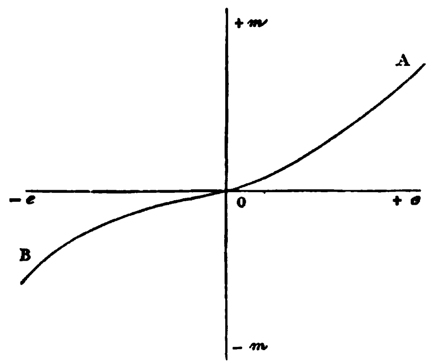


Fig. 232.

force of contact, is not modified; the difference observed represents then the variation of the electromotive force of contact of the second. If we take this latter alternately as positive or negative electrode—that is to say, charging it with oxygen or hydrogen, experiment gives the same value for the initial capacity defined as above. This then is independent of the direction of the polarization. If the electromotive forces and the corresponding charges are counted as positive or negative according as the active electrode is itself positive or negative, we obtain two curves which run into one another at the origin, as seen in Fig. 232.

Experiment shows that the capacity of a given electrode only depends on the difference of potential between this electrode and the liquid with which it is in contact, and not on the nature of the

liquid; it is the extension to a solid surface in contact with mercury of the law mentioned above (651) for mercury. The curve of Fig. 232 for water acidulated by sulphuric acid, agrees for any electrolyte which gives oxygen and hydrogen. Each electrolyte has on this curve its own origin corresponding to the normal electromotive force between it and the electrode.

In order to demonstrate this law, M. Blondlot places the two electrodes in different liquids separated by a porous diaphragm. As the electromotive forces of contact are unequal, the system forms a couple; but if this is closed until there is no current, the exact effect of polarization is to reduce the falls of potential on either side to equality. If one of the electrodes is very large, and remains without appreciable change, it is found that the initial capacity of the small one is then the same as when it is immersed in the same liquid as the large one. If the electrodes are equal, M. Blondlot shows from the following considerations that the capacities are equal.

Let e at a given instant be the difference of potential between the electrode and the liquid in contact; in the time dt this difference undergoes a total variation

$$(1) \qquad de = \frac{\partial e}{\partial t} dt + \frac{\partial e}{\partial q} dq ,$$

the first term representing the spontaneous depolarization by diffusion, the second the passage of a quantity of electricity dq across the electrode. If it be observed that the ratio $\frac{\partial e}{\partial q}$ represents the inverse of the capacity, and $\frac{dq}{dt}$ the intensity of the current i which traverses the electrode, we may write

$$(2) \qquad de = \left(\frac{\partial e}{\partial t} + \frac{i}{c}\right) dt .$$

When the couple is short-circuited a very feeble current is produced, which ultimately becomes constant; we have in that case $de = 0$, and for each electrode we may write

$$\frac{\partial e}{\partial t} + \frac{i}{c} = 0 .$$

$$\frac{\partial e'}{\partial t} - \frac{i'}{c'} = 0 .$$

If $c=c'$, as required by the law in question, it follows that

$$\frac{\partial(e+e')}{\partial t}=0\,;$$

that is to say, that when the circuit is broken, the algebraical sum of the electromotive forces should be constant—in other words, they should vary by quantities which are equal and of opposite signs; this is what is shown by experiment if each of the electrodes is connected with the inner coating of two identical condensers united by their external armatures, as shown in 1049. The outer coatings and one of the liquids being connected by a galvanometer, or, still better, by an electrometer, the connection of the short circuit is opened. If the two electrodes are equal, the needle is stationary, which proves that the two armatures experience equal and opposite changes of potential. The needle is deflected as soon as there is an appreciable difference of potential between the surfaces of the two electrodes.

Let us observe, in passing, that equation (2) would furnish a method of measuring the capacity of polarization c. The voltameter being connected with a battery of less than the maximum electromotive force, the intensity of the current of polarization i is measured by a galvanometer, and, connection being broken, the loss of electromotive force $\frac{\partial e}{\partial t}$ is measured by an electrometer.

CHAPTER VI.

CONSTANTS OF COILS.

1086. Construction of Coils.—We have seen, in Chapters IV. and V. of the First Part, how the properties of cylindrical and circularly-wound coils may, apart from the resistance, be deduced from the geometrical data of the coil alone; but these calculations presuppose a regularity of winding which is very difficult to obtain, and which is only realised under exceptional circumstances. The most minute precautions are necessary, for instance, that the radius of each winding of the same layer be the same, and that there be the same distance between each of them.

The first condition is that the core is truly cylindrical; this may be ascertained by means of calliper compasses, or by a suitable gauge. The frame is usually wood or metal. Metal frames may be the seat of induction currents, which are very detrimental in experiments where variable currents are made use of. The effects of induction are attenuated by making a slit along a diametral plane, which is filled up with an insulating plate of ivory or of ebonite. Wooden frames are apt to get out of shape in the course of time; but if a hard dry wood be used, such as mahogany soaked with hot paraffine—particularly if care be taken to construct the frame from a block consisting of a great number of pieces glued together, and with fibres in different directions—frames are obtained in which the changes of shape may be altogether neglected.

If the diameter of the wire, with its insulating envelope, is constant, careful winding readily gives equidistant spirals. The successive layers are usually separated by an insulating plate, such as one or more sheets of paraffine paper. The use of a counter enables us to ascertain the number of windings in each layer during the coiling, and it is advisable to verify this number directly before the layer is covered. The inclinations of the windings of successive layers are obviously in opposite directions; the effect of inclination is neutralised by making the coil of an even number of layers.

The length of the wire is determined either directly or by measuring the diameter of the coil before and after coiling each layer, or by two corresponding circumferences. A flexible metal band, wound on the coil, measures the circumference with great approximation. We may assume that the mean line of the band retains the same length, notwithstanding its curvature; so that, if a is the radius of the cylinder, the observed length corresponds to the radius a, added to half the thickness of the strip, and from it will be deduced the length of the circumference of radius a.

It must, however, be observed that if the coil contains several layers, separated by an insulating substance, a measurement of the diameters or circumferences before and after the coiling of a layer does not give the exact value of the mean diameter, for the pressure of the wires crushes the insulating substance.

On the other hand, we cannot, without special precautions, deduce the length of a wire from a measurement made before winding, because the wire elongates to a varying extent during winding, especially in consequence of the strain which it must undergo to obtain regular layers. According to Werner Siemens,* this elongation may, with fine wires, amount to 5 or 6 per cent.; it is less, but still considerable, even with thicker wires. If we wish to measure the wire before winding, it ought to be subject to the same strain as during the winding; the change of length due to the fact of winding itself, may be neglected if the radius of curvature is very large in comparison with the diameter of the wire.

All formulæ for coils imply the condition that the current is uniformly spread in each section of the wire, or, what amounts to the same thing, that the current may be considered as concentrated on the axis of the wire itself. This condition is not rigorously fulfilled, especially with variable currents; the calculation will only give then a first approximation, and the error cannot be neglected with rather thick wires.

There is, lastly, an evident condition, but one which experiment shows cannot be easily realised, which is, that the different windings be completely insulated from each other. In order to ascertain whether a coil is properly insulated, Lord Rayleigh † interposes its two insulated ends between the two coils of a Hughes' induction

* WERNER SIEMENS. *Pogg. Ann.*, Vol. CXVII., p. 327. 1866.

† Lord RAYLEIGH and Mrs. SIDGWICK. *Transactions of the Royal Society for 1884*, p. 419.

balance (987). Silence is broken if there is any connection between the windings.

Only coils of a very special form and construction can be submitted to calculation; and such coils should, so to speak, serve as standards. For ordinary coils, the constants can only be determined by comparison with the former.

1087. Magnetic Field of a Coil.—We shall obtain the magnetic action of a coil at a point, for unit current, by the general methods employed in determining the intensity of a magnetic field, and to which we shall afterwards return; but in comparative experiments we ought to seek for rapid methods, which require neither a constant current nor a measurement of the intensity.

In order to determine the direction of the force which a coil B exerts at a point P, it is sufficient to place at this point a magnetised needle of very small dimensions, subject at the same time to the action of an external field, such as the terrestrial field, and to turn the coil about the point P, until the passage of a current, or its change of direction, produces no deflection. The magnetic axis of the needle then gives the desired direction.

In the case of very powerful currents, the use of magnetic images often gives valuable indications; the successive elements of the iron filings trace out the lines of force, and give a direct idea of the form and intensity of the field.

The ratio of the actions F and F′ for unit current of two coils B and B′, at two points P and P′, can be obtained by the method used for comparing two galvanometers (875). The coils are placed so that their actions at the points P and P′ are sensibly perpendicular to the external fields—or, at least, that the projections X and X′ of the actions of the coils, on a plane perpendicular to the axis of rotation of the needles, are perpendicular to the projections H and H′ of the external fields, and very small needles are placed at these points. The deflections δ and δ', corrected for graduation, produced by the same current, which traverses the two coils, give

$$(1) \qquad \frac{X}{X'} = \frac{H}{H'}\frac{\delta}{\delta'}.$$

1088. In order to eliminate the ratio of the fields H and H′, we shall make the two coils act on the same needle (876).

Suppose, for instance, that the coils are traversed by the currents I and I′ respectively. Let δ and δ_1 be the deflections which the

needle experiences from the same currents, when the actions of the coils are in the same or contrary directions; we shall have

$$(2)\qquad \begin{aligned} H \tan \delta &= XI + X'I', \\ H \tan \delta_1 &= XI - X'I'. \end{aligned}$$

From this follows, if β is the ratio $\dfrac{\tan \delta_1}{\tan \delta}$,

$$(3)\qquad \frac{X'}{X} = \frac{I}{I'}\,\frac{1-\beta}{1+\beta}.$$

The ratio $\dfrac{I}{I'}$ is equal to unity if the same current traverses both coils. If the coils are shunted on a circuit, this ratio is equal to the inverse ratio of the resistances g and g' of the corresponding branches. We may then modify these resistances, so that the current is null in the second experiment, and we have

$$\frac{X}{X'} = \frac{I'}{I} = \frac{g}{g'}.$$

If the coil B to be compared is a tangent galvanometer of known dimensions, which enables us to calculate X in absolute value, equation (3) will give X'.

When the needle is not infinitely small in comparison with its distance from the windings which the current traverses, the values of X and X'—the ratio of which follows from experiment—are not exactly those corresponding to the points P and P'. We may then introduce a correction (746 and 796) for the length of the needle.

1089. Surface of a Coil.—Magnetic Moment.—The value of the electrodynamical surface of a coil, or its magnetic moment for unit current, in the case of a cylindrical coil of mean radius a (727), is

$$S = n\pi a^2\left(1 + \frac{1}{3}\frac{c^2}{a^2}\right) = n\pi a^2(1+\gamma),$$

γ being the term of correction.

The value of the surface S may be deduced from the intensity of the field of the coil at a point situated in a principal position, and sufficiently far from the centre; the component X of the field will be

determined by comparison with the value X′ of a standard coil, as in the preceding paragraph.

In the case of a cylindrical coil, the channel of which is of such small dimensions that we may neglect powers higher than the square of the ratios of these dimensions to the mean radius, the action X at a point P on the axis, at a distance x from the mean plane, may be written

$$X=\frac{2S}{x^3(1+\gamma)}\left[1+\frac{c^2}{4a^2}+\frac{2b^2}{x^2}-\frac{3a^2}{2x^2}+\frac{3.5}{2.3}\frac{a^4}{x^4}-\dots\right];$$

or, denoting by $1+\epsilon$, the quotient of the bracket by $1+\gamma$,

$$X=\frac{2S}{x^3}(1+\epsilon).$$

If the point P were in the mean plane at a distance r from the centre, we should have

$$X=\frac{S}{r^3(1+\gamma)}\left[1+\frac{c^2}{3a^2}+\left(\frac{3}{2}\right)^2\frac{c^2}{r^2}-\frac{3b^2}{2r^2}+\frac{1}{2}\left(\frac{3}{2}\right)^2\frac{a^2}{r^2}+\frac{1}{3}\left(\frac{3.5}{2.4}\right)^2\frac{a^4}{r^4}+\dots\right].$$

which may be written

$$X=\frac{S}{r^3}(1+\eta).$$

Taking, as term of comparison, a single circle of radius R, with a small magnetic needle at the centre, and traversed by the same current as the coil, we find*

$$S=\frac{\pi x^3}{R}\frac{1+\beta}{1-\beta}(1-\epsilon).$$

These two arrangements have the drawback of requiring a very exact measurement of the distances x and r, the cubes of which occur in the principal factor. The difficulty might be got over by making two observations at two different distances from the needle, as the displacement of the frame could be determined with great accuracy. The experimental errors acquire then great importance.

* F. KOHLRAUSCH. *Göttinger Nachrichten*, No. 20, p. 654. 1882.

If the support of the needle be turned through 180° about an axis which passes through the middle of the frame, the mean of the results obtained on either side corresponds to the semi-distance of the two positions of the needle; but the experiment will then be very complicated because the telescope and the scale must also be moved.

1090. It is better to arrange the experiment so as to get rid of any measurement of the distance. The two coils, the surfaces S and S′ of which are to be compared, are arranged concentrically, the axes coinciding. They are made to act on the same needle placed at a great distance, and the deflections δ and δ_1 are observed which are produced when a current passes through the coils in one direction or in opposite directions. Putting $\beta = \frac{\tan \delta_1}{\tan \delta}$, we have thus

$$\frac{X'}{X} = \frac{1-\beta}{1+\beta}.$$

By taking the means, a series of alternate experiments enables us to eliminate the variations of the principal current and the changes of declination. If ϵ and ϵ', or η or η', are the terms of corrections for the two coils in the two principal positions,

$$\frac{S}{S'} = \frac{1-\beta}{1+\beta}\frac{1+\epsilon'}{1+\epsilon},$$

or

$$\frac{S}{S'} = \frac{1-\beta}{1+\beta}\frac{1+\eta'}{1+\eta}.$$

The distance of the needle from the centre of the coils only comes then into the corrections, and may easily be determined with sufficient accuracy.

The method is particularly accurate when the surfaces to be compared are near one another, the ratio β being then very small.

In order to eliminate the error arising from want of centring, they are mounted on a frame which can turn about a vertical axis; and several series of experiments are made by turning through 180° the system of the two coils.

In this case it is not even necessary that the coils have different radii, so that one fits in the other. If the distance of the needle is great enough, we may place them on either side of the axis. The mean of the deflections observed for the positions gives very

appreciably that which would correspond to coils centred on the axis of rotation.

When the coils are eccentric, it is evidently better to place the needle in the second principal position—that is to say, in the mean plane of the coils.

1091. The ratio of the surfaces of two coils is also obtained by comparing the discharges induced under the action of the same external field. Suppose, for instance, that the terrestrial field is utilised. The coils are mounted on the same vertical axis, and situate first in a plane perpendicular to the magnetic meridian. When the system is suddenly turned through 180° about a vertical axis, the flow of magnetic forces across the two coils are then $2HS$ and $2HS'$ (**258**). Their ratio is that of the surfaces S and S′, and this ratio is independent of the angle of rotation. The case is the same if the system turns about a horizontal axis in the magnetic meridian, the variation of the flow of force is produced then by the vertical component Z of the terrestrial field.

Instead of rotating the system of coils, they may be left stationary, and the direction of the field changed. For this it is sufficient that the field be produced by an external current, the direction of which is reversed. The ratio of the flow of force across the two coils is then equal to the ratio of the surfaces multiplied by the ratio of the mean actions (**753**). If the coils are not of exactly the same diameter, the corrections for calculating the ratio of the mean actions are rather complicated, even in the case in which the induced coils are in the plane of the inducing current. It is better then to use as inducing circuit, a system of two, three, or four frames (**749** to **751**), the mean diameters of which, the distances, and the numbers of windings are so chosen as to form a sensibly uniform field.

1092. Whatever be the method used, the problem reduces in all cases to comparing two flows of force Q and Q′.

We might in the first case, as with permanent currents, connect the two induced coils in the same circuit containing a ballistic galvanometer, and observe the swings α and α' which correspond to the cases in which the two flows of induction act in the same or in contrary directions. The quantities of induced electricity m and m', being proportional to the swings as well as to the flows of force, we shall have, when all corrections are made (**883**),

$$\frac{\alpha}{\alpha'}=\frac{Q+Q'}{Q-Q'},$$

or, if β is the ratio $\frac{a'}{a}$,

$$\frac{Q'}{Q}=\frac{1-\beta}{1+\beta}.$$

This, however, would in general be only a very insufficient approximation, owing to the difficulty of measuring the swings, especially with movable frames, as the times of rotation should be very small in comparison with the time of oscillation of the ballistic galvanometer.

1093. By means of the differential galvanometer, the observation can be brought to that of a state of equilibrium. The coils B and B′ communicate separately with the two coils of a differential galvanometer, and the total resistances R and R′ of the two circuits are regulated by trial, so that the needle is stationary when induction is produced, either by the rotation of coils or by the inversion of an inducing circuit. As the quantities of induced electricity m and m' are then equal, it follows that the flows of induction are proportional to the corresponding resistances (515), and we have

$$\frac{Q'}{Q}=\frac{R'}{R}.$$

It is sufficient then to determine the ratio of the resistances as soon as equilibrium is obtained, so as to avoid the influence of change of temperature.

This method would require an absolutely symmetrical differential galvanometer; but if the defect of adjustment is very slight, it is sufficiently eliminated provided the connection of the coils with the frames can be interchanged. The resistances which re-establish equilibrium being R_1 and R_1', we have sensibly

$$\frac{Q'}{Q}=\frac{1}{2}\left(\frac{R'}{R}+\frac{R_1}{R_1'}\right).$$

A series of coils of graduated surfaces would enable us to act by opposition. The coils B and B′ being placed in the same circuit in such a manner that their flows of induction are in opposite directions, the auxiliary coils of increasing surfaces are successively interposed until the deflection is zero or changes its sign. If Q_1 and Q_2 are

the flows of force relative to the surfaces which should be added to the coil, B′ for instance, to obtain the weakest deflections in opposite directions, we have sensibly

$$Q = Q' + \frac{Q_1 + Q_2}{2}.$$

In strictness, the reading of the deflections δ_1 and δ_2 relative to the two nearest combinations, would enable us to obtain a closer approximation; for we may write

$$\frac{Q - Q' - Q_1}{Q_2 + Q' - Q} = \frac{\delta_1}{\delta_2},$$

which gives

$$Q = Q' + Q_1 \frac{\delta_2}{\delta_1 + \delta_2} + Q_2 \frac{\delta_1}{\delta_1 + \delta_2}.$$

The methods of reduction to zero make the observations much more precise, because we may multiply the swings by conveniently interchanging the induced circuits, and render evident deflections which are too weak to be directly observed.

Nevertheless, the use of the differential galvanometer with graduated surfaces does not give the degree of sensitiveness which might be expected.

The experiments are very regular, and the state of equilibrium is obtained with great accuracy in the case of coils which are almost identical with each other as to their form and their dimensions; but this is not the case if there is a great difference in the mode of coiling or in the diameters. In this case, if induction is produced by the rotation of the system of coils in the terrestrial field, and compensation is nearly established, the needle receives two very distinct swings of different kind in the same or in opposite directions, and it is impossible to obtain a condition of equilibrium. If the needle is finally at rest, it has not been stationary during the rotation. Analogous effects are seen even when the induction is produced by reversing an inducing current.

1094. Coefficients of Mutual Induction.—A coefficient of induction is an instantaneous electromotive force—or more exactly, the integral of an electromotive force in respect of time. Two coefficients may be compared by the corresponding electrical discharges, as the permanent electromotive forces are by continuous currents.

The surface of a coil is nothing but the coefficient of mutual induction of this coil with a circuit capable of giving with unit current, a uniform field equal to unity. The methods pointed out previously for comparing surfaces may then be used for the coefficients of mutual induction.

Let us consider, for instance, two systems of coils A and *a*, A′ and *a*′, the coefficients of mutual induction of which are respectively M and M′, and let us suppose the two systems without action on each other. The coils A and A′ are placed in the circuit of the same inducing current I, the coefficients M and M′ are then proportional to the flows of force Q and Q′, from the inducing current into the coils *a* and *a*′ respectively. If a ballistic galvanometer is interposed in a circuit which contains the coils *a* and *a*′, the connections are so arranged that the induced currents due to the flows

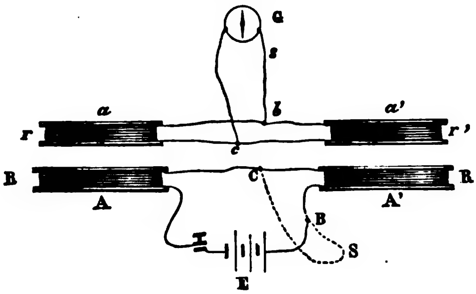


Fig. 233.

of force Q and Q′, are alternately in the same and opposite directions. The ratio of the swings α and α' (**1092**) will give then the ratio $\frac{Q'}{Q}$ or $\frac{M'}{M}$.

In like manner, a differential galvanometer, the two frames of which are in the circuits of the coils *a* and *a*′ respectively, will give the ratio of the coefficients M and M′, by the ratio of the total resistances of the two circuits, when the condition of equilibrium is realised.

These methods present special difficulties, and possess no great accuracy. It is better to attempt to produce a compensation of induced currents analogous to that which Wheatstone's bridge gives for permanent currents.

1095. In the method described by Maxwell,* the inducing

* MAXWELL. *Electricity and Magnetism*, Vol. II., p. 355.

circuit comprises the coils A and A′ (Fig. 233). The wires are connected, so that the induced currents on either side add themselves, and between the two points *b* and *c* of the induced circuit a shunt of resistance *s*, containing a galvanometer G, is inserted. The resistances *r* and *r*′, on either side of the shunt, are varied until the needle is stationary, when the inducing circuit is opened or closed.

If I is the inducing current, *i* and *i*′ the induced currents in the resistances *r* and *r*′, the coefficients of self-induction of which are L and L′, *l* the coefficient of self-induction of the shunt *s*, which contains the galvanometer, and $i' - i$ the corresponding current, we have, in general, equations

$$ir + (i' - i)s + L\frac{di}{dt} + l\frac{d(i' - i)}{dt} + M\frac{dI}{dt} = 0,$$

$$i'r' - (i' - i)s + L'\frac{di'}{dt} - l\frac{d(i' - i)}{dt} + M'\frac{dI}{dt} = 0.$$

Integrating these equations for the whole duration of the variable state, and calling I_0 the variation of the inducing current, *m* and *m*′ the induced discharges for the currents *i* and *i*′, we get

$$mr + (m' - m)s \pm MI_0 = 0,$$

$$m'r' - (m' - m)s \pm M'I_0 = 0.$$

The sign + in front of the latter terms corresponds, for instance, to the production of the inducing current, and the sign − to its suppression. The coefficients of self-induction of the various parts of the induced circuit have disappeared, which might be foreseen, as the induced current is null at the two limits.

If there is no discharge in the galvanometer, we have $m = m'$, and therefore

$$\frac{M}{M'} = \frac{r}{r'}.$$

1096. M. Brillouin* has arranged the experiment in a somewhat different manner. The coils *a* and *a*′ are connected, so that the induced currents are in opposition; a shunt of resistance *s* is placed between two points *b* and *c* of an induced circuit; the galvanometer, whose resistance is *g*, and coefficient of self-induction *l*, is interposed

* BRILLOUIN. *Ann. de l'École Normale* [2], Vol. XI., p. 352. 1882.

in one part of the circuit—for instance, on the side of the coil a'; the intensity of the current in the shunt is $i+i'$. The general equations

$$ir+(i+i')s+\mathrm{L}\frac{di}{dt}+l\frac{d(i+i')}{dt}+\mathrm{M}\frac{d\mathrm{I}}{dt}=0,$$

$$i'r'+(i+i')s+(g+\mathrm{L}')\frac{di'}{dt}+l\frac{d(i+i')}{dt}+\mathrm{M}'\frac{d\mathrm{I}}{dt}=0,$$

give, for the total duration of the variable state,

$$mr+(m+m')s\pm\mathrm{MI}_0=0,$$

$$m'r'+(m+m')s\pm\mathrm{M}'\mathrm{I}_0=0.$$

If the resistance s be adjusted so that the discharge m' in the galvanometer is null, we have

$$\frac{\mathrm{M}}{\mathrm{M}'}=\frac{r+s}{s}.$$

In order that the discharge be null, we must have $\mathrm{M}>\mathrm{M}'$; the galvanometer should be placed on that side of the coil on which the induction is weakest.

As the coils a and a' are always in opposition, and the galvanometer is in the induced current, we may, lastly, place a shunt S between the two points of the inducing circuit (Fig. 233)—for instance, on the side of the coil A'. If, then, I and I' are the currents in the coils A and A',

$$(r+r'+g)i+(\mathrm{L}+\mathrm{L}'+\lambda)\frac{di}{dt}+\mathrm{M}\frac{d\mathrm{I}}{dt}-\mathrm{M}'\frac{d\mathrm{I}'}{dt}=0,$$

which gives, for the whole duration of the variable state,

$$(r+r'+g)m+\mathrm{MI}_0-\mathrm{M}'\mathrm{I}'_0=0.$$

If the resistance S is chosen, so that the discharge m is null, the equation reduces to

$$\mathrm{MI}_0=\mathrm{M}'\mathrm{I}'_0.$$

As the currents I_0 and I'_0 correspond to a permanent state, we have

$$\mathrm{S}(\mathrm{I}_0-\mathrm{I}'_0)=\mathrm{R}'\mathrm{I}'_0,$$

and therefore

$$\frac{M}{M'}=\frac{I'_0}{I_0}=\frac{S}{R'+S}.$$

We should have $M<M'$; the shunt will then be placed on that side of the coil on which the induction is greatest.

In all cases the determination of the ratio of the two coefficients of mutual induction M and M' is reduced to that of the ratio of the two resistances.

The only objection that can be raised against these methods is the want of sensitiveness. It is remedied either by multiplying the discharges by timing the opening and closing of the circuit with the motion of the needle, or substituting a permanent regime for a single throw, by means of a rotating break. As the successive induced currents are equal and opposite in direction, only one of the two effects of opening and closing should be used; the inducing circuit being always closed, it is sufficient that the break closes the galvanometer as long as the induced current lasts which is to be eliminated.*

This latter arrangement has the drawback that the determination of the ratio of the coefficients of induction depends on the ratio of resistances traversed by the current of the battery—a resistance the value of which is more difficult to ascertain, owing to heating, than that of the corresponding resistances of the induced circuit in the previous method.

We may remark again, *apropos* of the second method, that throughout the whole time of making or breaking the inducing current, the induced current is always in the same direction; the elements of the discharge m are then always of the same sign, and consequently, if the discharge is null, each of the elements must be null separately; the induced currents are then at each instant equal and of opposite signs, and we may advantageously use the telephone to ascertain the condition of equilibrium.

1097. COEFFICIENTS OF SELF-INDUCTION.—The coefficients of self-induction are compared by an analogous method of compensation.† If the branches of a Wheatstone's bridge contain conductors whose coefficients of self-induction are not null, the equilibrium of the needle in the bridge, for a very short variation

* BRILLOUIN. *Loc. cit.*, p. 367.

† MAXWELL. *Electricity and Magnetism*, Vol. II., p. 317.

the principal current, requires, besides the condition for the permanent state, that we have (951)

$$\frac{L_b}{b} - \frac{L_{b'}}{b'} = \frac{L_a}{a} - \frac{L_{a'}}{a'}.$$

Moreover, any given variations are in equilibrium if the resistance of each branch is proportional to its current of self-induction. In order to make use of this property, the two coils to be compared, whose coefficients are L and L′, are placed in the two branches a and a', for instance, the other branches being supposed to have no induction. Complete equilibrium requires that the two conditions

$$\frac{L}{L'} = \frac{a}{a'} = \frac{b}{b'}$$

be simultaneously realised.

Two resistance-boxes, which have no induction of their own, are necessary; one is placed in the same branch as one of the coils, the other on one of the free branches. The experiment is a long one, for the equilibrium of the permanent currents must first be realised, before trying the effect of the transient currents; we must also see if the conditions for the permanent and for the variable state are not independent. The final result can only be attained after a series of trials.

1098. Comparison of a Coefficient of Self-Induction and a Coefficient of Mutual Induction.*—The coil of coefficient L is introduced in the branch AC of Wheatstone's bridge (Fig. 234), and the second coil is introduced in the circuit of the battery, near the first. Let M be their coefficient of mutual induction. Two electromotive forces of induction are thus introduced in the branch AC, which, if made equal and opposite, the equilibrium of the bridge for the permanent condition will also satisfy the variable state, for each branch of the bridge is conjugate in respect of the battery, and therefore indifferent to the variations of the principal current.

In the variable state, the difference of potential between the summits A and C is

$$a\alpha + L\frac{d\alpha}{dt} + M\frac{dI}{dt},$$

* Maxwell. *Electricity and Magnetism*, Vol. II., p. 356.

while the difference of potential between the summits A and D is equal to $a'\alpha'$.

Complete equilibrium requires the two conditions

$$a\alpha = a'\alpha',$$

$$L\frac{d\alpha}{dt} + M\frac{dI}{dt} = 0.$$

As $I = \alpha + \alpha'$, it follows that

$$\frac{L}{M} = -\left(1 + \frac{a}{a'}\right).$$

As the coefficient L is essentially positive, M must be negative,

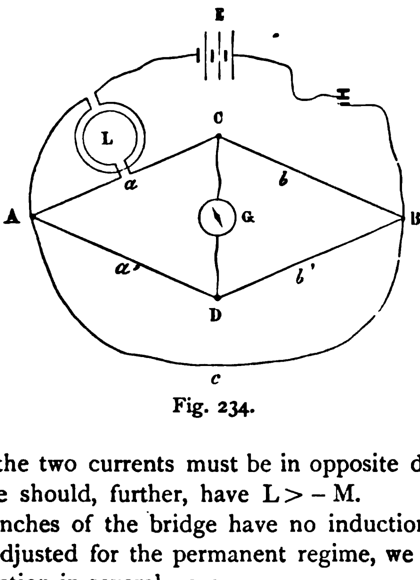


Fig. 234.

and therefore the two currents must be in opposite directions in the two coils. We should, further, have $L > -M$.

As the branches of the bridge have no induction of their own, and are first adjusted for the permanent regime, we may adjust the effects of induction in several ways:—

1st. The value of M is varied by changing the relative position of the two coils, until the variable currents are in equilibrium.

2nd. M being supposed constant, and smaller than L in absolute value, the two resistances are modified in the same ratio, until equilibrium is obtained.

3rd. Between the points A and B, at which the battery terminates, a shunt of resistance c is introduced which has no induction of its own.

The intensity γ of the current which passes through the shunt satisfies equations

$$I = \alpha + \alpha' + \gamma,$$
$$c\gamma = \alpha'(a' + b') = \alpha(a + b).$$

The condition of equilibrium for the variable state remains the same, and gives

$$a'(L + M) + a\left(1 + \frac{a' + b'}{c}\right) M = 0,$$

or

$$\frac{L}{M} = -\left(1 + \frac{a}{a'} + \frac{a + b}{c}\right).$$

The discussion of the equations shows that the relative error for the ratio $\frac{M}{L}$ is about 100 times that made in adjusting the resistances.* To obtain the greatest precision of which the method is susceptible, the galvanometer should be 100 to 1000 times more sensitive for transient than for permanent currents. It is advantageous to produce a permanent state by means of a rotating commutator.

1099. Comparison of a Capacity with a Coefficient of Induction. — By using repeated discharges, a capacity may be compared with a coefficient of mutual induction (**1063**). By the method of equilibrium we may also determine the ratio of a capacity to the coefficient of self-induction L of a coil. Place the coil in the branch AD of Wheatstone's bridge (Fig. 235) and the condenser F as a shunt on the branch CB. The condition that there is no current in the galvanometer is that the two points CD are at the same potential. If V is the difference of potential of the two armatures at a given time, this condition gives the two equations

$$a\alpha = a'\alpha' + L\frac{d\alpha'}{dt},$$
$$b\beta = b'\beta' = V.$$

* M. Brillouin. *Loc. cit.*, p. 348.

Further, since no electricity passes in the branch CD, we have also

$$C\frac{dV}{dt}=\alpha-\beta,$$
$$\alpha'=\beta'.$$

Eliminating α, β, β', and V, we get

$$\frac{d\alpha'}{dt}\left(Cb'-\frac{L}{a}\right)=\alpha'\left(\frac{a'}{a}-\frac{b'}{b}\right).$$

The second member is zero if the bridge is in equilibrium for

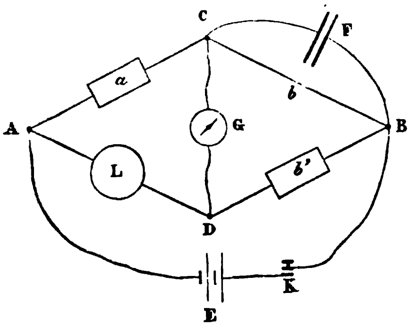


Fig. 235.

a permanent state. The first member will also be zero, and there will be equilibrium for the variable state, provided

$$L=ab'C.$$

The adjustment can only be made by trials. It requires three variable resistances, two of which, a and b, are standardised in the two branches which are free.

The formulæ of electrical oscillations, reduced, as in **1056**, to

$$T^2=\pi^2CL,$$

give also a relation between the duration, the capacity, and the coefficient of self-induction, which might be utilised in practice.

1100. COMPARISON OF A COEFFICIENT OF SELF-INDUCTION WITH A RESISTANCE.—Let us suppose the coil placed in the

branch b of Wheatstone's bridge, and the adjustment made for the permanent state. During the variable period the bridge is traversed by a current (949)

$$i = -\frac{aE_b}{M} = \frac{aL}{M}\frac{d\beta}{dt},$$

and the discharge, when the principal current is made, is

$$m = \int_0^\infty i dt = \frac{aL}{M}\beta_0.$$

The intensity β_0 of the permanent current is equal to $\frac{a'E}{N}$ (948), which gives

$$m = \frac{aa'L}{MN}E.$$

On the other hand, if the resistance b is made to vary by δb, the permanent current, which traverses the galvanometer, is (947)

$$i = E\frac{a'(b+\delta b) - ab'}{\Delta} = E\frac{a'}{\Delta}\delta b,$$

or, replacing Δ by the value $\frac{MN}{a}$, which represents equilibrium of the bridge,

$$i = E\frac{aa'}{MN}\delta b;$$

consequently

$$L = \frac{m}{i}\delta b.$$

If α be the swing for the discharge m, γ the deflection produced by the permanent current i, which corresponds to the defect of adjustment, we deduce

$$L = \frac{T}{\pi}\frac{\alpha}{\tan\gamma}\delta b.$$

It would have been the same thing if the change of resistance had been made in the branch b'.

Instead of breaking or making the current, it would be better to reverse it each time; twice the effect would then be obtained. Moreover, the circuit of the battery is permanently closed, except during the very short time of the reversal, and we have the advantage of obtaining a more uniform current. The principal difficulties of the method arise in fact from variations in the intensity of the principal current.

1101. Let us consider, in the same circuit, traversed by a sinusoidal current, the portions AB and A'B', formed respectively of the resistances R and R' with the coefficients of self-induction L and L'.* The differences of potential between A and B and between A' and B' are, at a given moment, if there were no mutual induction,

$$\mathrm{RI}+\mathrm{L}\frac{d\mathrm{I}}{dt} \quad \text{and} \quad \mathrm{R'I}+\mathrm{L'}\frac{d\mathrm{I}}{dt}.$$

If then the two ends of the resistance R and R' are successively connected with the electrodes of an electrometer arranged so as to measure the mean of the squares of potentials (**905**), the corsponding deflections satisfy the equation

$$\frac{\delta}{\delta'}=\frac{\mathrm{R}^2+\dfrac{4\pi^2\mathrm{L}^2}{\mathrm{T}^2}}{\mathrm{R}'^2+\dfrac{4\pi^2\mathrm{L}'^2}{\mathrm{T}^2}}.$$

Assume, as a particular case, that the resistances R and R' are equal and the coefficient L' is null, we shall have

$$\mathrm{L}=\frac{\mathrm{TR}}{2\pi}\sqrt{\frac{\delta}{\delta'}-1}.$$

1102. Apparatus with Variable Coefficient of Induction.—In certain circumstances it is useful to have apparatus with variable coefficient of induction, standardised like boxes of resistance or of capacity. The two following arrangements have been used by M. Brillouin.†

* Joubert. *Ann. de l'École Normale* [2], Vol. x., p. 131. 1881.

† Brillouin. *Ann. de l'École Normale* [2], Vol. xi., p. 352. 1882.

The first apparatus comprises an internal inducing coil with a somewhat thick wire, and an external coil on which is wound a tore formed of twenty insulated wires slightly twisted. The coefficient of mutual induction M between the internal coil and any one of the twenty wires is the same. By means of a commutator the induced current may be made to traverse $20 - p$ wires in one direction and p others in the opposite direction. The coefficient of mutual induction between the internal coil and the twenty wires thus joined is $2(10 - p)$ M. The wires, moreover, being joined end to end, the resistance of the external coil is constant.

In the second apparatus the coefficient varies continuously. It consists of an external inducing coil of thick wire in the form of a cylinder, and of an internal coil of fine wire placed in the centre of the first, and which can turn about an axis along a common diameter to the two coils. If the external coil was infinitely long, the internal field would be uniform, and the coefficient of mutual induction will be strictly proportional to the cosine of the angle of the two axes; but if the length of the great coil is ten times that of the small one, the error may be neglected.

The same condition will be approximately realised by placing the centre of the induced coil in the mean plane of a coil with any channel and of a much greater radius. An uniform internal field would finally be obtained with a spherical coil, the successive windings of which are arranged in equi-distant parallel planes.

CHAPTER VII.

MEASUREMENT OF RESISTANCES IN ABSOLUTE VALUE.

1103. Various Methods.—As the resistance of a conductor in electromagnetic units has the same dimensions as a velocity (**526**), the determination of a resistance in absolute value necessarily implies the measurement of a length and of a time, and the other quantities which come into play in the experiments always reduce to numerical ratios.

The principal methods used in measuring electrical resistance may be classed in two groups. In the first case, the resistance of a conductor is deduced from the thermal energy disengaged in this conductor in a certain time by a current of known intensity. In the second case, the resistance of a circuit is determined by the current which is produced by a known electromotive force, constant or variable. This electromotive force should itself be related to the fundamental units; but the only electromotive forces of which the value can be directly determined in electromagnetic units are those arising from the effects of induction.

We might still compare a resistance with a coefficient of self-induction (**1100**), and determine this latter either by direct calculation or by a comparison with a coefficient of mutual induction, which would be deduced from the dimensions of the conductor; but it can be conceived that the experiment would possess no great accuracy.

1104. Calorimetrical Measurements.—The thermal energy W developed by a current I in unit time t in a resistance R satisfies equation

$$R = \frac{W}{I^2 t}; \qquad (1)$$

according to Joule's law (**917**), a measurement of the quantities W, I, and t will give the value of R.

1105. Induced Discharges.—In a circuit which contains no permanent electromotive force, the current i induced by the variation of the flow of magnetic force Q satisfies the differential equation (518)

$$Ridt + d(Q + Li) = 0.$$

The quantity of electricity $m = \int idt$ which traverses the circuit in the time $t_2 - t_1$ is defined by the equation

$$Rm + [Q + Li]_{t_1}^{t_2} = 0.$$

When the current is zero at the two limits, or if it has resumed the same value as well as the coefficient of self-induction, we get simply

$$R = \frac{Q_1 - Q_2}{m}.$$

We may, first of all, displace the circuit in an invariable magnetic field; the difference $Q_1 - Q_2$ represents then the difference of the flow of magnetic force across the surface of the circuit in the two extreme positions.

Suppose, for instance, that the circuit is movable about an axis in a uniform field; if H is the component of the field perpendicular to the axis, S the maximum projection of the surface of the circuit on a plane passing through the axis, x_1 and x_2 the angles of the surface S with the meridian of the field in the two extreme positions, we have

$$Q_1 - Q_2 = HS(\sin x_1 - \sin x_2).$$

If $x_1 = \frac{\pi}{2}$, and $x_2 = \frac{3\pi}{2}$, that is to say, if the circuit is displaced through 180° from a position in which it is perpendicular to the field, we get

$$R = \frac{2HS}{m}. \tag{2}$$

Instead of displacing the circuit, we may displace the field itself. Suppose a magnet, of magnetic moment M, is placed in the centre of a frame. When the axis of the magnet makes an angle x with

the plane of the frame, the moment of the action of the frame on the magnet for unit current is GM $\cos x$; if the factor G may be considered as independent of the angle x, the flow of force from the magnet, and which traverses the circuit, is equal to GM $\sin x$.

Between two different positions, x_1 and x_2, the variation of the flow of force is then

$$Q_1 - Q_2 = MG(\sin x_1 - \sin x_2).$$

For a displacement of 180° from a position in which the magnet is perpendicular to the frame, we shall have still

$$R = \frac{2MG}{m}. \tag{3}$$

The flow of force Q may be produced by an external current I, which traverses a circuit fixed in respect of the former, and the intensity of which passes from the value I_1 to the value I_2. If M is the coefficient of mutual induction of the two circuits, we have

$$R = \frac{M(I_1 - I_2)}{m}. \tag{4}$$

The numerator is equal to $\pm$ MI if the phenomenon corresponds to the suppression or the establishment of the inducing current; it is equal to 2MI if the direction of the current is reversed.

1106. Damping of a Magnet or a Frame.—When a magnet is movable in the field of a closed circuit, or when a circuit is displaced in a magnetic field, the currents induced in the circuit tend to counteract the motion; the excess of the retardation caused by closing the circuit will give a measure of its resistance.

The problem relating to the oscillations of a magnet in the centre of a frame has been investigated in 845. When the deflections are very small, the resistance of the circuit is given by the expression

$$R = \frac{G^2}{2}\frac{M}{H}\frac{\pi^2 \phi_0}{\tau_0\left(\frac{\lambda}{\phi} - \frac{\lambda_0}{\phi_0}\right)} + L\frac{\lambda}{\tau}, \tag{5}$$

which includes the constant G of the frame on the magnet, the ratio of the magnetic moment M of the magnet to the horizontal component H of the external field, and the data furnished by the study of oscillations for the case in which the circuit is opened or closed.

The oscillations of the frame itself in a magnetic field, with an open or a closed circuit, will furnish an analogous expression.

Let us suppose that the frame of surface S is suspended by a wire whose couple of torsion is C, and is in equilibrium in the meridian; the flow of force which traverses it for a deflection x is HS $\sin x$, and the corresponding electromotive force HS $\cos x \frac{dx}{dt}$. The equation of the current is

$$L\frac{dI}{dt} + RI + HS\cos x\frac{dx}{dt} = 0.$$

The moment of the couple which acts on the current is

$$IHS\cos x - C_1\frac{dx}{dt} - Cx.$$

For very small deflections, the equations of the current and of the motion reduce to

$$L\frac{dI}{dt} + RI + HS\frac{dx}{dt} = 0,$$

$$K\frac{d^2x}{dt^2} + C_1\frac{dx}{dt} + Cx = IHS.$$

If these equations are compared with equations (12)′ and (14)′ of 845, it will be seen that it is sufficient if we replace MG by HS, and HM by C, the other letters retaining their signification.

We shall have then, for the resistance of the circuit,

$$R = \frac{H^2S^2}{2C}\frac{\pi^2\phi_0}{\tau_0\left(\frac{\lambda}{\phi} - \frac{\lambda_0}{\phi_0}\right)} + L\frac{\lambda}{\tau}. \tag{6}$$

1107. Alternating Currents.—The continuous rotation of a frame in a magnetic field, or of a magnet in a frame, produces a periodical electromotive force; and when the permanent regime is established, the current is itself periodic and alternating.

If the frame turns in a uniform magnetic field, with a constant angular velocity $\omega=\frac{2\pi}{T}$, the electromotive fôrce of induction for a deflection x, provided we choose conveniently the origin of the time t, may be written

$$HS\cos x\frac{dx}{dt}=\omega HS\cos x=\omega HS\sin 2\pi\frac{t}{T}.$$

When the permanent regime is established (535), the current is of the form

$$I=A\sin 2\pi\left(\frac{t}{T}-\phi\right),$$

and we have

$$A^2=\left(\frac{2\pi}{T}\right)^2\frac{H^2S^2}{R^2+\frac{4\pi^2L^2}{T^2}}=\frac{\omega^2H^2S^2}{R^2+L^2\omega^2}.$$

The mean square I''^2 of the intensity is

$$I''^2=\frac{A^2}{2}=\frac{\omega^2H^2S^2}{2(R^2+L^2\omega^2)},$$

which gives

(7) $$R^2=\frac{\omega^2H^2S^2}{2I''^2}-L^2\omega^2.$$

The measurement of the mean square of the intensity I''^2 by an electrodynamometer, or by a calorimetrical method, will give the resistance R of the circuit. Two observations, made with different velocities, will enable us to eliminate the coefficient L of self-induction.

Instead of rotating the frame, we may rotate, at a uniform speed, a magnet placed in the centre. For an angle x of the magnet with the plane of the frame, the value of the electromotive force of induction is

$$GM\cos x\frac{dx}{dt}=\omega GM\cos x.$$

It is sufficient if, in expression (7), we replace the product HS by GM; consequently

(8) $$R^2=\frac{\omega^2G^2M^2}{2I''^2}-L^2\omega^2.$$

1108. MEAN FIELD OF A ROTATING FRAME.—We may observe, in the experiment of the rotating frame, that the current has always the same direction in a given azimuth, although its direction changes in reference to the circuit at each half-turn; the magnetic action of the current induced at each point varies periodically, but the resultant action is not null. As the intensity of the current is a maximum, except for a retardation due to secondary effects, when the frame is parallel to the external field, this resultant is almost perpendicular to the component H, of the field normal to the axis of rotation; a magnetic needle placed in the interior of the frame will then be deflected from its original position.

Let α be the deflection of the needle, the magnetic moment of which is M, and x that of the frame. If we take into account the true direction of the current and of the field of the needle, the differential of the flow of force across the circuit is

$$\frac{dQ}{dt} = -[\mathrm{HS}\cos x + \mathrm{MG}\cos(x-\alpha)]\frac{dx}{dt}$$

$$= -\omega\mathrm{HS}\left[\cos x + \frac{\mathrm{MG}}{\mathrm{HS}}\cos(x-\alpha)\right],$$

and the equation of the current is

$$\mathrm{L}\frac{d\mathrm{I}}{dt} + \mathrm{RI} = \mathrm{HS}\omega\left[\cos x + \frac{\mathrm{MG}}{\mathrm{HS}}\cos(x-\alpha)\right].$$

We have further $\frac{d\mathrm{I}}{dt} = \frac{d\mathrm{I}}{dx}\omega$; putting $\frac{\mathrm{MG}}{\mathrm{HS}} = k$, we get

$$\mathrm{L}\frac{d\mathrm{I}}{dx} + \frac{\mathrm{R}}{\omega}\mathrm{I} = \mathrm{HS}\left[\cos x + k\cos(x-\alpha)\right].$$

If the movement is uniform, and rapid enough, in comparison with the time of oscillation of the needle, for this to be permanently deflected when the condition is established, the quantities ω and α are constants, and the integral of the equation is of the form

$$\mathrm{I} = \mathrm{A}\cos x + \mathrm{B}\sin x.$$

The differential equation then becomes

$$\frac{R}{\omega}(A \cos x + B \sin x) - L(A \sin x - B \cos x)$$
$$= HS[\cos x + k \cos (x - \alpha)].$$

As this equation should be satisfied, whatever be x, it follows that

$$\frac{RA}{\omega} + LB = HS(1 + k \cos \alpha),$$

$$\frac{RB}{\omega} - LA = HSk \sin \alpha,$$

and therefore

$$A = HS\omega \frac{R(1 + k \cos \alpha) - L\omega k \sin \alpha}{R^2 + L^2\omega^2},$$

$$B = HS\omega \frac{Rk \sin \alpha + L\omega(1 + k \cos \alpha)}{R^2 + L^2\omega^2}.$$

The couple produced by the induced current on the needle is the mean of the actions IGM $\cos (x - \alpha)$, relative to the different positions which the frame occupies during a half-turn—that is to say,

$$\frac{GM}{\pi}\int_0^\pi [A \cos x + B \sin x] \cos (x - \alpha) dx = \frac{GM}{2}(A \cos \alpha + B \sin \alpha).$$

As the needle is at the same time under the action of the field, and, if need be, allowance can be made for the torsion of the wire, the condition of equilibrium is

$$G(A \cos \alpha + B \sin \alpha) = 2H \sin \alpha.$$

Replacing the constants A and B by their values, we get

$$R^2 + L^2\omega^2 = R\frac{GS\omega}{2}\frac{k + \cos \alpha}{\sin \alpha} + L\omega\frac{GS\omega}{2}$$

As the terms which contain the coefficient of self-induction are very small, we might express the value R in the form

$$R = \frac{GS\omega}{2}\left[\frac{k + \cos\alpha}{\sin\alpha} + \frac{L\omega}{R}\right] - \frac{L^2\omega^2}{R},$$

or, replacing k by $\frac{MG}{HS}$, and the resistance R in the terms of correction by its approximate value $R_0 = \frac{GS\omega}{2\tan\alpha}$,

$$(9) \qquad R = R_0 + \frac{G^2}{2}\frac{M}{H}\frac{\omega}{\sin\alpha} + L\omega\tan\alpha - \frac{L^2\omega^2}{R_0}.$$

1109. We may also make the frame turn about a horizontal axis; the component to be considered is then the projection of the terrestrial action on a plane perpendicular to the axis. If α is the angle of this plane with the magnetic meridian, the value of the effective component is (305)

$$F_1 = \sqrt{Z^2 + H^2\cos^2\alpha} = Z\sqrt{1 + \cot^2 I \cos^2\alpha},$$

and it makes, with the horizontal plane, an angle I′ such that

$$\cot I' = \cot I \cos\alpha.$$

The simplest conditions are those in which the axis of rotation is in the meridian. In that case $\alpha = \frac{\pi}{2}$, $I' = \frac{\pi}{2}$, and the effective force reduces to the vertical component. Apart from the effects of self-induction, the change of the direction of the induced current is in the horizontal plane; and for an observer outside the frame, the current retains the same direction in all azimuths. On the other hand, the electromagnetic action on a pole placed at the centre being at each instant perpendicular to the plane, the resultant is a horizontal force perpendicular to the axis. So long as the horizontal needle is in the meridian, there is no variation in the flow from the needle relative to the frame; and therefore there is no induction on the part of the needle, whatever be its magnetic moment. The effect of induction may be regarded as negligable if the deflection continues to be small.

The conditions of the experiment appear then more simple than in the case of rotation about a vertical axis. But the effects of self-induction introduce a fresh complication. The change in the direction of the current does not really take place in the horizontal plane, and the effective component should be multiplied by the cosine of the angle of the two planes, the determination of which angle presents certain difficulties. The great advantage of the former method, in which the induced currents only give a horizontal component, is thus lost.

1110. TRANSIENT ELECTROMOTIVE FORCES.—If the rotating coil is open, the only effect of the induction of the field is to establish a difference of potential between the ends of the wire. As the capacity of this wire is very small if it is not connected with the condensers, the current is always very feeble, and the term relative to the coefficient of self-induction may be neglected. In this case the difference of potential at the ends of the wire is constantly equal to the electromotive force $\omega HS \cos x$ of the field (1107), and it acquires its maximum value when the frame is in the plane of the meridian.

Comparing this electromotive force, by a method of opposition, with the difference of potential of two points separated by a resistance R, on a circuit which is traversed by a constant current I, we shall have

$$R = \frac{\omega HS \cos x}{I}. \tag{10}$$

The same is the case if a magnet be rotated about the centre of a stationary frame, the circuit of which is open. The difference of potential at the two ends of the wire $\omega GM \cos x$ might also be determined at each instant by an opposition method, which will give the equation

$$R = \frac{\omega GM \cos x}{I}. \tag{11}$$

In both cases the constant elctromotive force RI should be opposed for a very short time, by an instantaneous contact, to the variable electromotive force of induction. It is important to observe that we do not introduce either the resistance of the circuit or the resistance of the points of contact, which is necessarily variable.

1111. Constant Electromotive Forces.—Finally, the use of a sliding contact enables us to obtain a constant electromotive force of induction.

Suppose, for instance, as in Faraday's experiment (530), that a conducting disc is rotated in a magnetic field, the centre of the disc being joined by a connecting wire with a spring which presses against the edge.

If the field is uniform, and if F is the component parallel to the axis, a the radius of the disc, the electromotive force for a constant angular velocity ω is

$$e=\frac{\omega a^2}{2}\mathrm{F}.$$

If the field is not uniform, F being the value of the component in question at a point at a distance r from the centre, we have

$$d\mathrm{Q}=\omega dt\int_0^a \mathrm{F}r dr,$$

and therefore

$$e=\frac{d\mathrm{Q}}{dt}=\omega\int_0^a \mathrm{F}r dr.$$

When the distribution of the forces F forms a system of revolution about an axis of rotation, the total flow of force across the disc is expressed by

$$\mathrm{Q}=2\pi\int_0^a \mathrm{F}r dr,$$

which gives

$$e=\frac{\omega \mathrm{Q}}{2\pi}=\frac{\mathrm{Q}}{\mathrm{T}}.$$

If the field is produced by a current of strength I, which traverses a coil having the same axis as the disc, the flow of force Q is the product MI of the current by the coefficient of mutual induction of the coil with the circumference of the disc, and we have

$$e=\frac{\mathrm{Q}}{\mathrm{T}}=\frac{\mathrm{MI}}{\mathrm{T}}.$$

The electromotive force e is measured by compensating it with the difference of potential RI′ which exists between the two ends A and B of a resistance R traversed by a constant current, so that there is no current in the disc. The result is also independent of the necessarily variable resistance of the sliding contacts, and we have

$$(12) \qquad R = \frac{M}{T} \frac{I}{I'}.$$

Any measurement of intensity is avoided if the current which traverses the resistance R is that which traverses the coil. The formula then becomes

$$(12)' \qquad R = \frac{M}{T}.$$

It is sufficient then to calculate the coefficient M and to determine the period of rotation T which establishes equilibrium.

1112. The calorimetric method (1) was employed by Joule* in 1866.

The first absolute determination of a resistance by induction is due to Kirchhoff,† who measured the discharge produced by the action of an auxiliary current.

W. Weber‡ suggested methods (2), (4), and (9), and applied the first two.

To the British Association, on the proposition of Sir W. Thomson, is due the credit of having undertaken a series of experiments with a view of establishing a standard of resistance; and the permanent deflection (9) produced by a rotating frame was used. §

The determination of an instantaneous electromotive force (10) was proposed by Professor G. C. Foster. ||

The last method (12), based on the production of a continuous current by induction, is due to Lorenz.¶

* J. P. JOULE. *British Association Report.* Dundee, 1867.—*Reprint of Reports of the Committee on Electrical Standards*, p. 175.—JOULE. *Scientific Papers*, Vol. I., p. 542.

† KIRCHHOFF. *Pogg. Ann.*, Vol. LXXVII. 1849.—*Gesamm. Abhand.*, p. 118.

‡ W. WEBER. *Elektrodyn. Maasbestimmungen.* — *Abh. d. König. Sächs. Gesellschaft*, Vol. I., p. 199. 1851.

§ *British Association Report.* Newcastle, 1863.—*Reprint*, p. 39.

|| G. C. FOSTER. *Electrician*, Vol. VII., p. 266. 1881.

¶ LORENZ. *Pogg. Ann.*, Vol. CXLIX., p. 251. 1870.

We shall review the principal experiments.

1113. CALORIMETRIC METHOD.—The uncertainty which prevails as to the true value of the mechanical equivalent of heat does not enable us to determine with sufficient accuracy the absolute value of a resistance by a calorimetric measurement. The law of Joule should be considered as furnishing a relation between J and R which would enable us to calculate one of these quantities when the other is exactly known. If, in the fundamental equation (1), we put JQ for the calorific energy W, corresponding to the disengagement of a quantity of heat Q, and make Q, I, and t equal to unity, R = J. The numerical expression of the mechanical equivalent of heat is then the absolute value of the resistance in which a current equal to unity liberates a thermal unit per second.

We need not return to the details of the calorimetric experiment (917). Joule measured the intensity of the currents by a tangent galvanometer with a circular frame; and, in order get at each instant the value of H, he observed at the same time an electrodynamometer placed in the same circuit (866). We have

$$I = \frac{H}{G}\tan\alpha = \frac{1}{2l}\sqrt{\frac{Sgp}{\pi}}(1 + A) = k\sqrt{p}.$$

The constant k was determined once for all by a series of observations made with two instruments, while the value of H was determined by Gauss' method.

Expressing the resistance in B.A. units, Joule found $J = 4{\cdot}212.10^7$. Experiments on the friction of water had given $4{\cdot}1624.10^7$. Considering this latter number as exact, he concludes that

$$\text{B.A.U} = \frac{4162}{4212}10^9 = 0{\cdot}9881.10^9.$$

According to this experiment the British Association unit is too small by 0·0119.*

The principal source of error inherent to the method is the uncertainty as to the temperature of the wire, which should always

* JOULE. *Phil. Mag.* [3], Vol. XXXI., p. 173. 1847.—*Scientific Papers*, p. 277.—*British Association Report.* Dundee, 1867.—*Reprint*, p. 175.—*Scientific Papers*, p. 542.—*Phil. Trans.*, *Roy. Soc.*, *February, 1878*, Pt. ii.—*Scientific Papers*, p. 632.

be higher than that of the calorimeter. Fletcher* turns the difficulty by placing a shunt of great resistance R′ communicating with the galvanometer at the ends of the wire R, which plunges in the calorimeter. If I′ be the intensity of the external current, we have $IR = I'R'$, and therefore

$$JQ = I^2Rt = II'R't.$$

We thus replace the measurement of R′ by that of R; we measure the current I′, and the total current $I + I'$.

In order to eliminate the determination of the equivalent of heat, Lippmann † proposed to place in the same calorimeter the motor which heats the liquid by friction and the wire which heats it when the current passes. If, then, the motor is first made to act by the pull of the weight, and then by the current, the experiment may be made so that the same final heating, equilibrated in respect of radiation, is attained in both cases. A simple thermometer is then sufficient to show that the temperatures are equal. The mechanical energy W expended in unit time, is equal to the electrical energy I^2R corresponding to the same heating, which gives the value of R.

1114. MEASUREMENT OF DISCHARGES.—In order to obtain an induced discharge by the displacement of a frame, recourse is usually had to the terrestrial field. The frame may be turned through 180°, either about the vertical or about a horizontal line situated in the meridian, starting from a position in which it is almost perpendicular to the component of the field normal to the axis. Thus either the horizontal component H, or the vertical Z of the field is used.

In the former case, which is that most generally employed, the axis must be carefully adjusted, or at any rate situate in a vertical plane, perpendicular to the meridian. If, being situate in the meridian, it made an angle ϵ with the vertical, the vertical component Z would affect the phenomenon; by a rotation of 180°, if I is the inclination, the variation of the flow of force will be

$$2HS \cos\epsilon + 2ZS \sin\epsilon = 2HS \cos\epsilon \left[1 + \tan I \tan\epsilon\right].$$

The defect of adjustment being very small, $\cos\epsilon$ does not appreciably differ from unity; but it cannot be assumed that the

* LAWRENCE FLETCHER. *Phil. Mag.* [5], Vol. xx., p. 1. 1885.

† LIPPMANN. *Comptes rendus*, Vol. xcv., p. 634. 1882.

second term is negligable; it is proportional to the error of adjustment ϵ, and to the tangent of the inclination.

The discharge m is determined by a ballistic galvanometer; if g is the constant of the galvanometer, h the component of the terrestrial field at the point at which it is placed, and α the angle of throw, all corrections made we have

$$m = \frac{h}{g} \frac{\mathrm{T}}{\pi} \alpha, \qquad \text{or} \qquad \mathrm{R} = 2\mathrm{S}g \frac{\mathrm{H}}{h} \frac{\pi}{\mathrm{T}\alpha}.$$

In his first experiments,* as well as in those which he made twice with Zöllner,† and which were completed by Wiedemann,‡ Weber used two coils of large dimensions, almost identical, one of which served as inductor, and the other as ballistic galvanometer.

The values of S and of g are determined directly from the dimensions of the coils. If A is the mean radius of the former, a that of the second, N and n the number of turns, A_1 and a_2 the values given by the formulas (9) of 727 and (14) of 729, we have

$$\mathrm{GS}g = 2\mathrm{N}n\pi^2 \frac{\mathrm{A}_1^2}{a_2}.$$

We must still introduce in the factor g a correction for the length of the needle (746).

Weber and Zöllner used two bars, the one of 10 cm. and the other of 20 cm. The results obtained with the first exceeded by 0·02 those which correspond to the second; they attribute this difference to the difference in length of the needles, and, regarding this error as proportional to the square of the length, they take one-third of this difference for the value of the correction which must be introduced into the results obtained with the bar of 10 cm. This mode of correction seems arbitrary, and but little in agreement with the degree of accuracy of the other measurements.

The ratio $\frac{\mathrm{H}}{h}$ is determined by making the same needle oscillate in the centre of the two instruments.

* W. Weber. *Elektrodyn. Maasbestim.—Abhandl. der Königl. Sächs. Ges.*, Vol. I., p. 219. 1846.

† W. Weber and Zöllner. *Berichte der Kön. Sachs. Ges.*, Vol. II., p. 77. 1880.

‡ G. Wiedemann. *Versuche zur Bestimmung des Ohms.* 1884.

The time of oscillation of the bar was 30 seconds in the experiments of Weber and Zöllner, and 55 to 65 in those of Wiedemann; the swing was determined by the recoil method, or by the mixed method.

1115. F. Kohlrausch* has introduced a modification into the experiment which had been pointed out by Weber. Instead of taking as multiplier a frame in which the value of g might be directly calculated, he uses an ordinary galvanometer of great sensitiveness, with astatic needles, and even with correcting magnets, and calculates the value of g by the difference of dampings, according as the circuit is open or closed.

Neglecting the coefficient of self-induction of the system, and the small difference between the times of oscillation of the magnet in the two cases, we have sensibly

$$\frac{M^2 g^2}{2RK} = \frac{\lambda - \lambda_0}{\tau} = \frac{\lambda - \lambda_0}{\pi}\sqrt{\frac{Mh}{K}},$$

$$g^2 = \frac{2RK}{M^2}\frac{\lambda - \lambda_0}{T} = h^2 \frac{2RT^3}{K\pi^4}(\lambda - \lambda_0),$$

and the expression of the resistance becomes

$$R = 8\frac{S^2H^2}{K}\frac{\lambda - \lambda_0}{\pi^2}\frac{T}{a^2}.$$

Besides determining the surface S of the inductor, and the time of oscillation, both the moment of inertia of the needle, and the horizontal component of the terrestrial magnetism, must be determined.

As the square of this latter quantity comes into the final formula, the relative error due to its determination is doubled; and this is also the case with that which is due to an inclination of the axis in the plane of the meridian.

It is simpler to determine the constant g of the galvanometer, by comparison with an absolute galvanometer.† If, further, the inducing frame is itself transformed into a tangent galvanometer, the ratio of the intensities of the two fields is at once eliminated.

* F. KOHLRAUSCH. *Ann. Pogg. Ergänzungs Band* VI., p. 1. 1874.

† MASCART, DE NERVILLE, and BENOÎT. *Resumé d'Expériences sur la Détermination de l'Ohm.* 1884.—*Ann. de Chim. et de Phys.* [6], Vol. VI., p. 5. 1885.

It is sufficient to turn the inducing frame through 90° from its original direction, to bring it into the meridian; to arrange at the centre a magnetised needle, and to pass the same current through both instruments. If the degrees of sensitiveness of the two instruments are very different, the use of a shunt, of a known value, on the ballistic galvanometer will enable us to obtain deflections of suitable magnitude. Let G be the galvanometric constant of the frame, Δ and δ the deflections produced by the common current, and μ the multiplying power of the shunt, if there is one; equation

$$\frac{\mathrm{H}}{\mathrm{G}} \tan \Delta = \frac{h\mu}{g} \tan \delta$$

gives

$$\mathrm{R} = 2\pi \mathrm{SG} \frac{\mu \tan \delta}{\tan \Delta} \frac{1}{\mathrm{T}\alpha}.$$

The ratio only of the deflections δ and α, relative to the ballistic galvanometer, affect the formula; it is unnecessary, therefore, to determine accurately the distance of the scale, and no correction need be made for the needle.

If a is the mean radius, n the number of turns, and l the length of the wire, we have, to within terms of correction,

$$\mathrm{S} = n\pi a^2, \qquad \mathrm{G} = \frac{2n\pi}{a},$$

$$\mathrm{SG} = 2n^2\pi^2 a,$$

or, if l is the total length of the wire,

$$\mathrm{SG} = n\pi l.$$

The quantities which must be exactly determined are, the length l of the wire coiled on the frame, and the distance of its mirror from the corresponding scale, the time T of the oscillations of the ballistic galvanometer, and the angle of swing α.

1116. When voltaic induction is used to produce the discharge, the intensity of the inducing current I, is determined by the deflection β of a tangent galvanometer, the elements of which are H and G; we have then

$$\mathrm{I} = \frac{\mathrm{H}}{\mathrm{G}} \tan \beta.$$

If the inducing current has been reversed, the resistance of the induced circuit is

$$R = 2\pi M \frac{H}{h} \frac{g}{G} \frac{\tan\beta}{T\alpha},$$

and we should calculate, by the dimensions of the coils, the value of their coefficient of mutual induction M.

Most of the experiments were made with equal coils of small section of the channel, compared with the mean radius, and placed at a certain distance from each other, the axes coinciding. In these conditions M is calculated by elliptic functions. The relative resultant error is a function of relative errors made on the mean radius a, and on the distance x of the two mean planes.

Apart from the terms of correction, the value of M (763) is equal to the product of $4\pi\sqrt{aa'}$, by a function of the angle γ defined by the equation

$$\sin\gamma = \frac{2\sqrt{aa'}}{\sqrt{(a+a')^2 + x^2}}.$$

As the coefficient M is a length, or a homogeneous function of the first degree of the variables a, a', and x, making $a = a'$, we may put

$$\frac{dM}{M} = \lambda \frac{da}{a} + \mu \frac{dx}{x},$$

with the condition

$$\lambda + \mu = 1.$$

Equation

$$M = f(a,x)$$

being in fact homogeneous, and of the first degree, we have

$$M = f(a,x) = af'_a + xf'_x,$$

$$dM = f'_a da + f'_x dx,$$

$$\frac{dM}{M} = \frac{af'_a}{af'_a + xf'_x} \cdot \frac{da}{a} + \frac{xf'_x}{af'_a + xf'_x} \cdot \frac{dx}{x}.$$

The following table has been calculated by Lord Rayleigh :*

γ	$\frac{x}{2a}$	λ	μ	$\frac{M}{a}$
60°	0·577	2·61	− 1·61	0·316
70°	0·364	2·18	− 1·18	0·597
75°	0·268	1·98	− 0·98	0·828
80°	0·176	1·76	− 0·76	1·186

In order to diminish the error made in the distance x of the mean planes, the coils are successively turned about, so that in the experiments they occupy all the relative positions. This has led observers to use coils with cores of bronze, turned with care on the outside.

M. Mascart used unequal coils, placed concentrically, with their mean planes in coincidence. The calculation of M is then made by formula (18) of 762.

M. Roïti † has proposed to use a closed solenoid as induced coil. The coefficient of mutual induction between a circular solenoid, containing n_1 windings for unit of arc, and a wire coiled n' times round the first, is expressed by

$$M = 4\pi n_1 n' \int \frac{dS}{x},$$

x being the radius of the circumference on which is the element dS of the section.

If the section of the ring is a circle of radius a, and if R is the mean radius of the ring, we have (502)

$$M = 8\pi^2 n' n_1 (R - \sqrt{R^2 - a^2});$$

if it is a rectangle of sides $2a$ and $2b$,

$$M = 8\pi^2 n' n_1 bl . \frac{R + a}{R - a}.$$

These expressions are very simple, but the regular coiling of the wire would prevent very great difficulties.

* Lord RAYLEIGH. *Comparison of Methods for the Determ. of Resist.* 1884.

† ROÏTI. *Atti dell'Ac. di Torino.* April 30th, 1882.—*Nuovo Cimento* [3], Vol. XV., p. 97. 1884.

An equally simple result is obtained if the closed solenoid is replaced by a cylindrical coil so long that the correction for the bases is very small. If n_1 is the number of turns of the inducing wire for unit length, and n' the number of turns of the induced wire, we have (551)

$$M = 4\pi n_1 n' S;$$

or, introducing the length l of the wire, the total number n of windings, the length L of the coil, the mean radius a and the radius a_1 of the mean circle,

$$M = \frac{l^2}{L}\frac{n'}{n}\left(\frac{a_1}{a}\right)^2.$$

The effect of the bases will be calculated as in 769.

1117. The simplest method of measuring the auxiliary quantities contained in the expression for the resistance R, is to eliminate the constants of the galvanometers, and the ratio of the terrestrial components, by a common current, giving a deflection Δ in the galvanometer, and a deflection δ in the ballistic galvanometer, provided with a shunt μ. We have then

$$R = 2\pi \frac{M}{T}\frac{\tan\beta}{\tan\Delta}\frac{\mu\tan\delta}{\alpha}.$$

The distance of the scales need not be very exactly measured, since only the ratios of the angles β and Δ on the one hand, δ and α on the other, appear in the formulæ; and the corrections for the length of the needles remain very feeble.

A shunt is then necessary for comparing galvanometers. The experiments of Mascart and of Glazebrook* made by this method show that, with suitable precautions, it may give very exact results.

In order to avoid the use of a shunt, Professor Rowland† and F. Weber‡ have adopted a more complicated arrangement. The ballistic galvanometer is formed of two symmetrical coils, and of such dimensions as to enable the constant g to be calculated. An independent frame, consisting of a single wire and placed in the

* Glazebrook. *Phil. Trans. 1883*, p. 223.

† Rowland. *American Journal of Science and Art*, Vol. XV., pp. 281, 325, 429. 1878.

‡ F. Weber. *Vierteljahrsschrift der Naturforsch. Gesellschaft in Zürich*, p. 273. 1877.—*Phil. Mag.* [6], Vol. V., pp. 30, 127, 189. 1877.

plane of symmetry, forms with the same needle a tangent compass the constant of which, g', is readily obtained with great accuracy.

The same current is passed through the galvanometer which serves to measure the inducing current and through the frame with a single wire. The deflections Δ and δ' give

$$\frac{H}{G}\tan\Delta = \frac{h}{g'}\tan\delta',$$

and the expression for the resistance becomes

$$R = 2\pi\frac{M}{T}\frac{g}{g'}\frac{\tan\beta}{\tan\Delta}\frac{\tan\delta'}{a}.$$

1118. In the experiments in which this method was first applied, Kirchhoff* used the same galvanometer for measuring the two quantities I and m. The coils A and A′ (Fig. 236) being placed in two

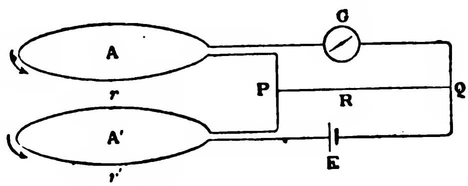


Fig. 236.

parts of the same circuit, one containing the battery E and the other the galvanometer G, the resistance to be measured, forms a bridge PQ between these two parts. The experiment consisted in measuring the swing of the needle when the induced coil A was moved from the position in which it was parallel to A′, and for which the coefficient of mutual induction is a maximum, to the perpendicular position in which it is null. If r and r' are the resistances of the two circuits A and A′, E the electromotive force of the battery, and I and I′ the permanent intensities of the currents, we have

$$I'(R+r') - IR = 0,$$
$$I(R+r) - I'R = E.$$

* Kirchhoff. *Pogg. Ann.*, Vol. LXXVI., p. 412. 1849.—*Gesamm. Abh.*, p. 118.

If $I+i$ and $I'+i'$ are the currents during the variable state, and if account be taken of the preceding equation, and if L and L′ are the coefficients of self-induction of the two circuits, we have

$$i'(R+r')-iR+\frac{d}{dt}\big[M(I+i)+L'(I'+i')\big]=0,$$
$$i(R+r)-i'R+\frac{d}{dt}\big[M(I'+i')+L(I+i)\big]=0.$$

The quantities m and m' of electricity put in motion by induction, throughout the entire duration of the variable state satisfy equations

$$m'(R+r')-mR-MI=0,$$
$$m(R+r)-m'R-MI'=0;$$

it follows that

$$m'=M\frac{(R+r)I+RI'}{(R+r)(R+r')-R^2}=\frac{M}{R}\frac{(R+r)(R+r')+R^2}{(R+r)(R+r')-R^2}I'.$$

If the resistance R of the bridge is very small in comparison with the resistances r and r' of the two circuits, we may expand this expression in a very converging series, which gives

$$\frac{m'}{I'}=\frac{M}{R}\left[1+\frac{2R^2}{rr'+R(r+r')}+\ldots\right],$$

or sensibly

$$\frac{m'}{I'}=\frac{M}{R}.$$

The current I' is given by the permanent deflection δ of the galvanometer, and the discharge m' by the swing α. The resistance R is then

$$R=M\frac{\pi}{T}\frac{\tan\delta}{\alpha}\left[1+\frac{2R^2}{rr'+R(r+r')}+\ldots\right].$$

If the deflection δ is very feeble, we may assume that the arc of the swing from the position of equilibrium is the same as starting from zero.

1119. M. Roïti, instead of a single swing, measures the deflection produced by a series of discharges corresponding to breaking

or making the circuit; a commutator breaks the inducing circuit n times in a second, and only closes the induced circuit by the galvanometer during a single one of the variable phases of the inducing current, making or breaking. A deflection α is thus obtained corresponding to the equation

$$i = nm = \frac{h}{g} \tan \alpha.$$

By stopping the commutator, and putting the galvanometer in the inducing current, we find

$$\mathrm{I} = \frac{h}{g} \tan \delta;$$

from which is deduced

$$\mathrm{R} = n\mathrm{M} \frac{\tan \delta}{\tan \alpha}.$$

The time to be measured is thus that which flows between two breaks. Graphical methods give this time with great accuracy.

The method is thus reduced to the greatest simplicity as regards the number of quantities to be measured; but it introduces some uncertainty, arising from the rapidity with which the breaks succeed. It may be feared that, from the effects of polarization on the one hand and of extra currents on the other, the intensity of the inducing current on breaking does not differ from the intensity relative to the permanent regime. It is, finally, possible that the break causes a loss of part of the induced current. This latter source of error in particular would have the effect of increasing the number found for the resistance R, and therefore of diminishing the value of the unit.

1120. Method of Damping.—This method was first used by W. Weber in observing the oscillations of a magnetised bar in a galvanometric frame, which was so good a conductor and so near the bar as to produce a rapid damping.

In calculating the resistance by the data furnished by experiment, Weber did not bring in the term for the coefficient of self-induction, which moreover is very small.

Taking into account equations (845)

$$\frac{\mathrm{HM}}{\mathrm{K}} = \frac{\pi^2}{\mathrm{T}^2} = \frac{\pi^2 \phi_0^2}{\tau_0^2},$$

the expression for the resistance may be put into different equivalent forms—

$$R=\frac{G^2M}{2H}\frac{\pi^2\phi_0}{\tau_0\left(\frac{\lambda}{\phi}-\frac{\lambda_0}{\phi_0}\right)}+L\frac{\lambda}{\tau},$$

$$R=\frac{G^2M^2}{2K}\frac{\tau_0}{\phi_0\left(\frac{\lambda}{\phi}-\frac{\lambda_0}{\phi_0}\right)}+L\frac{\lambda}{\tau},$$

$$R=\frac{G^2K}{2H^2}\frac{\pi^4\phi_0^3}{\tau_0^2\left(\frac{\lambda}{\phi}-\frac{\lambda_0}{\phi_0}\right)}+L\frac{\lambda}{\tau}.$$

The square of the galvanometric constant G appears in all cases. Weber, and after him F. Weber, calculated this constant from the dimensions of the frame; but in that case the multiplier should have dimensions which are inconsistent with rapid damping, and it will be seen that it is important to increase as much as possible the effect to be measured—that is to say, the difference of the two dampings with open and closed circuit.

It is better therefore to determine the constant G by comparison with another galvanometer G′ of known dimensions, using a shunt if necessary. The deflections δ and δ' produced by the same current give

$$\frac{G^2}{H^2}=\frac{G'^2}{H'^2}\frac{\tan^2\delta}{\tan^2\delta'}.$$

If this value be substituted in the expressions for the resistance it will be seen that, besides the times of oscillations τ and τ_0, the logarithmic decrements λ and λ_0, and the constant G′, we should determine either the ratios $\frac{H}{H'}$ and $\frac{M}{H'}$ (first form) or the moment of inertia K, and the horizontal component H′ of the terrestrial field (third form).

The experiment amounts, in fine, to using either the first of the values of R or the latter—that is to say, we should know one of the two expressions

$$\frac{G^2M}{2H}=\frac{G'^2}{2}\frac{H}{H'}\frac{M}{H'}\frac{\tan^2\delta}{\tan^2\delta'},$$

$$\frac{G^2K}{2H^2}=\frac{G'^2}{2}\frac{K}{H'^2}\frac{\tan^2\delta}{\tan^2\delta'}.$$

The second, used by M. Dorn,* does not bring in the magnetic moment of the bar, but the square of the component H′ of the terrestrial field appears in the formula, which doubles the relative error of the result.

In the first case, the ratio $\frac{M}{H'}$ is in general determined, as we shall see, by the deflection which the bar produces on a tangent galvanometer when it is put in a direction perpendicular to the magnetic meridian. This bar is not then in the same condition as during the experiments of oscillation, and we may have to allow for the influence of the earth's field on the value of the magnetic moment.

In order to get over this difficulty, Professor Wild† places the plane of the multiplier almost perpendicularly to the magnetic meridian, and keeps the magnet there by a bifilar suspension (721). A copper bar being brought into this position without torsion, it is replaced by the magnetised bar, and the torsion θ necessary to keep it there is measured.

Let C be the coefficient of the bifilar, α the angle which the bar makes with the meridian if it were under the influence of the bifilar alone, the angle $(\alpha - \theta)$ being near 90°. The condition of equilibrium is

$$C \sin\theta - MH \sin(\alpha - \theta) = 0.$$

When the bar makes small oscillations about its position of equilibrium, the couple which tends to bring it there is expressed by

$$\left[C \cos\theta + MH \cos(\alpha - \theta)\right] d\theta = MH\left[\cot\theta \sin(\alpha - \theta) + \cos(\alpha - \theta)\right] d\theta,$$

or virtually

$$\left[C \cos\theta + MH \cos(\alpha - \theta)\right] d\theta = MH \cot\theta \, d\theta.$$

Hence, in the equation for the motion (845), the couple MH must be replaced by MH $\cot\theta$, or H by H $\cot\theta$—that is to say, divide the expression for R previously found, by $\cot\theta$. The value of θ determined by the equation of equilibrium varies moreover with α—that is to say, with the declination, and especially with H.

* DORN. *Wied. Ann.*, Vol. XVII., p. 773. 1882.

† H. WILD. *Mémoires de l'Académie des Science de Saint Pétersburg* [7], Vol. XXXII. 1884.

It is necessary therefore to follow the variations of the horizontal component during the experiments.

1121. It is worthy of notice that experiments made by the method of damping, have always given higher numbers for the resistances, and therefore smaller values for the unit.

The calculation assumes that the deflections are very small, while we are led by experiment to observe considerable deflections, so as to facilitate the measurement of the damping, which should be rapid. In these conditions we cannot assume that the constant G is independent of the elongation. Approximations made by calculation on the hypothesis of small deflections do not seem to furnish a sufficient explanation of the disagreement between this method and the others.

The magnetisation of the bar by induced currents themselves (**885**) plays a part which it seems cannot be neglected.* These currents are very strong when the current is considerable, and they impart to the bar a transverse temporary magnetisation.

When the bar oscillates on either side of the meridian, the temporary intensity of magnetisation of the bar is proportional to the action GI of the current, and may be represented by fGI. If V be the volume of the magnet, the corresponding moment is fGIV. This magnetisation scarcely modifies equation (12)′ of **845** relative to the induction, for the work of the current on the transverse magnet is almost null; but the action of the earth brings into equation (14)′ a couple HfGIV of the opposite sign to the couple MGI, so that the second member of this equation should be replaced by

$$\mathrm{MGI} - \mathrm{H}f\mathrm{GIV} = m\mathrm{GI}\left(1 - \frac{\mathrm{H}f\mathrm{V}}{\mathrm{M}}\right);$$

it follows, finally, that the second member of the value of R should be multiplied by the factor $1 - \dfrac{\mathrm{H}f\mathrm{V}}{\mathrm{M}}$.

Now, if I_a is the mean intensity of the principal magnetisation of the magnet, I_t that of the earth, and if we suppose the experiment made in the magnetic latitude of 45°, we have (**153**)

$$\mathrm{M} = \mathrm{VI}_a,$$

$$\mathrm{H} = \frac{4}{3}\pi\mathrm{I}_t \cos 45° = \frac{2\sqrt{2}}{3}\pi\mathrm{I}_t;$$

* MASCART. *Comptes rendus*, Vol. C., p. 313. 1885.

consequently

$$\frac{\mathrm{H}fV}{\mathrm{M}}=\frac{2\sqrt{2}}{3}\pi f\frac{\mathrm{I}_t}{\mathrm{I}_a}.$$

This correction is not always negligable. It often happens, especially for magnets of appreciable size, that the mean intensity of magnetisation is not 2000 times greater than that of the earth; we may then take $\frac{\mathrm{I}_t}{\mathrm{I}_a}=\frac{1}{2000}$.

The value of the coefficient f depends on the shape of the magnet and the nature of the metal.

If the magnet were a very long cylinder parallel to the plane of the coil, which would be the best condition, we should have (387)

$$f=\frac{k}{1+2\pi k}.$$

As the coefficient k is between 30 and 40, it follows that in effect $2\pi f=1$.

For a sphere (385) we should have

$$f=\frac{3}{4\pi}, \qquad \text{or} \qquad 2\pi f=\frac{3}{2}.$$

If the magnet is a flat rectangular bar, of the usual form, it may be compared to an ellipsoid. Let us suppose that the ratios of the smaller axis c to the mean axis b, and of this latter to the major axis a are small, the coefficient M (387) for the mean axis has approximately the value

$$\mathrm{M}=4\pi\frac{b}{a}\left[1-\frac{b}{a}-\frac{1}{2}\frac{c^2}{b^2}l.4\frac{b}{c}\right].$$

If $a=10b$ and $b=10c$, we find

$$\mathrm{M}=1\cdot107;$$

we have then

$$f=\frac{k}{1+k\mathrm{M}}=\frac{1}{\mathrm{M}} \text{ about,}$$

or

$$2\pi f=5\cdot6.$$

With a magnet of this latter form the term of correction will be

$$\frac{H/V}{M} = \frac{\sqrt{2}}{3} \frac{5 \cdot 6}{2000} = 0 \cdot 0013.$$

The degree of magnetisation is often feebler than that which we have assumed, so that the correction may easily amount to several thousandths. The effect of this source of error is to increase the number obtained for the resistance and to diminish the value of the unit.

Another source of error, the influence of which it is more difficult to eliminate, is due to the induced currents which are developed in the mass of the magnet by the fact of its displacement in a magnetic field, and from variations of the current in the multiplier.*

1122. Continuous Rotation of a Frame.—The Committee of the British Association utilised the action, which currents induced by the earth in a frame rotating uniformly, exert on a magnetic needle at the centre of the frame. The experiments of the Committee were made in 1863 and 1864.† As some sources of error were observed in this first series of experiments, they were resumed in 1881 with special care by Lord Rayleigh and Professor Schuster,‡ and in 1882 by Lord Rayleigh.§

The rotating frame consists of two identical coils with a space between them in which is placed the apparatus for suspending the wire. Each of the coils has 156 windings of a wire 1·37 mm. in diameter, the mean radius being 15·8 cm. The frame is of copper, but made up of two parts separated by ebonite, so as to offer an obstacle to currents induced in the mass.

In the first experiments a regulator and a counter were used; in the latter Lord Rayleigh measured the velocity by a kind of phenakistoscope. On the axis is a cardboard disc on which are drawn five concentric circles divided into teeth which are alternately black and white, to the number of 60, 32, 24, 20, 16 respectively. This disc is read from a distance by means of a telescope, and through a system of parallel slits supported by one prong of a tuning-fork, which oscillates in front of a fixed screen provided with a system

* Lord Rayleigh. *Wiedemann's Annalen*, Vol. xxiv., p. 214. 1885.

† *British Association Reports for 1863, Newcastle; for 1864, Bath.*—Reprint, pp. 96, 115.

‡ Lord Rayleigh and A. Schuster. *Proceedings of the Royal Society, 1881.*

§ Lord Rayleigh. *Phil. Trans. Roy. Soc. 1882.*

of identical slits. At each oscillation of the tuning-fork, which makes 127 in a second, the rotating disc may be seen; and if the teeth of one circle appear at rest, it is because they replace each other in $\frac{1}{127}$ of a second. The observer who has the eye on the telescope may keep the velocity absolutely constant by the simple friction of the hand on one of the cords by which the motion is transmitted.

The needle, which is supported by a cocoon fibre about 130 centimetres in length, is protected against air currents by a glass tube. This needle has an extremely small magnetic moment, so that its inductive action on the frame does not come in as a term of correction. In the experiments of the Committee it was a small steel sphere 0·8 cm. in diameter and weighing about 2 grammes, and its magnetisation was only about the fortieth of that which steel can acquire. Its moment was equal to that which a soft iron wire, 10 grammes in weight, would take under the action of the earth's magnetism alone. The weight of the stirrup and of the mirror was relatively considerable, and the time of oscillation was about 10 seconds—that is to say, at least 30 times that of the needle alone.

In measuring a deflection, the smallness of the needle and the weak magnetic moment have theoretically no influence on the exactitude of the result; but the directive action is then very weak, and the torsion of the wire may have considerable influence. Moreover the slightest causes, such as air currents produced by the smallest changes of temperature in the bell-jar itself, which contains the movable system, are sufficient to destroy equilibrium.

It is particularly important that the magnetic axis be absolutely invariable, and from this point of view the spherical shape is not the best. It was chosen because a sphere uniformly magnetised exerts the same external action as an infinitely small magnet placed at its centre (**157**); but the same result is virtually obtained with a cylinder whose length is to its diameter as $\sqrt{3}$ is to $\sqrt{2}$. Lord Rayleigh replaces the sphere by a system of four small needles 0·5 cm. in length, which satisfy the preceding condition, and are mounted on the four edges of a small cube of cork.

1123. The experiments of the Committee present some anomalies, the most serious of which is that the differences in the values obtained amount in the mean to 3 per cent., according as the frame turns in one or the other direction. This result cannot be due to a previous torsion of the suspension wire; for this torsion should have

been such that the position of equilibrium of the needle had made an angle of 12°, and even of 26°, with the magnetic meridian in certain experiments.

Anomalies of the same kind, but much smaller, have shown themselves in the experiments of 1881. Kohlrausch had pointed out, as a possible source of error, the currents induced in the metal frame of the apparatus. To meet this objection the various parts have been cut and insulated from each other, and connection could be again restored. Experiment has shown that the effect of the currents could almost be neglected, but it appeared to diminish the deflection.

In the expression for the resistance (9)

$$R = \frac{GS\omega}{2 \tan \alpha} + \frac{G^2}{2} \frac{M}{H} \frac{\omega}{\sin \alpha} + L\omega \tan \alpha - \frac{L^2\omega^2}{R_0},$$

the principal term comprises, besides the observed deflection and the angular velocity, the product SG which is given by the dimensions of the frame. The second term requires that we know the ratio of the magnetic moment of the needle to the terrestrial field; this ratio is small, and may be readily determined with any desired approximation. The two other terms contain the coefficient of self-induction of the frame. We could eliminate this coefficient by two experiments made with different velocities, but it is better to calculate it directly or determine it by comparison with a coefficient of mutual induction (1098). In consequence of an error in calculating the coefficient of self-induction, the value of the ohm adopted by the Committee of the British Association is somewhat different from the true value.

H. F. Weber* has used the same method by making the frame rotate about a horizontal axis in the magnetic meridian.

1124. Measurement of Instantaneous Electromotive Forces.—This method has been used by Professor G. C. Foster† in some trial experiments with a rotating coil, which also served as tangent galvanometer. The current I, the velocity of rotation or the resistance R between the points of contact, were adjusted so that there was equilibrium when the electromotive force is at its maximum—that is to say, when the frame passes the meridian. A current of the same value passing this in the frame gave a deviation δ.

* H. F. Weber. *Der absolute Werth der S. Q. U.* Zurich, 1884.

† Carey-Foster. *British Association Report for 1881*, p. 2.

Formula (10) gives thus

$$\omega HS = R\frac{H}{G}\tan\delta, \qquad \text{or} \qquad R = \frac{\omega GS}{\tan\delta}.$$

The method is very simple. It is not, however, quite true that the maximum electromotive force corresponds exactly to the passage of the frame through the meridian. Let us assume that the ends of the wire, instead of being free, communicate separately with a condenser C. The problem is thus solved by the equations of 993, in which $R' = \infty$. The electromotive force of induction being

$$E = \omega HS \sin 2\pi\frac{t}{T} = \omega HS \sin \omega t,$$

and V the difference of potential of the armatures, the current I which traverses the circuit, is equal to $C\frac{dV}{dt}$. The equation of induction becomes in that case

$$CL\frac{d^2V}{dt^2} + CR\frac{dV}{dt} + V = E.$$

We may write

$$V = A\sin 2\pi\left(\frac{t}{T} - \phi\right),$$

and following the usual course, we find

$$\tan 2\pi\phi = \frac{CR\omega}{1 - CL\omega^2},$$

$$A^2 = \frac{H^2S^2\omega^2}{C^2R^2\omega^2 + (1 - CL\omega^2)^2} = \frac{H^2S^2}{C^2R^2}\sin^2 2\pi\phi.$$

There is consequently a difference of phase, or a retardation, which strictly only vanishes provided the capacity is zero. This retardation, as well as the maximum value of V, are functions of the capacity, or of the coefficient of self-induction. The moment the frame passes through the meridian, the difference of potential of

the armatures is $A \cos 2\pi\phi$; for a contact made at this instant the error is

$$HS - A \cos 2\pi\phi = \omega HS \left[1 - \frac{\sin 4\pi\phi}{2CR\omega}\right].$$

The condition $C = 0$ is in reality never fulfilled; for the open wire has a capacity of its own which may be appreciable if the coil consists of a great number of windings. The influence of the capacity of the wire may, however, generally be considered as negligable.*

This method has the great advantage that the wire whose resistance is measured is not, as in previous methods, the wire of the coil experimented with, but a separate wire, the temperature of which is more easily ascertained.

1125. Instead of making an instantaneous contact, the ends of the wire might be connected with an electrometer arranged as in 816.† We shall have in this way the mean square $\frac{A^2}{2}$ of the electromotive force, or sensibly $\frac{\omega^2 H^2 S^2}{2}$, which would be compared by the same instrument to the difference of potential existing between the two extremities of a resistance R traversed by a constant current. The frame itself used as a galvanometer giving a deflection δ for this current, we shall have

$$\frac{\omega HS}{\sqrt{2}} = R \frac{H}{G} \tan \delta,$$

or

$$R = \frac{\omega GS}{\sqrt{2} \tan \delta}.$$

The greatest difficulty of this method is in getting an electrometer of small capacity but sufficiently delicate.

1126. Measurement of a Constant Electromotive Force.—The method of M. Lorenz,‡ in which the electromotive force of the disc is compensated by the fall of potential of a resistance traversed

* Lippmann. *Comptes rendus*, Vol. xciii., pp. 813, 955. 1881.—Brillouin. *Ibid.*, pp. 845, 1069.

† Joubert. *Comptes rendus*, Vol. xciv., p. 1519. 1882.

‡ Lorenz. *Pogg. Ann.*, Vol. cxli., p. 251. 1873.—*Wied. Ann.*, Vol. xxv., p. 1. 1885.

by the current itself which passes through the coil, is evidently that which presents the highest degree of simplicity, as it only requires the measurement of a coefficient of induction—that is to say, of a length; and of a velocity of rotation—that is to say, of a time.

The difficulties special to it arise on the one hand from the smallness of the force of induction, and on the other hand from the relative magnitude of the thermoelectric forces produced by the contact of the sliding pieces. These effects are diminished by using contact springs of the same metal as the disc; and if they are sensibly constant, their effect is eliminated by taking the means of the results obtained for two opposite directions of the field. The smallness of the electromotive force of induction necessitates that the resistance at which the electrodes terminate shall be very small. In order to avoid errors arising from the comparison of two unequal resistances, Lorenz works directly on columns of mercury contained in tubes of 2 cm. to 3 cm. in diameter, carefully calibrated, so as to be able to deduce the value of the mercurial unit by a simple calculation. The resistances used varied from 0·0002 ohms to 0·0015 ohms. These mercurial columns should moreover be placed in baths which are kept at a constant temperature.

Lord Rayleigh and Mrs. Sidgwick* turn the difficulty by a kind of multiplication. Two points A and B of the principal circuit being separated by a resistance R which is greater than that suitable for equilibrium, they are joined by a shunt of considerably greater resistance r. The point on the shunt with the resistance ρ, starting from A, is then sought which must be interposed between the electrodes connected with the spring of the disc so as to produce equilibrium.

The principal current being I, the branch circuit is equal to $I\dfrac{R}{R+r}$, and the difference of potential which corresponds to the resistance ρ is

$$I\frac{R\rho}{R+r}.$$

The condition of equilibrium is then

$$\frac{M}{T}=R\frac{R+r}{\rho}.$$

* Lord Rayleigh and Mrs. Sidgwick. *Phil. Trans. Lond. Roy. Soc. for 1883*, p. 245.

If the ratio $\frac{\rho}{R+r}$ is equal to $\frac{1}{100}$, for instance, we may operate on a resistance R 100 times greater than with a direct contact.

The error in this calculation of M arises especially from the estimation of the mean radius of the coil. The radius a' of the disc may be ascertained with far greater exactitude. In his first experiments Lorenz had given to a' a value too near a; as the intensity of the field increases very rapidly near contact with the coil, the least error in the radius introduces a considerable error in the coefficient M. In his latter experiments he placed the disc in what is practically the uniform field of a long coil. This consisted of a single layer of wire comprising 472 turns wound on a brass cylinder 100 cm. in length and 33 cm. in diameter. The calculation was made by a formula analogous to that of 771.

Lord Rayleigh used two coils which are sensibly identical, placed either in contact or at a distance $2x = a\sqrt{2}$. With this latter arrangement, the relative error of M is almost independent of the error made in the valuation of the mean radius, and depends more particularly on the distance $2x$, a quantity far more easy to measure. For if each of the coils contains n turns, and we take the approximate value

$$M = 4n\pi^2 \frac{a^2 a'^2}{u^3},$$

we deduce from it

$$\frac{dM}{M} = 2\frac{da}{a} + 2\frac{da'}{a'} - 3\frac{du}{u}.$$

Taking equations $u^2 = a^2 + x^2$ into account, we get

$$\frac{dM}{M} = \frac{da}{a}\left(2 - 3\frac{a^2}{u^2}\right) + 2\frac{da'}{a'} - 3\frac{x^2}{u^2}\frac{dx}{x};$$

now, the first term of the second member vanishes for the condition $2u^2 = 3a^2$, or $2x^2 = a^2$.

1127. SUMMARY OF THE EXPERIMENTS.—In the following table we shall collate the numbers furnished by various methods for the length of the column of mercury at zero, a square millimetre in cross section, which represents the value of the ohm, or a resistance of 10^9 absolute units (C.G.S.).

In certain cases the conductor, the absolute resistance of which is determined, has been compared with a copy of Siemens' unit, either with a column of mercury of known dimensions; in other cases the comparison has been made with a copy of the British Association

unit, the value of which in mercury should then be known. The ratio of the mercurial unit to that of the British Association is

According to Lord Rayleigh	... ...	0·95412
„ Mascart, de Nerville, Benoît	...	0·95374
Mean	... ...	0·95393

These two results differ by 0·0004. We shall take the mean value to express in a column of mercury the experiments which refer simply to the unit of the British Association.

To the results of the experiments cited previously we shall add a few numbers due to other observers: MM. Rowland and Kimball by induced discharges, M. Himstedt* by the mean current of a series of induced discharges, M. Baille† by damping, M. Lenz‡ and MM. Rowland, Kimball, and Duncan by the method of M. Lorenz.

TABLE OF RESULTS.

Calorimetric Method.

Date and Observer.	Value of the Ohm in column of Mercury. Cm.
1866. Joule	106·22
1867. „	106·10
1877. H. F. Weber	105·88
1885. Fletcher	105·95
Discharge Induced in a Frame by a Rotation of 180°.	
1874. F. Kohlrausch	105·91
1884. Mascart, De Nerville, and Benoît ...	106·37
1884. G. Wiedemann	106·19
Discharge Induced by a Current.	
1878. Rowland	106·16
1882. Glazebrook	106·29
1883. Kimball	106·25
1884. Mascart, De Nerville, and Benoît ...	106·30
1884. H. F. Weber	105·37
1884. Rowland and Kimball	106·31

* HIMSTEDT. *Berliner Berichte.* Berlin, July, 1885.

† BAILLE. *Ann. Télegraph* [3], Vol. XI., p. 261. 1884.

‡ LENZ. *Conf. Internat. des Unités Électr.*, Second Session, p. 30. 1884.

Mean Current of a Series of Induced Discharges.

Date and Observer.	Value of the Ohm in Column of Mercury. Cm.
1884. Roïti	105·89
1885. Himstedt	105·98

Damping of Magnets.

1882. Dorn	105·46
1884. Wild	106·03
1884. H. F. Weber	105·26
1884. Baille	105·67

Mean Action of the Current Induced in a Rotating Frame.

1865. Committee of the British Association ...	104·83
1881. Lord Rayleigh and Schuster	105·95
1882. Lord Rayleigh	106·25
1882. H. F. Weber	106·16

Current of Continuous Induction.

1873. Lorenz	107·10
1883. Lord Rayleigh and Mrs. Sidgwick ...	106·22
1884. Lorenz	106·19
1884. Lenz	106·13
1884. Rowland, Kimball, and Duncan	106·29
1885. Lorenz	105·93

1128. These results do not perfectly agree, and there is something arbitrary in the choice of a definite value; nevertheless the causes of error inherent in some of the methods are sufficient to explain the most divergent results.

The calorimetric method is excellent in principle, but the uncertainty that may still exist as to the mechanical equivalent of heat which enters into the calculation, leaves some doubt as to the accuracy of the results.

The damping of magnets, and successive induced discharges, have given the smallest numbers; we have dwelt sufficiently on the difficulties which these two methods present.

The necessity of making the axis of rotation perfectly vertical, and especially the time of displacement of a frame of large dimensions, are reasons for some doubts as to the results furnished by Weber's first method.

The three other methods seem open to no objections, in the present state of our knowledge.

If we eliminate from the third series the number of F. Weber (which is manifestly too small), the two first numbers of the sixth series, and the first of the seventh, they give respectively as means:

3rd Series	106·25 ± 0·05
6th "	106·20 ± 0·05
7th "	106·15 ± 0·10
Mean	106·207 ± 0·069

It will be thought, perhaps, from these means, and from the examination of isolated experiments, that the true value of the ohm is between 106·2 and 106·3. We know, further, that the International Commission, not being fixed as to the value of the fourth cipher, has adopted as legal ohm the round number 106 centimetres.

The standard of the British Association, referred to the legal ohm, would then be

$$\frac{104{\cdot}83}{106} = 0{\cdot}98895 = \frac{1}{1{\cdot}0112}.$$

A circumstance, which has not been sufficiently taken into account in the experiments, and to which Lord Rayleigh has called attention, is the want of insulation of the windings of the coils. This cause of error may be very important in experiments in which, the electromotive force of induction being variable, may at a given moment attain a high value; it is possible that the insulation in that case leaves much to be desired, while it would be sufficient for weak and continuous currents.

In the simplest case, in which two or more windings of the induced circuit are in contact, the inaccuracy of the result would obviously represent an error of one or more units in the number of windings; its effect is to increase the numerical value of the resistance observed, and to diminish that of unity. An effect of the same kind is produced when the want of insulation is variable with the electromotive force; so that, from this point of view, the higher numbers are the more probable.

CHAPTER VIII.

RATIO OF UNITS.

1129. Various Methods.—The ratio a of the electromagnetic unit of electricity to the electrostatic unit (**610**) is the ratio of a length to a time, and is therefore of the same nature as a velocity; it is a quantity the absolute value of which is independent of the choice of the fundamental units of length, mass, and time. The experimental determination of this ratio leads then to the comparison of a length and of a time; and in particular, as a resistance is equal in electromagnetic measure to a velocity, it may be expressed as a function of the resistance of a circuit.

It is evident, moreover, from the series of equalities

$$a=\frac{q}{Q}=\frac{i}{I}=\frac{E}{e}=\sqrt{\frac{c}{C}}=\sqrt{\frac{R}{r}},$$

that there are as many methods of determining this constant a as there are quantities which can be measured at once in electrostatic and in electromagnetic units. Nevertheless, these methods are not entirely distinct, and often lead to the same experimental measurements; we may reduce them to three principal methods, corresponding to the only quantities which can be measured directly in electrostatic units—that is to say, an electrical charge, a difference of potential, or a capacity.

1130. Measurement of a Quantity of Electricity.—The former researches on this subject are due to Weber and to Kohlrausch;* we shall mention them in detail, in consequence of their

* Weber and Kohlrausch. *Electrodyn. Maasbestim.*, *Abh. der Königl. Sächs. Gesellschaft der Wissensch.*, Vol. v., p. 219. 1856.

historical importance. The charge of a Leyden jar is measured in electrostatic units, by determining the repulsion of two balls of a Coulomb's electrometer when they had received a *known fraction* of the total charge; the measurements in electromagnetic units was given by discharging the jar through a ballistic galvanometer.

In the electrostatic measurement, when once the jar is charged, the knob is touched with a conducting sphere of radius R, and this with a ball of radius r, which is then insulated, and which serves as fixed ball in a Coulomb's balance. The movable ball is electrified by contact with the first, and their repulsion at a given distance is determined.

A preliminary experiment gives the fraction of the total charge worked with. After having electrified the jar, an insulated sphere R is put successively four times in contact with the knob, being each time reduced to the neutral state. Before the first contact, and after the last, the jar is connected with an electrometer which is systematically graduated—a sine electrometer, for instance—which gives the ratio of the potentials in the two cases. Let C be the capacity of the jar, C′ that of the sphere at the moment of contact, V, V_1, V_2, V_3, and V' the successive potentials; we have

$$\frac{C+C'}{C}=\frac{V}{V_1}=\frac{V_1}{V_2}=\frac{V_2}{V_3}=\frac{V_3}{V'},$$

and consequently,

$$\left(1+\frac{C'}{C}\right)^4=\frac{V}{V'}, \quad \text{or} \quad \frac{C'}{C}=\sqrt[4]{\frac{V}{V'}}-1.$$

Two observations made on the electrometer before the contacts, and two others afterwards, enable us to allow for the leakage. The ratio $\frac{C'}{C}$ was found equal to 0·03276.

The electrostatic capacity C′ of the sphere in the conditions of the experiment is not equal to its radius, but it need not be known exactly; and it may be admitted, without any great error, that the division of the electricity between this sphere and the small knob is in the same ratio as if the system were withdrawn from any external influence; this ratio is given by Plana's tables,* calculated from the formulæ of Poisson.

* See MASCART. *Traite d'Électricite Statique*, Vol. I., p. 281.

In the experiment

$$R = 7{\cdot}973 \text{ cm.}, \qquad \frac{r}{R} = 0{\cdot}07234;$$
$$r = 0{\cdot}5768 \text{ cm.},$$

the ratio of the capacity c of the small ball to that of the sphere is thus

$$\frac{c}{C'} = 0{\cdot}007875.$$

It follows that

$$\frac{c}{C} = \frac{C'}{C} \cdot \frac{c}{C'} = \frac{1}{3876};$$

this is the fraction of the charge q which the small ball takes on contact.

As the movable ball of the balance has sensibly the same radius as the first (0·5798 cm.) the common charge, after contact, is equally divided between them, and each of them has the quantity of electricity

$$q' = \frac{q}{7752}.$$

After this preliminary determination, two observers are necessary for the course of the experiments.

The jar being charged, one of the observers makes the contacts mentioned, puts the small ball in the Coulomb's balance, and measures the repulsion; while the other observer discharges the residue q in a ballistic galvanometer, and makes the corresponding readings.

The balls of the balance were hollow, carefully turned, gilt, and polished; the torsion necessary to keep them at 90° was measured—or, more exactly, after having given slightly too weak a torsion, the period is noted at which the angle of deflection passes through 90°, owing to leakage, and the necessary corrections are then made.

The distance of the fixed ball from the axis was 9·353 cm., that of the movable one 6·17 cm., hence for a deflection of 90° the distance of the centres is 11·205; but this distance should be increased by 0·0124 cm., to allow for the distribution. The repulsion of the balls is then

$$f = \left(\frac{q'}{11{\cdot}2174}\right)^2.$$

According to these data, the distance h from the axis to the line of the centres is equal to 5·1502 cm., and the expression for the couple of the electric action is

$$fh = 5{\cdot}1502\left(\frac{q'}{11{\cdot}2174}\right)^2 = \frac{q'^2}{24{\cdot}43}.$$

The torsion of the wire determined by the method of oscillations gave a couple of 0·0050615 for an angle of 1′. T being the number of minutes observed, the correction made for leakage, we have

$$\frac{q'^2}{24{\cdot}43} = 0{\cdot}0050615\mathrm{T},$$

or

$$q' = \frac{q}{7752} = 0{\cdot}35164\sqrt{\mathrm{T}},$$

$$q = 2726\sqrt{\mathrm{T}}.$$

The same charge, represented by Q in electromagnetic units, was given by the swing a of the galvanometer, all corrections being made.

The dimensions of the frame were

$$a = 13{\cdot}324 \text{ cm.},$$
$$2c = 4{\cdot}080 \text{ cm.},$$
$$2b = 7{\cdot}204 \text{ cm.},$$
$$n = 5{\cdot}635 \text{ cm.},$$

and the needle was 2 cm. in length; from it was deduced G = 2·621.

Investigation of the oscillations of the needle had given

$$\tau = 9{\cdot}244'',$$
$$\lambda = 0{\cdot}070''.$$

Taking for the terrestrial component H = 0·17983, and allowing for a correction of 0·0014 for the torsion of the wire, we have finally

$$\mathrm{Q} = \frac{\mathrm{H} \times 1{\cdot}0014 \times \tau}{\mathrm{G}\pi \times 1{\cdot}0024}\, a = \frac{a}{4937};$$

and consequently

$$a = \frac{q}{\mathrm{Q}} = 13{\cdot}46 \times 10^6 \frac{\sqrt{\mathrm{T}}}{a}.$$

The means of five series of experiments gave the following numbers:—

Q	q	$a=\frac{q}{Q}$
$1{\cdot}194.10^{-6}$	$36{\cdot}06.10^{3}$	$30{\cdot}20.10^{9}$
1·300	41·94	32·26
1·568	49·70	31·70
1·480	44·35	29·96
1·586	49·66	31·25
	Mean . .	$31{\cdot}07.10^{9}$

This experiment presents the greatest difficulties. The law of distribution assumed for the contact of the small ball and of the sphere would only be exact provided the two conductors were withdrawn from any extraneous action, a condition which cannot be realised.

The rectangular shade was 144 cm. high by 116 cm. in length and 87 cm. in breadth. These dimensions are so large that the influence of the sides need not be allowed for, especially in measuring charges (**803**).

The greatest source of error is probably due to phenomena of absorption and to residues of the Leyden jar. Kohlrausch,* in a preliminary investigation, had endeavoured to determine the law of the residue and the loss in air for the jar used for the experiment. For a charge q_0 of the jar, the charge q disposable after a time t, is equal to the excess of the initial charge over the loss by air a_t and the residue b_t. Kohlrausch found for b_t the expression

$$b_t=p\left[q_t-q_0e^{-\frac{k}{m+1}t^{m+1}}\right],$$

in which $q_t=q_0-a_t$, and in which the values of the coefficients are $p=0{\cdot}04494$, $k=0{\cdot}1834$, $m=0{\cdot}4255$, this latter only depending on the dimensions of the jar.

We could also allow for the variations of the charge due to the loss and the residue, first during an interval of 40 seconds, which elapsed between the two electrometric measurements, and then during a second interval of 3 between the period of this

* KOHLRAUSCH. *Pogg. Ann.*, Vol. XCI., p. 56. 1854.

latter electrometric observation and the discharge of the jar through the galvanometer. This latter correction was quite insignificant.

Kohlrausch and Weber never considered their experiments exact to more than 2 per cent. The difference between the extreme values amounts to 7 per cent.

1131. Measurement of an Electromotive Force.—This method was first employed by Sir W. Thomson.* Consider two points A and B of a conductor traversed by a permanent current. The difference of electromagnetic potential E between the two points, is equal to the product IR of the intensity of the current, and of the resistance between them measured in the same system, and its electrostatic value e may be given by an absolute electrometer (**806**) the plates of which are in connection with the two points A and B.

The resistance is measured by comparison with standards.

In order to avoid the determination of H, the intensity was measured by an electrodynamometer with unifilar suspension (**863**) which gives, for a deflection θ,

$$I^2 = \frac{C}{GS'}\theta.$$

The difference of potential of the two points is then

$$E = RI = R\sqrt{\frac{C\theta}{GS'}}.$$

At the same time, each of the points A and B is put alternately in connection with the plate of the electrometer, the other point being in connection with the shade of the instrument. If F is the force necessary to bring the movable disc to the fiducial point, D and D′ the two readings for the distance of the plates, A the corrected surface of the movable plate, we have (**807**)

$$e = V - V' = (D - D')\sqrt{\frac{8\pi F}{A}};$$

consequently

$$a = \frac{E}{e} = \frac{R}{D - D'}\sqrt{\frac{C\theta}{GS'} \cdot \frac{A}{8\pi F}}.$$

* Sir W. Thomson. *British Association Report, 1869.*

The fixed coil of the electrodynamometer (**865**) is formed of two distinct coils. The current arrives by the wire by which the movable coil is suspended, and emerges by a very light wire wound in a spiral suspended to the coil, and then plunging in a mercury cup.

Magnets conveniently arranged nullify almost completely the action of the terrestrial field on the coil. As the compensation thus obtained is never complete, the current was passed alternately in opposite directions, and the mean of two readings which differed very little from each other was taken as final deflection.

The movable coil was made of 3000 turns of a very fine wire; its radius being too small for the surface S to be calculated, this being measured by a standard surface.

The resistance of the electrodynamometer was about 15,600 ohms. In order to have greater differences of potential on the electrometer, a supplementary resistance of 10,000 ohms was introduced between the points A and B, and in consequence the number of the couples was varied. This was sometimes 90 and sometimes 180 Daniell in series.

The first experiments were made by Mr. King* in 1869 under the direction of Sir W. Thomson. They were then resumed in 1870 to 1872 by Mr. Dugald M'Kichan.†

1132. The same method was applied by M. Shida‡ with some modifications.

It was not attempted to bring the charge of the electrometer to a fixed value by using a *replenisher* and a *gauge;* but putting alternately the plate and the cage in connection with the two poles A and B of a battery closed by a resistance R, and working at equal intervals, readings D_1, D'_1; D_2, D'_2; D_3, D'_3.... were obtained, and, as a mean of the differences, the values

$$\frac{D_1 + D_2}{2} - D'_2, \qquad \frac{D'_1 + D'_2}{2} - D_2 \ldots,$$

were taken, which should be equal to within errors of reading.

The intensity of the current was determined by a tangent galvanometer arranged like that of Joule (**837**). If the deflection Δ

* KING. *Report of the Committee on Electrical Standards*. 1869.—Reprint, p. 186.

† DUGALD M'KICHAN. *Phil. Trans. for 1879*, pp. 409, 427.

‡ SHIDA. *Phil. Mag.* [5], Vol. x., p. 401. 1880.

of the needle is observed at the moment the electrostatic measure is made, we shall have

$$E = IR = \frac{H}{G} R \tan \Delta.$$

Instead of working in this way, Shida determined then the deflection δ obtained by closing the battery the resistance of which is R_0, and the electromotive force E_0, by an auxiliary resistance r. The intensity of the current gives

$$E_0 = i(R_0 + r) = \frac{H}{G}(R_0 + r) \tan \delta.$$

If the electromotive force of the battery is unchanged, the product $(R + r) \tan \delta$ is constant, when the external resistance is made to vary, and the conditions can be chosen so that the deflection is near 45°, which corresponds to the maximum sensitiveness (833). We have then

$$E = \frac{H}{G} \frac{R}{R_0 + R}(R_0 + R) \tan \Delta = \frac{H}{G} \frac{R}{R_0 + R} C.$$

This mode of arranging the experiments does not appear to be very advantageous, because the observation for the two systems of measurements are not simultaneous, and the resistance of the battery comes into the calculations, the determination of which always presents some difficulties.

1133. In the preceding experiments two forces are in reality compared—the attraction between the two plates of the electrometer and the mutual action of the two coils of the electrodynamometer. Instead of measuring these two forces separately, Maxwell* arranged the experiments so as to balance one by the other. Two systems were arranged, each consisting of a conducting disc and an opposed coil; one of the systems is fixed and the other movable.

The two plates being charged with opposite electricities with a constant difference of potential, attract each other; the two coils, which are traversed by currents in contrary directions, repel. The first force is sensibly inversely as the square of the distance of the plates, while the second varies according to a less rapid law; there

* MAXWELL. *Transactions of the Royal Society for 1868*, p. 643.

is always a position of the two systems for which there is equilibrium. It must be observed that this is a position of unstable equilibrium, which makes the experiments somewhat difficult.

Fig. 237 shows the arrangement of the two systems: C and C′ are the two plates, B and B′ the two coils. The plate C′ is insulated, and with the corresponding coil B′ may be moved parallel to itself by means of a micrometric screw V. The plate C, which is in connection with the earth, is placed at the end of a beam movable about the point O. It passes exactly through the aperture of a box S, also in connection with the ground, which contains the plate C′, and serves as guard-ring for the plate C.

In observing, the plate C should be kept in the plane of the guard-ring, the distance of the two systems being regulated by the

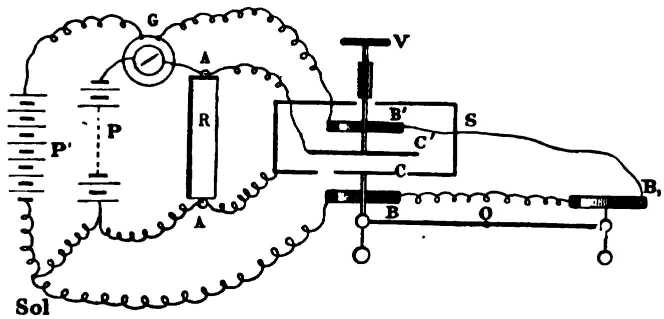


Fig. 237.

play of the micrometric screw V. This position is defined by means of two small silvered mirrors, one of which is on the posterior part of the plate C, and the other on the ring itself; the disc and the ring are in the same plane when the positions of the images of the same straight line given by two mirrors are exactly in the prolongation of each other; this position is marked on a divided plate supported by the movable system, and which is observed with a microscope. The distance of the two plates C and C′ is deduced from comparative measures with the micrometer and divided scale; while the two discs C and C′, put in contact with each other, are simultaneously displaced.

In order to compensate the action of the earth on the coil B, when it is traversed by the current, there is an identical coil B_1, through which the current circulates in the same direction. The system constituted of these two coils is perfectly astatic.

The beam is suspended to a copper wire, by which the current reaches the coils; this emerges thence by a wire dipping in a mercury cup. The time of oscillation of the system was 7 seconds; these oscillations are rapidly extinguished, owing to the changes in pressure which they determine in the box S. The torsion of the wire need not be allowed for, as the position of equilibrium corresponds to zero torsion.

The difference of potential established between the two plates C and C′, is that of two points A and A′, of a circuit traversed by the current of a battery P of 2600 bichloride of mercury elements. The resistance R, which separates the two points A and A′, is formed of a standard coil with about a million of ohms, and the current is determined by a galvanometer G.

A second battery P′ furnishes a current which traverses the three coils B, B′, and B_1. This current traverses, in the galvanometer G, a second frame, formed of a small number of turns, and superposed on the first, so as to form a differential galvanometer. A portion of the current is diverted through a shunt, the resistance of which is modified so that the needle remains at zero.

The two currents are closed for a very short time, the moment the oscillation of the beam brings it into the position of equilibrium; the distance of the two plates is then varied, so that the movable plate is neither attracted nor repelled; and, on the other hand, the shunt is modified until the needle remains at zero.

Two circumstances make the experiment difficult—the instability of the equilibrium of the beam, and the rapid variation of the electromotive force of the battery P, from the moment the circuit is closed.

The attraction of two discs for the distance x, and with an electrostatic difference of potential equal to e, is $\frac{A}{8\pi}\frac{e^2}{x^2}$; the repulsion of the two coils is $I'^2\frac{\partial M}{\partial x}$, where M is the coefficient of mutual induction, and I′ the intensity of the current. As these two forces are equal,

$$e = I'x\sqrt{\frac{8\pi}{A}\frac{\partial M}{\partial x}}.$$

The intensity I of the current of the battery P gives the difference of electromagnetic potential $E = IR$ of the two plates. As, moreover, the needle of the galvanometer is at zero, when a shunt s is

added to the resistance of the wire g, we have, if G and G′ are the constants of the two coils,

$$\frac{s}{s+g}\,\mathrm{IG} = \mathrm{I'G'},$$

$$\mathrm{E} = \mathrm{IR} = \mathrm{RI'}\frac{\mathrm{G'}}{\mathrm{G}}\,\frac{s+g}{s};$$

consequently

$$a = \frac{\mathrm{E}}{e} = \frac{\mathrm{R}}{x}\,\frac{\mathrm{G'}}{\mathrm{G}}\,\frac{s+g}{s}\sqrt{\frac{\mathrm{A}}{8\pi\dfrac{\partial \mathrm{M}}{\partial x}}}.$$

In order to determine the ratio $\frac{\mathrm{G}}{\mathrm{G'}}$, the same current is passed through the two coils, with a suitable shunt on that with the fine wire, so as to bring the needle to zero.

The coefficient of self-induction M is determined by elliptic functions. For two circles of radius a and a', whose currents are in opposite directions, we have, as a function of quantities which have been defined above (763),

$$\frac{\partial \mathrm{M}}{\partial x} = -4\pi\sqrt{aa'}\left[\frac{2\mathrm{F}}{k^2} + \frac{k^2-2}{k^2(1-k^2)}\,\mathrm{E}\right]\frac{\partial k}{\partial x}$$

$$= \frac{\pi}{\sqrt{aa'}}\left[\frac{2}{k^2}\,\mathrm{F} + \frac{k^2-2}{k^2(1-k^2)}\,\mathrm{E}\right]k^3 x,$$

and the calculation for the two coils may be made by the method of Lord Rayleigh (765).

1134. Measurement of a Capacity.—The measurement of a capacity may be determined directly in electrostatic units. The simplest and most certain method is that of two parallel plates, one of which is surrounded by a guard-ring, in order to avoid the influence of the edges.

In order to determine the same capacity in electromagnetic units, the method will be employed which we have pointed out in 1053 and the following. The measurement reduces itself to that of a resistance and of a time. With a single discharge, let T be the time of oscillation of the needle of a galvanometer, a the swing produced by the discharge of a condenser, the armatures of which

have a difference of potential E; and let δ be the deflection, corrected for the graduation, which would be produced in the same galvanometer by the current due to an electromotive force E in a resistance R; the capacity C is expressed by

$$(1) \qquad C = \frac{1}{R}\frac{T}{\pi}\frac{a}{\delta}.$$

If c is the value of the same capacity in electrostatic units, it follows that

$$a^2 = \frac{c}{C} = cR\frac{\pi}{T}\frac{\delta}{a}.$$

The advantage of the method is that the ratio a only depends on the square root of a resistance R, so that the relative error in this latter quantity only affects the value of a by less than one-half.

1135. In the experiments of Professors Ayrton and Perry,* the condenser was square, and the area of the plate A comprised within

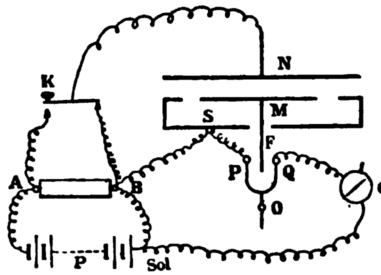


Fig. 238.

the guard-ring was 1325·14 square centimetres. The distance of the two plates was 0·7728 cm.

The pile P, composed of 382 Daniell's elements, was closed by a resistance AB (Fig. 238) of about 10,000 ohms. The difference of potential of the ends of this resistance was used to charge the condenser. With this view, the key K was pressed down, and the insulated fork PQ, movable about the point O, was turned so as to connect the branch P with the rod F, which supports the disc M,

* AYRTON and PERRY. *Journal of Soc. Tel. Engineers*, Vol. VIII., p. 126. 1879.

and the guard-ring; the condenser is charged. P is then immediately separated from F, and the key K is raised; the plate N and the box S are then connected with B. F is touched by the branch Q of the fork; the charge of the disc runs to the earth by the galvanometer.

The difference of potential E of the two points A and B remains to be measured. A greater resistance ρ is put in the galvanometer circuit, and one end of the resistance AB is attached at A, and the other at a point D, such that the resistance AD is a fraction m of the resistance AB, so that the difference of potential between A and D is equal to mE. Moreover, the galvanometer being shunted, the current i, measured by the deflection δ, is a fraction $\frac{s}{s+g}$ of that which the electromotive force mE would give, in a resistance $\rho+\frac{gs}{g+s}=\frac{\rho(g+s)+gs}{s+g}$. This, then, is the current which the electromotive force E would produce in the resistance $\frac{\rho(g+s)+gs}{ms}$; and from equation (1) we have

$$C=\frac{ms}{\rho(s+g)+sg}\frac{T}{\pi}\frac{a}{\delta}.$$

The galvanometer was a Thomson's astatic one, in which, in order to increase the time of oscillation and diminish the damping, the ordinary needles had been replaced by two systems of 20 bars each, mounted in a kind of small leaden shell, so as to represent a small sphere. The time of oscillation attained 39·5 s., and the logarithmic decrement was reduced to 0·1565.

1136. Instead of a single discharge, a series of discharges may be used. If β is the permanent deflection due to the discharges, δ that which corresponds to the current produced in a resistance R, by the electromotive force established between the armatures, and n the number of discharges in a second, we have

$$a^2=ncR\,\frac{\delta}{\beta}.$$

If the conditions are chosen so that the deflections δ and β are equal, we have simply $a^2=ncR$. In these conditions, it is unnecessary that the apparatus should have as large dimensions as for a single discharge, and the battery may be reduced to a small number of cells.

This method is that with which most experiments have been made; we may mention, in particular, those of M. Stoletow,* of Professor J. J. Thomson,† and of M. Klemencic.‡ The condenser of M. Stoletow was constructed of two circular plates, very near each other. Professor Thomson uses the arrangement of **1062**, and uses cylindrical condensers with guard-rings.

M. Klemencic used a differential galvanometer. The current of the battery bifurcates; one of the parts passes through a resistance box and one of the coils, the other goes to a make-and-break, and passes through the second coil in the form of discharges. The resistances are adjusted so that the needle is at zero. The condenser is formed of two circular plates the distance of which can be varied; the effect of the edges is allowed for by the following formula of Kirchhoff,§ in which A is the radius of the plates, b their thickness, and e the distance between them,

$$c=\frac{A^2}{4e}+\frac{A}{4\pi}\left[l.\frac{16\pi(e+b)}{e^2}+\frac{b}{e}l.\frac{e+b}{b}+2\right].$$

1137. A last method would consist in determining the electromagnetic capacity C of a condenser by the time t necessary for a difference of potential of the two armatures connected by resistance R, to pass from a value V_0 to the value V, the ratio $\frac{V_0}{V}$ being determined by an electrometer. We shall then have (**985**)

$$\frac{1}{C}=\frac{R}{t}l.\frac{V_0}{V}.$$

The electrostatic capacity c of the condenser being measured directly, it follows that

$$a^2=\frac{c}{C}=\frac{cR}{t}l.\frac{V_0}{V}.$$

* STOLETOW. *Journal de Physique* [1], Vol. x., p. 468. 1881.

† J. J. THOMSON. *Transactions of the Royal Society for 1883*, p. 707.

‡ KLEMENCIC. *Wiener Berichte* [3], Vol. LXXXIII., p. 88. 1884.

§ KIRCHHOFF. *Berliner Monatsberichte, 1887.—Gesammelte Abhandlungen*, p. 101.

1138. Summary of the Experiments.—Apart from the method of Weber and Kohlrausch, in which the value of a is taken solely from the numbers given by the experiment itself, the other methods bring in the numerical value of a resistance, which has often been determined by the British Association. If we assume (1128) that the most probable value of the ohm is represented by 106·25 cm. of mercury, the unit (B.A.U.) is equal to 0·98664. By correcting the results for the error made in estimating the resistances, we find as means of the numbers obtained by various experimenters:

Date and Observer.	Value of a	
	Found.	Corrected.
1856. Weber and Kohlrausch...	$31·07.10^9$	$31·07.10^9$
1869. W. Thomson and King...	$28·46.10^9$	$28·08.10^9$
1872. Dugald McKichan	$29·35.10^9$	$28·96.10^9$
1880. Shida	$29·95.10^9$	$29·55.10^9$
1879. Ayrton and Perry	$29·80.10^9$	$29·60.10^9$
1883. J. J. Thomson	$29·20.10^9$	$29·20.10^9$
1884. Klemencic	$30·19.10^9$	$30·19.10^9$
Mean		$29·52.10^9$

The most recent researches have given for the velocity of light:

1862. Foucault	$29·80.10^9$
1874. Cornu	$30·04.10^9$
1879. Michelson	$29·98.10^9$

It will be seen that, in all probability, or at any rate with an error which is less than one per cent., the velocity of light in a vacuum, and the ratio of the electromagnetic and electrostatic units of electricity, are represented by the same number.

PART III.

MAGNETIC MEASUREMENTS.

CHAPTER I.

MAGNETIC FIELD.

1139. The methods used to determine the field of a current are also applicable for any given magnetic field. But the measurement of magnetic fields, and particularly that of the terrestrial field, which comes so often into electrical measurements, is of such importance that it is necessary to resume the question in some detail more particularly from this point of view.

1140. Oscillations of a Magnetic Needle.—The most ancient method is that of oscillations. If K is the moment of inertia of a magnetic needle movable about an axis, M that component of its magnetic moment which is normal to its axis, N the number of oscillations, reduced to infinitely small angles, which it makes in unit time under the action of a magnetic field which is sensibly uniform in the space occupied by the needle, and the component perpendicular to the axis of which is H, we have

$$(1) \qquad \pi^2 K N^2 = MH.$$

This equation furnishes the product HM of the field by the magnetic moment of the needle, or the directing couple, if we knew the moment of inertia of the needle.

If the observation is made successively in two different fields H and H′, their ratio is equal to the square of the ratio of the numbers of oscillations which the same needle would make under their influence in the same time.

Nevertheless the magnetic state of the needle is modified by the field itself, but the induced magnetisation is not strictly parallel to

the magnetising force (398). As long as the oscillations are very small, it may be assumed that the induced magnetism consists of two terms; one parallel to the field, which is displaced in reference to the needle during the oscillations and does not produce a couple; the other, parallel to the greatest length of the needle, which increases its magnetic moment by an almost constant quantity, and which is proportional to the intensity of the field. In order to calculate the oscillations, we should add to the magnetic moment M_0, corresponding to the rigid magnetism, a term of the form iH, and equation (1) then becomes

$$(1)' \qquad \pi^2 K N^2 = (M_0 + iH)H = M_0 H\left(1 + \frac{i}{M_0}H\right) = M_0 H(1 + \phi H).$$

The coefficient ϕ depends on the shape of the needle and on its initial magnetisation. It is very small for highly magnetised steel bars, such as are used in observations on terrestrial magnetism. The equation

$$(2) \qquad \frac{N^2}{N'^2} = \frac{H(1 + \phi H)}{H'(1 + \phi H')}$$

may then be written

$$\frac{N^2}{N'^2} = \frac{H}{H'}\Big[1 + \phi(H - H')\Big].$$

This simplification is no longer possible in the case of very strong fields, in which temporary magnetisation may acquire a predominant influence. If the needle is of soft iron, or, better, of a substance which is but little magnetic, there remains in each case only the magnetisation proportional to the field. Equation (2) gives then sensibly

$$\frac{N}{N'} = \frac{H}{H'}.$$

1141. In order to compare any two fields F and F′ which may be displaced at will, like those of a current or a magnet, each experiment is arranged so that the field observed is parallel to the terrestrial field in the same or in the opposite direction—that is to say, does not deflect the needle, and the action of the earth

is eliminated by difference. This is the experiment of Biot and Savart (**444**).

For greater strictness, it may be observed that the effective magnetic moment of the needle, which is $M_0(1+\phi H)$ in the terrestrial field, becomes $M_0[1+\phi(H+F)]$ in the field $H+F$. If this latter is of the order of the terrestrial field, the numbers of oscillations N_0 and N, relative to to the two experiments, give

$$(3) \qquad \frac{N^2-N_0^2}{N_0^2}=\frac{F}{H}\,\frac{1+\phi(2H+F)}{1+\phi H}=\frac{F}{H}\left[1+\phi(H+F)\right],$$

or, replacing $H+F$ in the term of correction by its approximate value,

$$\frac{F}{H}=\frac{N^2-N_0^2}{N_0^2}\left[1-\phi H\frac{N^2}{N_0^2}\right].$$

The number N' of oscillations in the field $H+F'$ gives an analogous equation, and we finally get

$$(4) \qquad \frac{F}{F'}=\frac{N^2-N_0^2}{N'^2-N_0^2}\left[1+\phi H\frac{N'^2-N^2}{N_0^2}\right].$$

When the field F, instead of being parallel to the terrestrial field, makes with it a small angle α, the resultant field R is given by the equation

$$R^2=F^2+H^2+2HF\cos\alpha=(H+F)^2\left[1-\frac{HF}{(H+F)^2}\alpha^2\right].$$

The error made in taking $R=H+F$ is of the same order as the square of the angle of deflection, and may therefore be neglected provided the sum $H+F$ is not very small, which would correspond to two opposite fields.

1142. Methods of Torsion. — The magnetic needle being suspended by a metal wire the coefficient of which is C, and situate at first in the magnetic meridian, the torsion ω is measured which is necessary to bring it into a determinate direction. If it is near the transverse position (**721**) the induced magnetisation gives from symmetry a zero couple, and rigid magnetism alone comes in. If ω is the rotation of the milled head, and θ the

deflection of the needle, the directive couple M_0H will be given as a function of the coefficient C, by the equation

$$(5) \qquad C(\omega - \theta) = M_0H \sin \theta.$$

Some precautions are still necessary to eliminate the defects of adjustment. If the needle in its original position, instead of being strictly parallel to the magnetic meridian, makes with this plane a small angle α, the wire itself having an initial torsion ϵ, we have

$$C\epsilon = M_0H \sin \alpha,$$

and the equation of equilibrium (5) should be replaced by

$$(5)' \qquad C(\omega - \theta + \epsilon) = M_0H \sin(\theta + \alpha).$$

The same experiment, repeated on the other side, gives

$$C(\omega_1 - \theta_1 - \epsilon) = M_0H \sin(\theta_1 - \alpha).$$

When the two positions of equilibrium are opposed—that is to say, when the sum of the angles θ and θ_1 is almost π, these angles being sensibly equal—it is easy to prove, by adding and subtracting two equations of equilibrium, that the error of adjustment ϵ is $\frac{\omega_1 - \omega}{2}$. If this difference is small, the angles ω and θ may be replaced by the means of the readings right and left.

If ω_0 and θ are the mean values of the angles observed in a first experiment under the influence of the earth alone, ω the torsion necessary to obtain the same deflection θ in the field $H + F$, we have

$$C(\omega - \omega_0) = M_0F \sin(\theta + \alpha);$$

so that for two different fields F and F′,

$$(6) \qquad \frac{F'}{F} = \frac{\omega' - \omega_0}{\omega - \omega_0}.$$

If the direction of the fields F and F′, instead of being parallel to the terrestrial field, is arranged so that the needle is not deflected, they themselves make an angle α with the meridian. It is sufficient then to produce deflections on one side; the errors of adjustment

disappear in the difference of torsion, and the ratio of the fields is still given by equation (6).

More generally, if the directions of the fields F and F' and of the terrestrial fields are very close, it will readily be seen that the square of the error is simply of the order of the angle of deflection.

A bifilar suspension would give similar results. The apparatus being properly adjusted, we should have

$$C \sin(\omega - \theta) = M_0 H \sin\theta. \tag{7}$$

The defect of adjustment will be eliminated by taking the mean of the readings of two observations right and left, with positions of equilibrium which are sensibly opposed; if the difference of the angles is very small, we may replace the angles ω and θ in equation (7) by the means of two readings to the right and left.

For a mean deflection θ is 90°, we get simply

$$MH = -C \cos\omega.$$

In this case, if the mean torsion is ω_0 with the terrestrial field, ω and ω' with the fields $H + F$ and $H + F'$, we have

$$\frac{F'}{F} = \frac{\cos\omega_0 - \cos\omega'}{\cos\omega_0 - \cos\omega}.$$

1143. Let us now assume that the apparatus is carefully adjusted with a bifilar or unifilar suspension. For a small displacement $d\theta$, starting from the position of equilibrium defined by equations (5) or (7), the value of the couple which brings the needle into its position of equilibrium is given by one of the expressions

$$[M_0 H \cos\theta + C]\, d\theta = M_0 H \left(\cos\theta + \frac{\sin\theta}{\omega - \theta}\right) d\theta,$$

$$[M_0 H \cos\theta + C\cos(\omega - \theta)]\, d\theta = M_0 H \frac{\sin\omega}{\sin(\omega - \theta)}\, d\theta,$$

and the corresponding numbers of oscillations N and N' give

$$\pi^2 K N'^2 = M_0 H \left[\cos\theta + \frac{\sin\theta}{\omega - \theta}\right],$$

$$\pi^2 K N''^2 = M_0 H \frac{\sin\omega}{\sin(\omega - \theta}.$$

It is to be observed that, for deflections near 90°, the oscillations only depend on the rigid magnetism M_0.

1144. We may still arrange the auxiliary field in a direction at right angles to the terrestrial field, and determine the torsion necessary for bringing the needle into its original direction. According to the mode of suspension, we shall have

$$C\omega = M_0 F, \qquad \text{or} \qquad C \sin \omega = M_0 F.$$

The experiment is the same as for measuring currents by the method of torsion (830), and the defects of adjustment will be eliminated by observations made in both directions, right and left.

1145. METHODS OF DEFLECTION.—When the field F makes a considerable angle α with the magnetic meridian, and the needle is suspended by a wire without torsion, the condition of equilibrium relative to the deflection δ is

$$F \sin(\alpha - \delta) = H \sin \delta,$$

and, if the two fields are rectangular,

$$F = H \tan \delta;$$

the defects of adjustment are eliminated as in the tangent galvanometer.

We may also work as with the sine galvanometer. The needle being in the magnetic meridian, the field is made to act at right angles to the original direction of the needle, and it is then turned until it resumes the same position in respect of the needle—that is to say, that it is perpendicular to its final direction. If δ is the rotation observed, we have

$$F = H \sin \delta.$$

These two methods, which were first utilised by Gauss for observing the action of a bar magnet, are equivalent to the use of galvanometers in observing currents. Here we need not allow for variations which the magnetism of the needle experiences, for its magnetic action is parallel to the direction of the resultant field.

In this case, as in the galvanometer, the dimensions of the needle would have to be allowed for when the field in question is not uniform in the space it occupies.

1046. If the direction of the field is unknown, the measurement of the deflections and that of the oscillations, in the two cases, give the direction of the field, and its value as a function of the terrestrial field. For let α be the angle of the field F with the meridian, δ the deflection, N_0 and N the numbers of oscillations which correspond to the original direction and to the deflected needle, R the resultant of the forces F and H, we have, neglecting the induced magnetisation,

$$\frac{F}{\sin\delta} = \frac{R}{\sin\alpha} = \frac{H}{\sin(\alpha - \delta)},$$

$$\frac{N^2}{N_0^2} = \frac{R}{H};$$

from which is deduced

$$\cot\alpha = \cot\delta - \frac{N_0^2}{N^2 \sin\delta},$$

$$F = H\frac{\sin\delta}{\sin(\alpha - \delta}.$$

1047. The method of deflections may also be utilised even when the fields to be compared are parallel to the terrestrial field.

Let us assume that the needle of a declinometer of moment M is supported by a bifilar suspension, and deflected through an angle θ. The equation of equilibrium is

$$\text{(7)} \qquad HM\sin\theta = C\sin(\omega - \theta).$$

ω being the torsion of the system from the meridian.

In these conditions, if the needle is displaced through an angle δ, the couple which tends to restore it to its primitive position is

$$\begin{aligned} P &= HM\sin(\theta + \delta) - C\sin(\omega - \theta - \delta) \\ &= \frac{C}{\sin\theta}\left[\sin(\theta + \delta)\sin(\omega - \theta) - \sin\theta\sin(\omega - \theta - \delta)\right]. \end{aligned}$$

Replacing the products of the sines by sums of cosines, we find finally,

$$\text{(8)} \qquad P = C\frac{\sin\omega}{\sin\theta}\sin\delta = HM\frac{\sin\omega}{\sin(\omega - \theta)}\sin\delta.$$

As the couple is proportional to the sine of the displacement, the needle behaves exactly as if it were situated in a uniform field, parallel to the direction of equilibrium, and of intensity

$$H' = H \frac{\sin(\omega - \theta)}{\sin \omega}.$$

Let us suppose that the displacement δ had been obtained by adding to the terrestrial field, a field F parallel and in the same direction, the deflection of the needle is $\theta - \delta$. If the couple produced by the fresh displacement is made equal to the couple P, the condition of equilibrium is

$$(9) \qquad FM \sin(\theta - \delta) = C \frac{\sin \omega}{\sin \theta} \sin \delta = HM \frac{\sin \omega}{\sin(\omega - \theta)} \sin \delta.$$

This equation gives the ratio of the fields F and H, if the angle ω is known, and defects of adjustment are eliminated by an experiment made on the other side.

If the original equilibrium is exactly in the transverse position, the angle θ being 90°, we have

$$F = \frac{C}{M} \sin \omega \tan \delta = -H \tan \omega \tan \delta.$$

Another field F′, parallel to the terrestrial field, will give in like manner a displacement δ', and we deduce

$$(10) \qquad \frac{F}{F'} = \frac{\sin \delta}{\sin \delta'} \cdot \frac{\sin(\theta - \delta')}{\sin(\theta - \delta)},$$

the deflections δ and δ' being of the same or different signs, according as the fields F and F′ are of the same or contrary directions.

For a first equilibrium in the tranverse position, we have simply

$$(10)' \qquad \frac{F}{F'} = \frac{\tan \delta}{\tan \delta'}.$$

It is important to observe, first, that the torsion ω does not enter into the equations (10); and then that the deflections δ and δ' are

given directly by experiment. Finally, the ratio $\frac{\sin(\theta - \delta')}{\sin(\theta - \delta)}$ is sensibly equal to $\frac{\cos \delta'}{\cos \delta}$ when the displacements are very small, even when the angle is not strictly 90°.

1148. We may even eliminate the angles θ and θ' by observation of oscillations; for when the needle is acted on by the field $H + F$, and is displaced through the angle ϵ, the couple Q which tends to bring it back is, from equations (8) and (9),

$$Q = C \frac{\sin \omega}{\sin(\theta - \delta)} \sin \epsilon = FM \frac{\sin \theta}{\sin \delta} \sin \epsilon,$$

and the corresponding number N of oscillations, satisfies the equation

$$\pi^2 KN^2 = C \frac{\sin \omega}{\sin(\theta - \delta)} = FM \frac{\sin \theta}{\sin \delta}.$$

An analogous expression for the number N′ of oscillations relative to the field $H + F'$ gives

$$\frac{F}{F'} = \frac{N^2}{N'^2} \frac{\sin \delta}{\sin \delta'}, \tag{11}$$

and this equation now only contains the angles δ and δ'.

Comparing these equations (10) and (11), it follows, as a condition to be verified by experiment,

$$\frac{N^2}{N'^2} = \frac{\sin(\theta - \delta')}{\sin(\theta - \delta)}.$$

1149. Use of Galvanometers.—The action of a magnetic field may be compared with that of a current by very varied experiments.

Suppose, for instance, that a galvanometer, whose constant is G, is placed in a field F, parallel to the frame, and that a current I, determined in absolute value by some other instrument, is passed through it, the deflection δ of the needle will give the field F by the equation

$$GI = F \tan \delta.$$

If the field F is perpendicular to the frame, the experiment may be adjusted so that there is no deflection, and then

$$F = GI.$$

Lippmann's galvanometer (862) is particularly suited for the investigation of parts of very intense fields, because it may be reduced to very small dimensions. By measuring the change of pressure corresponding to the action of the field on a current I, which traverses a layer of mercury of thickness e, we shall have the component F of the field perpendicular to the bath by the equation

$$F = \frac{ep}{I}.$$

The measurement of the thickness e and of the current I enables us to determine F in absolute value. For comparative experiments, the field F is proportional, in all cases, to the difference of pressure p.

M. Leduc* used this arrangement, and even made the apparatus more sensitive by estimating the difference of pressure by a column of mercury.

We have seen, finally (857), how the simultaneous use of a tangent galvanometer, and of a movable frame, enables us to determine separately the intensity of the current which traverses the two apparatus, and that of the field in which they are placed, by the deflections of the needle, and of the frame, when the dimensions of the instrument are known.

By a suitable arrangement (858 and 859), we may also obtain the same result by observing the deflection of the frame by the field, and that which it produces, on an adjacent needle, situate in a principal position in respect of the original plane of the frame. These latter methods are particularly suited for determining the terrestrial field.

1150. Induced Currents. — The measurement of induced charges determines variations of the flow of force in the surface of a circuit, and therefore the mean value F_m of the field on this surface. This method was first used by Verdet,† to measure the intensity of the field, in which he placed substances having rotatory magnetic power; this is particularly suitable for powerful fields, such as those produced by electromagnets or industrial machines.

* A. Leduc. *Comptes rendus*, Vol. xcix., p. 186. 1884.

† Verdet. *Ann. de Chim. et de Phys.* [3], Vol. xli., p. 395.

A small coil is placed on a support by which it can be rapidly turned through 180° about an axis parallel to the plane of the windings, and it is interposed in the circuit of a ballistic galvanometer. The axis of the coil being, first of all, parallel to the direction of the field, the swing is noted which corresponds to a rotation of 180°. For a coil of n windings, the mean surface of which is S, the quantity q of electricity induced in a circuit of resistance R is

$$q=\frac{nS}{R}F_m.$$

The experiment itself gives the means of verifying that the axis of the coil is parallel to the field, by the condition that the induced discharge is a maximum.

In order to make the constant of the ballistic galvanometer, care must be taken that the circuit contains a frame of known surface S'. By turning this frame face over, from a horizontal position, or from a vertical position perpendicular to the meridian, the induced discharge corresponds to the flow of force $2ZS'$ or $2HS'$; in this way, any measurement of resistance, of galvanometric constant, and of time of oscillation, is avoided.

When the terrestrial field has a considerable part in the effect produced, the action is eliminated by displacing the coil parallel to itself until it emerges from the limits of the field observed.

With very powerful fields the surface S may be so small as to permit of an exploration of the distribution of force.

With comparative experiments, and when the deflections of the galvanometer are small with a feeble damping, the ratio of the fields is equal simply to the ratio of the swings observed.

We ought in particular to insist on the fact that the method of induction applies to any field, and does not necessitate any of the corrections which we have mentioned in other cases.

1151. Field of a Magnet.—At a distance which is great compared with the dimensions of a magnet, the magnetic field is the same as that of two infinitely near masses of opposite signs.

Take as the x axis the magnetic axis of a magnet, or the polar line, and for the y axis a right line in the perpendicular plane which passes through the middle of the magnet—that is to say, on its magnetic equator. Let x and y be the co-ordinates of a very distant point P, R the distance $\sqrt{x^2+y^2}$ of this point from the centre of the magnet, ω the angle of the right line R with the polar axis, the components X and Y of the field at the point P, and the

components Z and H, one perpendicular and the other tangential to the sphere of radius R, are thus expressed (153)

$$X = \frac{M}{R^3}\left(3\frac{x^2}{R^2} - 1\right) = \frac{M}{R^3}(3\cos^2\omega - 1),$$

$$Y = \frac{M}{R^3}3xy = \frac{M}{R^2}3\cos\omega\sin\omega.$$

$$Z = 2\frac{M}{R^3}\cos\omega,$$

$$H = \frac{M}{R^3}\sin\omega.$$

The expressions of these different forces are much more complicated when the distance R is not very great compared with the length $2L$ of the magnet. If they are expanded in increasing powers of the ratio $\frac{L}{R}$, they will be formed of a principal factor given by the value which agrees for great distances, multiplied by a series the first term of which is unity, and which will only contain even powers of the ratio in question.

For all these forces retain the same values, to within the sign, when the magnet is reversed—that is to say, when the sign of L is changed; as the moment M, which exists in the principal factor, changes its sign, the terms of the series should not change.

1152. In order to get an idea of the form of these expressions, the magnet may be compared to two masses $\pm m$ situate at the poles, and defined by the condition $2Lm = M$. The distances from the point P to the masses $+m$ and $-m$ being r and r', the magnetic potential is

$$V = m\left(\frac{1}{r} - \frac{1}{r'}\right);$$

the components Z and H are respectively equal to

$$-\frac{\partial V}{\partial R} \quad \text{and} \quad -\frac{1}{R}\frac{\partial V}{\partial \omega}.$$

Putting

$$z = \frac{2LR}{R^2 + L^2}\cos\omega,$$

we easily find

$$Z = \frac{2M\cos\omega}{(R^2+L^2)^{\frac{3}{2}}}\,\frac{R^2}{R^2+L^2}\left[1 - \frac{L^2}{2R^2} + \frac{3.5}{2.4}z^2\left(\frac{2}{3} - \frac{L^2}{2R^2}\right) + \frac{3.5.7.9}{2.4.6.8}z^4\left(\frac{3}{5} - \frac{L^2}{2R^2}\right) + \frac{3.5\ldots 13}{2.4\ldots 12}z^6\left(\frac{4}{7} - \frac{L^2}{2R^2}\right) + \ldots\right],$$

$$H = \frac{M\sin\omega}{(R^2+L^2)^{\frac{3}{2}}}\left[1 + \frac{3.5}{2.4}z^2 + \frac{3.5.7.9}{2.4.6.8}z^4 + \ldots\right].$$

If we replace z by its value, putting R as a factor everywhere, and expand in series, we ultimately get only even powers of the

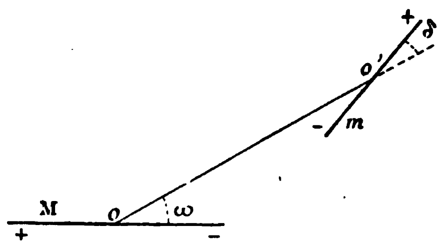


Fig. 239.

ratio $\frac{L}{R}$ in the bracket; but the law of the formation of successive terms is then very complicated.

We may observe, in particular, that in the equator of the magnet, where $z=0$, the value of H is simply inversely as the cube of the distance to the poles $\sqrt{R^2+L^2}$.

1153. Reciprocal Couple of Two Magnets.—Consider more generally two symmetrical magnets, the magnetic moments of which are M and m (Fig. 239), the lengths $2L$ and $2l$, and the magnetic axes of which, situated in the same plane, make the angles ω and δ with the right line $OO' = R$ which joins their centres. If the two magnets were very small compared with their distance, the moment D of their reciprocal couple will be

$$D = m\,(Z\sin\delta - H\cos\delta) = \frac{Mm}{R^3}\,(2\cos\omega\sin\delta - \sin\omega\cos\delta).$$

The couple of the two magnets is again equal to the product of this expression by a function $f(\mathrm{L}, l, \omega, \delta)$ which should only contain even powers of the lengths L and l. For if the sign of one of the lengths be changed, the corresponding magnetic moment changes its sign. As the couple D retains the same value to within the sign, the function f should not alter. We may then write

$$\mathrm{D} = \frac{\mathrm{M}m}{\mathrm{R}^3}\left(2\cos\omega\sin\delta - \sin\omega\cos\delta\right)\left[1 + \frac{h_2}{\mathrm{R}^2} + \frac{h_4}{\mathrm{R}^4} + \ldots\right];$$

the numerators $h_2, h_4, \ldots\ldots$ are respectively homogenous polynomials of the 2nd and 4th degree in L and l, only containing even powers of these two lengths.

If the magnets are reduced to four poles, we calculate the resultant couple either by the actions Z and H of the first on each pole of the second, or by the four actions in pairs. The expansion has been carried by Lamont* to terms of the 4th degree for the case in which the middle of the second magnet is in the principal position in respect of the first—that is to say, is situated on the line of the poles or in the plane of the equator. If α and β are the angles which the magnetic axis of the bar $2l$ makes with the equator of the former, the moments A and B of the couples relative to these two principal positions are

$$\mathrm{A} = \frac{2\mathrm{M}m\cos\alpha}{\mathrm{R}^3}\left[1 + \frac{2\mathrm{L}^2 - 3l^2(1 - 5\sin^2\alpha)}{\mathrm{R}^2}\right.$$
$$\left. + 3\,\frac{\mathrm{L}^4 - 5\mathrm{L}^2l^2(1 - 5\sin^2\alpha) + \frac{15}{8}l^4(1 - 14\sin^2\alpha + 21\sin^4\alpha)}{\mathrm{R}^4}\right],$$

$$\mathrm{B} = \frac{\mathrm{M}m\cos\beta}{\mathrm{R}^3}\left[1 - \frac{3}{2}\,\frac{\mathrm{L}^2 - l^2(4 - 15\sin^2\beta)}{\mathrm{R}^2}\right.$$
$$\left. + \frac{15}{8}\,\frac{\mathrm{L}^4 - 2\mathrm{L}^2l^2(6 - 23\sin^2\beta) + 8l^4(1 - 42\sin^2\beta - 21\sin^4\beta)}{\mathrm{R}^4}\right].$$

It may be observed that, in the expansion of the couple A, the terms of correction for the angle α are of the same form as for a coil acting on a needle centred on the axis (746), as should be

* LAMONT. *Handbuch des Magnetismus*, p. 281. 1867.

the case, since the action of the magnet is equivalent to that of the coil.

These expressions will be very important in determining the field of a magnet M by the action which it exerts on a second magnet m of smaller dimensions, and it is useful to investigate the importance of the terms of correction.

In all cases the expressions for the couples A and B may be put in the form

$$(12)\qquad \begin{aligned} A &= \frac{2Mm}{R^3}\cos\alpha\,[1 + p + p'], \\ B &= \frac{Mm}{R^3}\cos\beta\,[1 + q + q']. \end{aligned}$$

1154. Suppose, first, that the second magnet may be regarded as infinitely small in comparison with the distance to the centre of the first. Putting $\rho = \frac{L}{R}$, we have

$$p = 2\rho^2, \qquad p' = 3\rho^4;$$
$$q = -\frac{3}{2}\rho^2, \qquad q' = \frac{15}{8}\rho^4.$$

If the distance R, for instance, is four times the length of the magnet, or $R = 8L$, it follows that $\rho^2 = 0{\cdot}015625$, $\rho^4 = 0{\cdot}000244$, and the terms of correction are

$$p = 0{\cdot}031250, \qquad p' = 0{\cdot}000732;$$
$$q = -0{\cdot}023437, \qquad q' = 0{\cdot}000457.$$

1155. When the length of the second magnet is not very small, we shall assume that the two magnets are almost perpendicular to each other, or almost parallel.

If the magnets are almost perpendicular, the angles α and β may be neglected in the terms of correction. Putting $\frac{l}{L} = \lambda$, we have then

$$p = 2\rho^2\left(1 - \frac{3}{2}\lambda^2\right), \qquad p' = 3\rho^4\left(1 - 5\lambda^2 + \frac{15}{8}\lambda^4\right).$$
$$q = -\frac{3}{2}\rho^2(1 - 4\lambda^2), \qquad q' = \frac{15}{8}\rho^4(1 - 12\lambda^2 + 8\lambda^4).$$

In order that the second terms of correction, p' or q', should have the value null, the ratio λ of the lengths of the magnets must be given by the positive root less than unity of one of the equations

$$1 - 5\lambda^2 + \frac{15}{8}\lambda^4 = 0, \qquad \text{or} \qquad \lambda = \frac{1}{2\cdot15},$$

$$1 - 12\lambda^2 + 8\lambda^4 = 0, \qquad \lambda = \frac{1}{3\cdot36}.$$

The first condition is almost satisfied with the value $\lambda = \frac{1}{2}$, which annuls q; taking $\rho = \frac{1}{8}$, we have

$$p = 2\rho^2 \frac{5}{8} = 0\cdot019531, \qquad p' = -3\rho^4 \frac{17}{8\cdot16} - 0\cdot000097;$$

$$q = 0, \qquad q' = -\frac{15}{8}\rho^4 \frac{3}{2} = -0\cdot000685.$$

When the magnets are almost parallel, the angles α and β differ little from 90°. If α_1 and β_1 are the complementary angles $\frac{\pi}{2} - \alpha$, and $\frac{\pi}{2} - \beta$, the expressions for the couples A and B become

$$(13) \qquad A = \frac{2Mm}{R^3} \sin \alpha_1 [1 + p + p'],$$
$$B = \frac{Mm}{R^3} \sin \beta_1 [1 + q + q'].$$

The sines of the angles α and β being replaced by unity in the terms of correction, we have then

$$p = 2\rho^2 (1 + 6\lambda^2), \qquad p' = 3\rho^4 (1 + 20\lambda^2 + 15\lambda^4);$$

$$q = -\frac{3}{2}\rho^2 (1 + 11\lambda^2), \qquad q' = \frac{15}{8}\rho^4 (1 + 34\lambda^2 - 496\lambda^4).$$

In this case we cannot any longer nullify the second term p' for the first principal position, and the term would be null for the

second, making $\lambda = \frac{1}{3 \cdot 32}$. With the preceding values of λ and ρ, we should have

$$p = 0.078125, \qquad p' = 0.005078;$$
$$q = -0.087898, \qquad q' = -0.009825.$$

The best arrangement, if we propose solely to diminish the importance of the terms of corrections, is then to employ a small magnet the length of which is half that of the principal magnet, and to place it in the equator of the second.

1156. The *a priori* calculation of the terms of correction can, moreover, be of no use practically; for the action of the two magnets is not reducible to that of four poles. If the values of ρ and λ are not higher than those which have served for the previous calculations, when the deflected magnet is in a principal position in respect of the first and almost perpendicular or parallel to its direction, we may express the moment of the couple by one of the formulæ (12) or (13) in which only the second term of the series is taken in the form $\frac{h}{R^2}$, the coefficient h being determined by comparison of the results obtained for two different distances R and R'. It is not even necessary to assume that the term of the fourth power is negligable; for the mode of determination includes in the value obtained for the first term, the greater part of the correction for the next term.

1157. Law of Magnetic Actions.—The investigation of the field of a magnet may serve to determine the law of magnetic actions. Let us assume, in fact, that the action between two magnetic masses is inversely as the nth power of their distance.

The magnetic field of a mass m, at a distance r, is equal to $\frac{m}{r^n}$, and its potential to $\frac{1}{n-1}\frac{m}{r^{n-1}}$.

If we consider two very near masses $+m$ equal and of opposite signs, and at a distance $2a$, the potential at a point P, the distances of which to the masses in question are respectively r and r', is

$$V = \frac{m}{n-1}\left(\frac{1}{r^{n-1}} - \frac{1}{r'^{n-1}}\right) = \frac{m}{r^n}\, dr.$$

But the product mdr is the projection of the magnetic moment $2am = M$, of the two masses on the right line joining the middle of

their distance to the point P. The angle of these two directions being ω, we have

$$V = M\frac{\cos\omega}{r^n}.$$

The components of the force—one perpendicular and the other tangential to the sphere of radius r—are

$$Z = -\frac{\partial V}{\partial r} = n\frac{M}{r^{n+1}}\cos\omega,$$

$$H = -\frac{1}{r}\frac{\partial V}{\partial\omega} = \frac{M}{r^{n+1}}\sin\omega.$$

We shall see again, as before, that for any two masses, and therefore for a symmetrical magnet, the components of the field at a point, developed as functions of its distance from the middle of the magnet, will have the same expression, excepting terms of correction only containing even powers of the length of the magnet.

Let us observe, as a particular case, that for a great distance R, the values F_p and F_e of the field of a magnet, on the line of the poles and in the plane of the equator, are

$$F_p = n\frac{M}{R^{n+1}},$$

$$F_e = \frac{M}{R^{n+1}};$$

the ratio of these two expressions is equal to the index n of the power which defines the elementary action.

1158. Experiments of Gauss.—In a series of experiments, made with a view to verify the law of magnetic actions, Gauss* acted on a movable magnet by a second one, placed at varying distances of 1·3 to 4 metres.

The deflecting magnet was always perpendicular to the magnetic meridian, and the line of the centres perpendicular or parallel to the meridian. For each distance the deflection of the movable bar was obtained by the means of four readings, relative to two positions

* Gauss. *Intensitas vis Magneticæ Terrestris*, etc.—*Comment. Soc. Reg. Götting.*, Vol. VIII. 1841.—*Gauss' Werke*, Vol. V., p. 81.

of the deflecting bar on either side of the movable magnet, and of two reversals, so as to get rid of want of symmetry.

In each case, the tangent of deflection was equal, within a correction, to the ratio of the field of the deflecting magnet to the terrestrial field. The deflections δ and δ' for the distance R, for two different arrangements, satisfy very well the equations

$$\tan\delta = 0{\cdot}086870\,R^{-3} - 0{\cdot}002185\,R^{-5},$$

$$\tan\delta' = 0{\cdot}043435\,R^{-3} + 0{\cdot}002449\,R^{-5},$$

as is seen from the following tables:—

Distance R. Metres.	δ Observ. ° ′ ″	Obs.—Calc. ″	δ' Observ. ° ′ ″	Obs.—Calc. ″
1·3	2 13 51·2	+ 0·8	1 10 19·3	+6·0
1·4	1 47 28·6	+ 4·5	0 55 58·9	+0·2
1·5	1 27 19·1	− 9·6	0 45 14·3	−6·6
1·6	1 12 7·6	− 3·3	0 37 12·2	−3·2
1·7	1 0 9·9	− 5·0	0 30 57·9	−1·2
1·8	0 50 52·5	+ 4·2	0 25 59·5	−3·4
1·9	0 43 21·8	+ 7·8	0 22 9·2	+2·6
2·0	0 37 16·2	+10·6	0 19 1·6	+5·9
2·1	0 32 4·6	+ 0·9	0 16 24·7	+4·9
2·5	0 18 51·9	−10·2	0 9 36·1	−2·5
3·0	0 11 0·7	− 1·1	0 5 33·7	−0·2
3·5	0 6 56·9	− 0·2	0 3 28·9	−1·0
4·0	0 4 35·9	− 3·7	0 2 22·2	+1·7

For a great distance, the ratio of the tangents of the deflections—that is, of the fields of the deflecting bar on the line of the poles, and in the plane of the equator—is equal to the ratio of the first coefficients, which is exactly 2.

The law of the square of the distances is thus established directly with a degree of exactitude which the experiments of Coulomb would not furnish.

1159. Terrestrial Field.—Declination.—The value of the terrestrial field (**305**) is usually determined by the declination and the inclination, which give the directions, and by the value of one component. It would also be sufficient to know the declination and two of the components.

In order to have the declination, the geographical meridian must first be determined, and then the azimuth in which the magnetic axis

of a magnet sets movable about a vertical axis. This latter observation usually presents some difficulties, since the magnetic axis of a magnet is not generally parallel to its axis of figure; the error is corrected by turning the magnet upside down, observing in each position the direction of a line of sight; the mean of the two azimuths is the line which passes through the magnetic axis.

The line of sight is formed either by the ends of a needle, cut in the shape of an acute lozenge, or by two cross wires at the ends of a bar, as in Gambey's compass; or by two lines traced on the terminal faces of the magnet, and which are viewed with the microscope. We may also use hollow bars, converted into collimators by an object-glass let into one end, and a divided scale or a cross wire at the other.

1160. A declination compass, or declinometer, is a true theodolite, provided with accessory pieces for magnetic observations.

In the older instruments, such as those of Gambey, and Gauss' declinometer, the magnet was usually of great length, which has many inconveniences, the principal of which is the slowness of oscillation. In fact, for the same mean intensity of magnetisation, and for bars of the same shape, the magnetic moment is proportional to the volume, and the moment of inertia to the product of the volume by the square of the length; the time of oscillation is then proportional to the length. It is impossible to stop the bar in its position of equilibrium, and the slowness of the oscillations renders the observations very long.

We shall describe, as an example, the last model of a compass constructed by MM. Brunner (Fig. 240). The magnet is a square prism, furnished with two pins in the centre; at each end is a silver disc, on which is traced a division. This magnet is placed in a stirrup, suspended by a cocoon fibre, which passes over a pulley I at the top. The magnet moves in a metal box, terminated by two glass plates with parallel faces, and mounted on the movable part of the theodolite. The horizontal axis supports a microscope M, which can pass under the box, and read the ends of the magnet; a cross wire, with three equidistant wires, serves for the sighting.

A small plane, which can be raised or lowered by a screw outside, makes it possible to stop the oscillations.

Finally, the wheel is mounted on a small movable divided circle, and by lateral screws the wire of suspension can be centred; so that, for a given position of the magnet, the microscope points alternately on the middle lines of the division.

1161. The theodolite being adjusted in the usual way, in the geographical meridian, either by a polar star or by the corresponding heights of a star or of the sun, etc., the sun is observed by a blackened glass, and, instead of sighting one of the edges, it

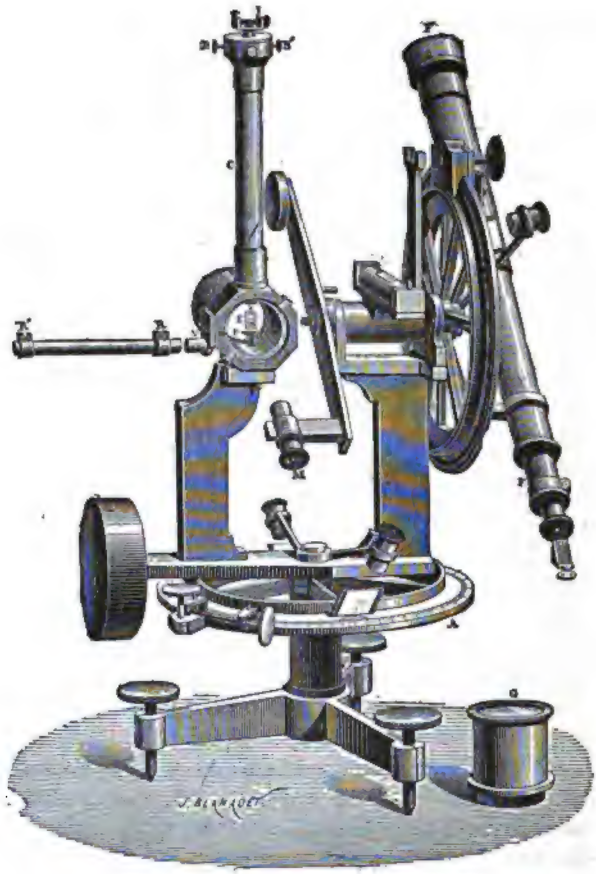


Fig. 240.

is better to use a network, so that the image comes within four rectangular tangent wires.

Suppose, for instance, that we merely know the latitude of the place. Let OP (Fig. 241) be the earth's axis, OA the vertical, OE the direction of the star at the moment of observation, l the latitude of the place or the height of the pole, h the height EQ of the star.

In the spherical triangle APE, the side PA is equal to $90° - l$, the side AE to $90° - h$, the side EP is at the polar distance Δ of the star, or the complement of its declination. The angle h is given by the vertical circle, and the corresponding reading of the verniers is made on the horizontal circle.

Putting

$$h + l + \Delta = 2S,$$

the angles A and P will be given by the formulæ

$$\cos\frac{A}{2} = \sqrt{\frac{\cos S \sin (S - \Delta)}{\cos l \cos h}},$$

$$\sin\frac{P}{2} = \sqrt{\frac{\cos S \sin (S - h)}{\cos l \sin \Delta}}.$$

A is the azimuth of the star—that is to say, the angle by which the arrangement must be turned, so that the telescope is in the

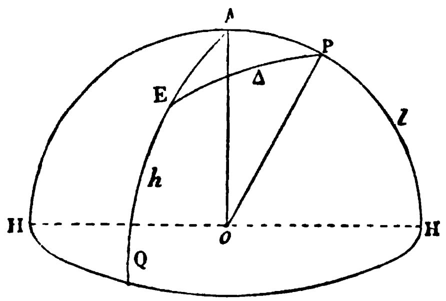


Fig. 241.

geographical meridian; the corresponding position of the verniers on the horizontal scale is then known. P is the hour-angle of the star, and from it is deduced the local time of observation.

Several corrections are necessary. As the zero of the graduation of the circle does not exactly correspond with the horizontal position of the telescope, two observations, with the telescope to the right and to the left, give twice the zenith distance of the star.

The height should, on the other hand, be corrected for atmospheric refraction. Lastly, if the body is not a fixed star, the astronomical declination changes with the time of day; but this variation does not amount to 1′ per hour for the sun. An approximate

knowledge of the time or of longitude is sufficient to calculate the declination by horizontal tables. Observation is especially accurate when the height of the star varies very rapidly—that is to say, when it is to the east or the west in the vicinity of the prime vertical.

1162. In magnetic observations we ought to be certain first that the wire has no initial torsion. A copper bar of the same weight as the magnet is suspended to it, and the circle of the pulley is turned until the bar is stationary in the plane of sight of the microscope. The magnet is substituted for the bar, one of its ends is sighted with the microscope, and, after having made the oscillations very small, the direction of the microscope is adjusted by the screw so that the oscillations of the median line are symmetrical in respect of the system of vertical lines; the verniers are then read, and the same observation is repeated at the other end. The mean of the two observations corresponds to the line of sight of the magnet. The magnet is then turned in its stirrup, and two similar observations are repeated. The mean of four readings corresponds to the magnetic axis, and from this is deduced the declination.

As the observer is placed very near the magnet, the smallest iron objects which he may carry, and even traces of iron which exist in certain materials used for clothing, may produce considerable perturbations. The precautions taken in this direction are rarely sufficient.

The instrument itself should be of copper or of bronze, free from iron; for there is no mode of correction by which this source of error can be eliminated. The magnet may be turned end over, and the observations repeated; but the concordance of the results is not an absolute guarantee, and the mean does not correct the error if it exists.

The series of four observations being of long duration, the magnetic declination may have varied in the interval of the readings. The indications of a variation apparatus, or, still better, of a registering apparatus, would enable us to correct each of the readings so as to refer them to the same period. In all cases it is better to make the observations when the declination passes through a maximum or a minimum. This precaution is especially necessary in summer, when the amplitude of the daily variation is greatest.

Instead of making the central line of the divisions coincide at each sighting with the central wire of the network, we can determine the angular value of the divisions, and that of the distance of the wires, which would sometimes enable us to make the

observations more rapidly, by noting the division which corresponds to the central wire.

It is, finally, well to know the torsion couple of the wire. For this it is sufficient to observe the deflection δ produced by a considerable torsion—for instance, half a turn—and we have

$$C\pi = HM \sin\delta, \qquad \text{or} \qquad \frac{C}{HM} = \frac{\sin\delta}{\pi};$$

the wire is not suitable if this ratio is not very small.

Very different forms have been given to declination compasses. Whatever be the mode of construction, the apparatus should be provided with all the necessary means of correction, and the course of the observations is always the same.

In the compass which we have described, the point at which the stirrup is attached to the wire of suspension is so high above the centre of gravity of the movable system, that the couple arising from the vertical component does not make the bar incline to any appreciable extent; but if the needle is supported on a pivot, it is sometimes necessary to allow for this source of error, and in different latitudes we are obliged to bring the needle into the horizontal by small counterpoises.

1163. INCLINATION.—We may determine the inclination of a needle which moves about an axis perpendicular to the magnetic meridian, either by the apparent inclination in a plane which makes a known angle with the meridian, or in two rectangular planes.

In inclination compasses, or *inclinometers*, as in that of Gambey, long needles were formerly used to facilitate the readings. Here again, besides that the apparatus is of too great dimensions, the time of oscillation makes the observations far too slow, without adding anything to their exactitude.

Fig. 242 represents an inclination compass of very small dimensions, in which the needle is only 7 centimetres in length. It is a steel plate in the form of a very acute lozenge, traversed in the centre by a steel axis the ends of which are truly cylindrical, and which rest on two agate knife-edges which are in the same horizontal plane. The agates are supported by a frame which rotates on a horizontal circle, and the needle moves in front of a vertical circle. The axis of the needle, in rolling on the agates, may leave the centre of the circle; it is brought back by raising by a screw a V-shaped stirrup, which lifts the axis and then lets this down gently on the agates.

The readings are made by a very sensitive method, described above (661); by reading with the lenses *l*, when the point of the needle is made to coincide with its real image in a concave mirror M supported by a vertical circle.

1164. The sources of error are here more numerous than in the declination compass. For in order that the inclination could be determined by a simple reading, it is necessary: 1st, to know the horizontal line on the vertical circle; 2nd, that the needle be centred

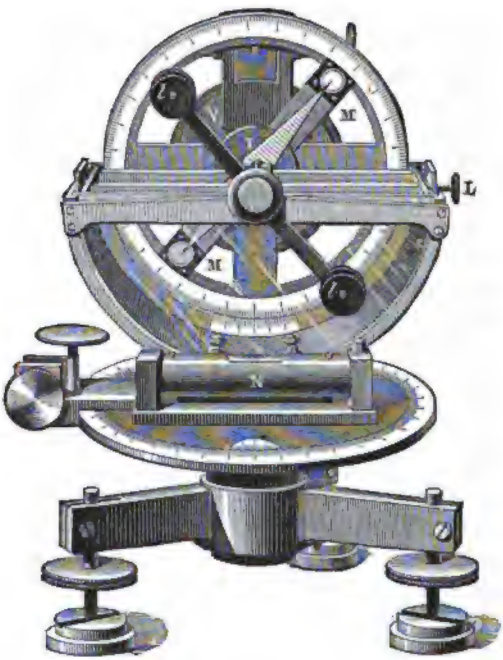


Fig. 242.

in the circle; 3rd, that the magnetic axis of the needle be parallel to the line of sighting; 4th, that the centre of gravity of the needle be on the axis of rotation; 5th, and lastly, that the plane of the agates be horizontal. All these conditions can only be approximately realised.

1st. In order to know the horizontal line or the vertical, the needle is replaced by a plumb-line situate in the same plane, and this wire is observed successively above and below in the two mirrors; the mean of the readings gives the division which is at

the zero of the vernier when the plane which passes through the axis of the mirror is vertical. This determination will be useful in making the needle vertical in certain cases; nevertheless, in ordinary observations it is sufficient to turn the mounting through 180°, when the difference of the readings give twice the angle of the needle with the vertical.

2nd. The defect of centring is eliminated by the mean of the readings for the two ends of the needle.

3rd. After having observed the needle in one position, it is inverted; the mean of the readings eliminates the defect of parallelism of the magnetic axis with the line of the points.

4th. The weight of the needle increases or diminishes the apparent inclination when the centre of gravity is eccentric and situate above or below the axis of rotation. The inverse effect is observed by reversing the magnetism of the needle, and the mean of the results is again taken.

5th. If the plane of the agates, although parallel to the vertical axis of the circle, is not itself horizontal, the needle has a tendency to roll in one direction or the other, and the error changes its sign when the apparatus is turned through 180°. This error is again eliminated by the mean of the readings. We are further certain that the difference does not exist if half the difference of the readings corresponds to the vertical given by the plumb-line.

1165. Nevertheless, as several sources of error act simultaneously, it is necessary to demonstrate that the compensation due to these reversals is exact, especially as regards the position of the centre of gravity.

Let us suppose that the needle moves in a plane which makes the angle α with the magnetic meridian. The apparent inclination I_α (**305**) satisfies the equation

$$\cot I_\alpha = \cot I \cos \alpha.$$

Let:

I' be the inclination given by the line of the points OA of the needle (Fig. 243);

G, the centre of gravity of the needle, situate at the distance d from the axis on the right line which makes the angle γ with the line of the points;

β, the angle of the line of the points with the magnetic axis OB;

F, the component of the terrestrial field in the plane of the needle, in C.G.S. units;

p, the weight of the needle in the same units;

M_1, its magnetic moment.

The plane of the agates being supposed horizontal, the condition of equilibrium is

$$(14) \qquad pd \cos (\mathrm{I}' - \gamma) = \mathrm{FM}_1 \sin (\mathrm{I}_a - \mathrm{I}' + \beta).$$

When the needle is inverted, the angles β and γ change their sign; the new inclination I″ observed gives

$$(15) \qquad pd \cos (\mathrm{I}'' + \gamma) = \mathrm{FM}_1 \sin (\mathrm{I}_a - \mathrm{I}'' - \beta).$$

Adding these two equations member to member, calling I_1 half the sum and i half the difference of the observed angles I′ and I″, we get

$$(16) \qquad pd \cos \mathrm{I}_1 = \mathrm{FM}_1 \sin (\mathrm{I}_a - \mathrm{I}_1) \frac{\cos (i + \beta)}{\cos (i + \gamma)}.$$

If the angles I′ and I″ differ very little from each other, by 1° at the maximum, it follows, as we shall see from the difference of

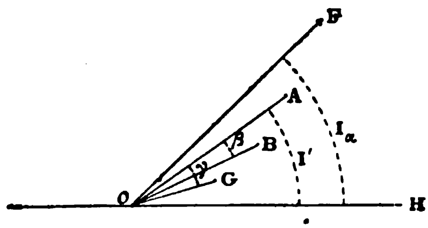


Fig. 243.

the equations (14) and (15), that the angle $\gamma - \beta$ of the direction of the centre of gravity with the magnetic axis is very small. The ratio of the cosines which comes into the second member of the equation (16) does not appreciably differ from unity, and we may write

$$(17) \qquad pd \cos \mathrm{I}_1 = \mathrm{FM}_1 \sin (\mathrm{I}_a - \mathrm{I}_1).$$

The needle having received a magnetic moment M_2 in the opposite direction but little different from the first, the centre of gravity is below the axis, and the inclination I_2, given by the mean of the observations on turning, is greater than I_a, which leads to the equation

$$(18) \qquad pd \cos \mathrm{I}_2 = \mathrm{FM}_2 \sin (\mathrm{I}_2 - \mathrm{I}_a).$$

We shall again assume that the angles I_1 and I_2 differ very little from each other, without which the needle would be too defective. The angles $I_a - I_1$ and $I_2 - I_a$ are then very small, and we may accordingly replace their sines by the corresponding angles in the ratio of the equations (17) and (18); we deduce from this

$$(19) \qquad \frac{I_a - I_1}{I_2 - I_a} = \frac{M_2}{M_1} \frac{\cos I_1}{\cos I_2}.$$

The second member of this equation differs little from unity; for the angles I_1 and I_2 are supposed to be very near, and the magnetisations also as equal as possible. It follows, to within an error of the second order, that

$$(20) \qquad I_a = \frac{I_1 + I_2}{2}.$$

The angle I' is really given by the mean ot two readings for the two points of the needle, and of two other readings after the frame is turned through 180°. The angles I_1 and I_2 are then each the mean of eight readings. The value I_a which is to be observed is given, without appreciable error, by the mean of sixteen readings corresponding to reversals and to change of magnetisation.

It will be seen that, in the same conditions, the want of homogeneity of the agates is eliminated by a second observation made by turning the frame through 180°.

If the magnetisations were very different, the ratio m of the moments M_1 and M_2 could be determined by the times of oscillation in the same azimuth. Equation (9) will then give

$$(20)' \qquad I_a = \frac{mI_2 + I_1}{1 + m}.$$

It is ascertained, in all cases, if the times of oscillation are materially different; but the introduction of this new correction, if it were necessary, would be a disadvantageous condition.

When the reversal of the needle, the inverse magnetisation, and the rotation of the frame, give variations of several degrees in the reading, the apparatus is manifestly imperfect, and no mode of correction will find an exact inclination.

In the compasses of MM. Brunner, all these reversals and inversions do not give variations exceeding 5′, but we should add that so great a perfection is rarely attained.

1166. In order to observe the inclination directly, the direction of the magnetic meridian must be known; it is determined by the property that a dipping needle stands vertical in a plane which is perpendicular to the meridian.

With a good needle, it is sufficient to take the mean of two azimuths in which the needle is vertical before and after being reversed. It would be sufficient to sight a point in any given azimuth, then to turn the arrangement until this point is in the same position in reference to the vertical circle. One of the bisections of these two corresponding positions gives the meridian, and the other the azimuth perpendicular to the meridian. Except when the inclination exceeds 70°, it is easy to find rapidly by this method the direction of the meridian to less than a few minutes, and it is sufficient that the error be less than 1°.

Compasses do not generally enable us to determine directly very small inclinations. Near the equator it is better to make the observation in two rectangular azimuths, or in any two azimuths on either side at the same distance from the meridian.

In this latter case we need not know exactly the direction of the meridian. The distance of the azimuth planes being 20°, for instance, the mean of the observations gives the inclination at 10° from the meridian. The true inclination is obtained by the equation

$$\tan I = \tan I_a \cos a.$$

1167. The measurement of the inclination is always the result of a great number of different readings. It is important to arrange the observations methodically in order to simplify manipulation, and touch the needle as little as possible. One of the faces of the needle has on one side a distinctive mark—for instance, a point marked on the mounting near the axis, so that the positions of the magnet can be defined in each case.

Suppose, for instance, that the observations are made in two rectangular azimuths, nearly 45° on either side of the meridian.

In order to magnetise the needle, it is placed in a cavity rather deeper than the thickness of the needle and is kept there by a screw. With two magnets the breadth of which is greater than that of the cavity, so that they do not rub against the needle, a number of determinate passes is made—10 or 20, for instance—

by the method of separate touch, and the same operation is made on the other face. We have thus a first magnetisation, which will serve for a series of observations.

The needle is placed in the compass and the vertical circle is placed in the first azimuth. We shall call that the *face* of the instrument at which the observer looks in making the readings. If the mark of the needle is above for this first magnetisation the various observations may be described as follows:—

FIRST MAGNETISATION.—MARK ABOVE.

First Azimuth.

MARK IN FRONT.		MARK BEHIND.	
Face to the S.E.	Face to the N.W.	Face to the S.E.	Face to the N.W
(1)	(2)	(8)	(7)

Second Azimuth.

Face to the N.E.	Face to the S.W.	Face to the N.E.	Face to the S.W.
(3)	(4)	(6)	(5)

By making the observations in the above order, the needle need only be touched once.

It is then magnetised in the opposite direction by the same number of passes: the mark will be found below, and the readings are made in the same order.

We may add that for each position we should not be satisfied with a single reading above and below. It is necessary to recommence several times, by raising the needle with its stirrup and letting it rest on the agates, and then taking the mean of the readings.

We may determine the magnetic declination with a less error than a minute; but, whatever be the perfection of the instruments, it is difficult to get the dip to within a minute.

1168. The inclination may be determined from the equation $Z = H \tan I$, if we know the ratio of the horizontal and vertical components of the terrestrial field.

We might determine, for instance, the ratio of the directive couples MH and MZ of a magnet movable successively about a vertical axis and a horizontal axis perpendicular to the meridian (**1191**).

Dr. Lloyd* acted on a declination compass by one end of a vertical soft iron bar, which acquires an induced magnetism proportional to the component Z. Observation gives thus a quantity proportional to the ratio of the components Z and H, and consequently to the tangent of the inclination.

This method was used by Lamont.† It has the advantage of giving all the elements of the earth's magnetism by a single instrument.

In Lamont's theodolite a bar of soft iron, or, better, two symmetrical bars, are arranged vertically in a fixed mounting in such a manner that the top of one and the bottom of the other are in the plane of the movable magnet. The action of the induced magnetisation on the movable magnet may be represented by CZ, and if the deflecting bars are in a plane perpendicular to the magnet, the mean deflection α gives

$$\sin \alpha = \frac{CZ}{H} = C \tan I,$$

the coefficient C being determined by comparison with a dip needle. A series of reversals render it possible to eliminate the defects of symmetry of the bars and their residual magnetism.

The properties of iron are, however, modified by so many causes, physical and mechanical, that we can never be certain of the bar being always in the same condition. Experiment shows, in fact, that the coefficient C is not invariable, and that we obtain thus only an approximate value for the inclination.

1169. Induction Method.—In Weber's‡ induction inclinometer, the ratio of the components Z and H is determined by discharges induced in a frame which turns through 180° about a vertical from a plane perpendicular to the meridian and about a horizontal parallel to the meridian starting from a horizontal plane (**529**).

With this arrangement it is necessary to measure the swings of the ballistic galvanometer, which corresponds to two induced discharges, and the relative error of the inclination of the same order as that of the two readings.

* Lloyd. *Account of the Magnetic Observatory in Dublin.* 1842.

† Lamont. *Handbuch des Erdmagnetismus*, p. 212. 1849.

‡ W. Weber. *Pogg. Ann.*, Vol. XLIII., p. 293. 1886.

The error may be greatly diminished by searching by trial for the initial position of the frame in the second case, so that the discharge is the same as in the first.* More generally, the frame being made movable about an axis perpendicular to the meridian, two initial positions are determined, such that the deflection of the ballistic galvanometer is the same as for a rotation of 180°; the bisection of this gives the direction of the inclinations.

Any measurement of the discharge is eliminated, as well as the regulation of the initial and final positions of the movable frame, by determining the right line about which rotation should take place so that no current is induced.† The apparatus is quite similar to a dip needle; the axis of rotation of the frame may be displaced on a vertical circle which itself is movable about a horizontal circle.

By means of a milled head the frame may be turned 180° to the right or left. The deflections observed in a ballistic galvanometer enable us by a series of methodical trials to put the axis of rotation first in the meridian, and then in the direction of the dip needle. The accuracy of the observations is much increased, for when the current is near null, we can combine the movements of the frame with those of the ballistic galvanometer, and multiply the angles of swing.

The adjustment for the magnetic meridian is so accurate that the apparatus can even serve as declination compass.

The sensitiveness may moreover be increased by putting soft iron in the frame, since the changes of magnetisation are null for any given rotation about an axis parallel to the field.

We may finally replace the ballistic galvanometer by a telephone, giving the frame a continuous motion. The periodic currents produced in these conditions do not vary suddenly, and can with difficulty be perceived in a telephone;‡ but they become more perceptible when a make-and-break is introduced in the circuit—for instance, a toothed wheel. In this way, if the frame turns continuously, the telephone is only silent when the rotation is parallel to the dipping needle, but without giving a sufficient approximation. Better results are obtained by putting soft iron in the frame.§

* H. WILD. *Bulletin de l'Acad. des Sc. de St. Pétersbourg*, Vol. XXVII., p. 320. 1881.

† MASCART. *Comptes rendus*, Vol. XCVII., p. 1191. 1883.

‡ J. STEPHAN. *Wiener Sitzungsberichte*, p. 262. 1880.

§ W. SCHAPER. *Meteorolog. Zeitschrift*, p. 71. 1886.

1170. Intensity.—The method of oscillations (**1137**) has often been applied to compare the horizontal component of the terrestrial field. In this case, the correction for the induced magnetism may generally be neglected, but the variations in the magnetic moment of the magnet with the needle should be allowed for. The observation is made at the temperature t; let M_0 be the magnetic moment at zero, and a the coefficient of variation, we have

$$\pi^2 K N^2 = M_0 (1 - at) H,$$

or

$$\pi^2 K \frac{N^2}{1 - at} = M_0 H.$$

Hence, for each observation the square of the number of oscillations N^2, should be divided by the binomial for the corresponding temperature.

The value of a is always less than a thousandth, so that it is unnecessary to know very exactly the temperature of observation.

1171. This coefficient should be determined directly for each needle. The numbers of oscillations N and N′ for the two different temperatures t and t' give the value of a by the equation

$$\frac{N^2}{1 - at} = \frac{N'^2}{1 - at'}.$$

The experiment is, however, rather delicate, for if we do not allow for the changes of the surrounding temperature, it is difficult to know the temperature of a needle which oscillates freely in a heated space.

It is better to arrange the experiment so as to observe directly variations in the magnetic field of the bar considered as to its action on another needle which serves as declinometer.

A sort of calorimeter is formed of two tubes of Bohemian glass, fitted in each other, which leave a stationary layer of air between them. The bar is contained in a glass or metal tube and placed in the centre of the inner vessel, which is filled with water and closed by a wooden lid. Hot water at a temperature of about 60° is first introduced; the liquid is then agitated by blowing air in at each observation. In these conditions the cooling is very slow, and the liquid takes more than six hours to revert to the surrounding temperature.

The bar being placed in a plane perpendicular to the magnetic meridian, the deflections δ, δ'..... produced at different temperatures are determined; we have then

$$\frac{\tan\delta}{1-at}=\frac{\tan\delta'}{1-at'}=\dots$$

As the experiment lasts some time, it is well to eliminate variations of declination. The calorimeter is placed on a platform movable about a vertical axis which passes almost through the centre of the magnet. When the frame is turned through 180°, the action of the magnet on the declinometer changes its sign, and the difference of the readings corresponds to twice the deflection δ. It is simply assumed that the horizontal component does not appreciably vary during the experiments; if that is the case, the necessary corrections are made from the indications of a variation apparatus.

The coefficient a may have very different values according to the nature and temper of the steel. Lamont* had conceived the idea of a magnetic system formed of two opposite magnets M and M′, the coefficients of which a and a' were very unequal. The magnetic moment $M_0(1-at)-M'_0(1-a't)$ of the system may be written $M_0-M'_0-(M_0a-M'_0a')t$; it is independent of the temperature if the coefficients a and a' are inversely as the corresponding moments. We may thus form a resultant magnet compensated for temperature, but the compensation seems difficult to realise.

1172. On the other hand, the magnetism of a needle becomes weaker in time: at the outset somewhat rapidly in the first few weeks after magnetisation, and then much more slowly.

This variation is very unequal according to the nature of the magnets, the degree of magnetisation, and the time that has elapsed since this operation was performed. Thus, for seven needles observed during the voyage of the *Recherche*,† the magnetic enfeeblement in a year varied from 0·0025 to 0·0486. The corresponding correction is made for the intermediate observations, assuming that this diminution has been proportional to the time.

1173. The magnetism of a needle may, lastly, be suddenly modified by a blow, the neighbourhood of a magnet or a piece of iron, or by any other unknown cause. These accidents are avoided,

* LAMONT. *Handbuch of Magnetismus*, p. 402. 1867.

† *Voyage de la Recherche en Islande et au Groenland.—Magnétisme*, Vol. II., p. 326.

at any rate partially, by using an uneven number of magnets, 3 or 5. If the ratios of the results obtained in two different stations are the same as for the three needles, it is probable that they are exact; when one of them disagrees with the two others it is to be assumed that its magnetism has undergone an accidental variation, and acquired a new value, which will serve for the following observations. In this way, provided that two at least of the needles have not been modified in the interval of the two stations, the series of observations is never interrupted whatever changes may have taken place in the state of the needles.

1174. Poisson's Method.—Poisson* was the first to point out how the horizontal component in a place may be determined in absolute measure independent of the magnetic moment of the magnet.

The oscillations which a bar makes under the influence of the earth determine the directing couple $MH = A$. On the other hand, the magnetic field of the bar is proportional to M, and the comparison of this field with the terrestrial field determines the ratio $\frac{M}{H} = B$. Two numbers thus formed by experiment enable us to calculate separately the values of M and H:

$$M^2 = AB,$$
$$H^2 = \frac{A}{B}. \tag{21}$$

Taking into account the induced magnetisation, the observation of oscillations gives

$$A' = MH(1 + \phi H) = A(1 + \phi H).$$

The bar is then made to act on a needle which is parallel to it, and at a distance R in the principal position in respect of the bar—for instance, in its prolongations. By fresh oscillations (1140) the ratio $\frac{F}{H} = B'$ of the field of the magnet is thus determined.

The bar being parallel to the magnetic meridian and in the same direction, its magnetic moment, increased by the induced

* Poisson. *Connaissance des Temps pour 1828*, p. 113.

magnetisation, may be represented by $M(1+fH)$; if p is the term of correction for the distance R (1155), we have

$$F=\frac{2M(1+fH)}{R^3}(1+p)=B'H,$$

and the value of p will be determined by two experiments made at different distances. We deduce from this

$$B=\frac{M}{H}=\frac{B'}{2}\frac{R^3}{(1+fH)(1+p)}$$

and consequently

$$\begin{aligned} M^2&=\frac{A'B'}{2}\frac{R^3}{[1+(f+\phi)H](1+p)},\\ H^2&=\frac{2A'}{B'}\frac{[1+(f-\phi)H](1+p)}{R^3}. \end{aligned} \tag{22}$$

The component H is then expressed as a function of the data of the experiment, and will be independent of the magnetisation induced in the bar, if it may be assumed that the coefficients f and ϕ are sensibly equal.

Corrections for temperature relative to the distance R and the moment of inertia of the bar, are not of great importance; for they are of the order of expansion of the metals—that is to say, less than 0·0002 for a variation of 10°.

In the case of comparative measurements, it is useless to know the moment of inertia of the bar, and the distance R, provided the quantities are the same in all the experiments.

1175. Method of Gauss.—Poisson had thus established the principles of the method; but the use of oscillations has the inconvenience of only giving the ratio of the fields F and H by the difference of the squares of the numbers observed, and does not carry with it great accuracy. The determination of the term of correction p, in particular, will be obtained by an expression containing differences of the numbers of oscillations, which are difficult to estimate exactly.

Poisson's method was very rarely used; it seems even to have been little known when Gauss* pointed out an experimental arrangement easier of application, which since then has been generally

* Gauss. *Intensitas vis Magnet.*, *Comm. S. R. Götting.*, Vol. VIII.

adopted. The comparison of the magnetic field of the magnet with the terrestrial field is made by a method of deflection (1145). The auxiliary needle of the declinometer is placed in a principal position in respect of the bar, either on its direction (first position), or in the plane of the equator (second position).

Let us consider the first case, for instance, and suppose that the deflecting bar is perpendicular to the magnetic meridian; its moment is not modified by the action of the earth, and the deflection α of the declinometer satisfies the equation

$$\tan\alpha = \frac{F}{H} = \frac{2M}{HR^3}(1+p). \tag{23}$$

At the distance R′, the deflection α' will give, in like manner,

$$\tan\alpha' = \frac{2M}{HR'^3}\left(1+p\frac{R^2}{R'^2}\right). \tag{24}$$

1176. It is important to investigate how we ought to choose the ratio of the distances R and R′, to determine as exactly as possible the term of correction p. Putting

$$\rho = \frac{R}{R'},$$

we deduce, from equations (23) and (24),

$$p = \frac{1 - \dfrac{1}{\rho^3}\dfrac{\tan\alpha'}{\tan\alpha}}{\dfrac{1}{\rho^3}\dfrac{\tan\alpha'}{\tan\alpha} - \rho^2}. \tag{25}$$

As the value of p is less than 0·02 when the distances are at least four times the length of the deflecting magnet, and the lengths of the bars in the ratio of 2 to 1 (1155), the fraction $\dfrac{1}{\rho^3}\dfrac{\tan\alpha'}{\tan\alpha}$ is near unity; if the deflections α and α' are themselves small, which is the ordinary case, we get as approximate value

$$p = \frac{1}{1-\rho^2} - \frac{1}{\rho^3(1-\rho^2)}\frac{\alpha'}{\alpha}.$$

Let us assume that, the distances R and R′ being exactly known, the errors made in measuring the angles α and α' are $d\alpha$ and $d\alpha'$; the corresponding error on p will be

$$dp = -\frac{\alpha'}{(1-\rho^2)\rho^3\alpha}\left(\frac{d\alpha'}{\alpha'} - \frac{d\alpha}{\alpha}\right).$$

Other things being equal, this error is the least possible when the product $(1-\rho^2)\rho^3$ is a maximum—that is, when

$$\rho^2 = \frac{3}{5}, \qquad \text{or} \qquad \frac{R'}{R} = 1{\cdot}29 \text{ about.}$$

With this value of ρ, the angles α and α' are almost in the ratio of 2 to 1, and we have sensibly $\rho^3\alpha = \alpha'$, which gives

$$dp = -\frac{5}{2}\left(\frac{d\alpha'}{\alpha'} - \frac{d\alpha}{\alpha}\right).$$

In order that the error dp be less than 0·001, the relative errors $\frac{d\alpha'}{\alpha'}$ and $\frac{d\alpha}{\alpha}$, which we shall suppose equal and opposite, are less than 0·0002. If the angle α is 10°, or 600′, the probable error $d\alpha'$ must then be less than 4″; these are conditions which are difficult to realise in practice.

The relative error on the determination of the ratio $\frac{M}{H}$ is equal to dp, and the corresponding relative error on the value of the component is half less. From this it will be seen that the principal difficulty consists in this term of correction.

In permanent observatories it is better to use a special apparatus for this kind of experiment.

1177. For travelling observations, magnetic theodolites have special arrangements, by which the field of the magnet may be compared with the terrestrial field. In Brunner's compass (Fig. 240) a lateral rod on the box supports two stirrups R and R′, on which a bar can be placed at two different distances, to make it act on another magnet, of half the length, and provided with copper ends.

The microscope being sighted for the movable magnet, the deflecting bar is placed on a support, and the instrument is turned so that the sighting of the microscope is unchanged; the same observation is then made after having reversed the deflecting bar. The

semi-difference of the readings gives the deflection α, and the condition of equilibrium is

$$\sin\alpha = \frac{F}{H} = \frac{2M}{HR^3}(1+p).$$

As the deflections are not large, the influence of the induced magnetisation on the deflecting magnet need not be taken into account. The series of the observations is the same as in the preceding case, with this difference—that the sines, and not the tangents, of the angles, come into the formulæ.

The distances R and R′, from the centre of the stirrup to the deflecting bar, have been determined once for all, by measuring with a microscope, on the slide of a dividing machine, the distance from the suspension-wire to the ends of the magnet placed on a stirrup, and then turned over. As the bar cannot be placed on the other side, to correct for the error of centering, the position of the wire must remain unchanged; it is centred each time with the screws B and B′, so that the microscope, when turned through 180° about the horizontal axis, sees the two fiducial lines of a bar without displacing the arrangement.

The change of side is possible when the rod T is replaced by another rod, parallel to the movable magnet, and carrying stirrups in which the deflecting bar may be placed, in a direction perpendicular to the rod; in this case, observations right or left eliminate the want of centering of the wire. The declinometer is then in the equatorial plane of the deflecting bar; the deflections are almost half what they are in the first case, and do not admit of the same accuracy; but, when the construction of the apparatus permits, it is useful to make the observations in the two positions, as a control.

It has been assumed that the terrestrial field has remained the same during the observation of the oscillations, and the series of readings of the declinometer; the data of an apparatus for variations observed simultaneously will enable us to reduce all readings to the same period. We ought, lastly, to ascertain whether the suspension-wire has an appreciable torsion, and a correction is introduced for the duration of the oscillations.

1178. Gauss determined the directing couple MH by the method of torsion (**1142**), with a unifilar or bifilar suspension, and measuring the coefficient C for the system of suspension by the ordinary methods. In this case the induced magnetisation does not come in, if the deflection is near 90°.

By using a current with the bifilar and the tangent galvanometer (858), we may arrange the apparatus so as to make the two observations at the same time.

Suppose, for example, that the deflecting bar M is placed above the declinometer, at a distance R, and is supported by a bifilar suspension. When this bar is perpendicular to the meridian, the deflection α produced on the declinometer is

$$\tan\alpha = \frac{M}{HR^3}(1+q),$$

and the couple is determined by the torsion of the bifilar. This arrangement enables us even to regulate with facility the position of the bifilar, for there should be no deflection when the deflecting bar is exactly in the meridian.

This would also be the case if the bar were placed at the same height as the declinometer, either in the meridian or in a plane perpendicular to the meridian; and this latter case would correspond to almost double deflections.

1179. Correction for the Magnetisation induced by the Earth.—When the couple MH is estimated by oscillations, as is usually done, it is necessary to determine experimentally the coefficient ϕ, which corresponds to induced magnetisation.

In strictness, this coefficient should be deduced from an oscillation experiment; and we might work in the following manner. The bar being supported by a bifilar suspension, and in equilibrium in the magnetic meridian, the oscillations of the system give the couple

$$C + HM(1+\phi H) = A_1.$$

The bar is next reversed end-wise; the induced magnetisation is then opposed to the magnetic moment, and, if we assume that it has sensibly the same effect, the fresh oscillations give the couple

$$C - HM(1-\phi H) = A_2.$$

It follows that

$$2HM = A_1 - A_2,$$
$$2C + 2HM\phi H = A_1 + A_2,$$

and consequently

$$\phi H = \frac{A_1 + A_2 - 2C}{A_1 - A_2}.$$

The oscillations of the bifilar in the meridian may again be combined with those which would be obtained in another azimuth. If the bifilar is deflected through an angle θ, near 90°, the director couple A′, determined by the oscillations, is (1147)

$$A' = C\frac{\sin\omega}{\sin\theta} = HM\frac{\sin\omega}{\sin(\omega-\theta)}.$$

We have then

$$A_1 = HM\left[1 + \phi H + \frac{\sin\theta}{\sin(\omega-\theta)}\right],$$

and consequently

$$\frac{A_1}{A'} = (1+\phi H)\frac{\sin\omega-\theta}{\sin\omega} + \frac{\sin\theta}{\sin\omega}.$$

When the angle θ is exactly 90°, it follows that

$$\phi H = \frac{1-\cos\omega}{\cos\omega} - \frac{A_1}{A'}\tan\omega,$$

and the ratio of the director couples A_1 and A' is given by the ratio of the squares of the corresponding numbers of oscillations N_1 and N'.

1180. It is generally assumed that the coefficients f and ϕ are equal, and the true influence of the magnetic moment under the influence of the earth is determined.

Lamont* arranged the magnet on the frame of a magnetic theodolite in a vertical position, so that one of its ends is near the horizontal plane which passes through the needle of the declinometer, and this bar is always in a plane perpendicular to the needle at the moment of observation.

The pole S of the magnet being alternately above and below, the resultant magnetic moment acquires the values $M(1+fZ)$ and $M(1-fZ)$; the corresponding deflections α and α' satisfy equation

$$\frac{1+fZ}{\sin\alpha} = \frac{1-fZ}{\sin\alpha'}.$$

If the induced magnetic moment remains sensibly the same when it is parallel or contrary to the rigid magnetism, then, ob-

* LAMONT. *Handbuch des Erdmagnetismus*, p. 152. 1849.

serving that the difference of the deflections is very small, we deduce

$$fH = \frac{H}{Z}\frac{\sin\alpha - \sin\alpha'}{\sin\alpha + \sin\alpha'} = \frac{\alpha - \alpha'}{2\tan I\tan\alpha}.$$

By a series of reversals the want of symmetry of the magnet may be eliminated.

With a series of magnets Lamont obtained for this term of correction fH very different values, from 0·00053 to 0·00198. We shall subsequently see that the coefficients f and ϕ have not the same meaning.

1181. Joule* has pointed out a method which enables us to eliminate the effect of induced magnetisation in the observation of oscillations. Two nearly identical bars M_1 and M_2 are taken. They are arranged parallel to each other, and at such a distance that the action of one on the centre of the other is sensibly equal to the terrestrial field. Each bar being then in a field which is almost null, only retains its rigid magnetisation. In this case the oscillations of the system only depend on the sum of the moments M_1 and M_2, or of the directive couple

$$(M_1 + M_2)H = A.$$

The oscillations of the two magnets separately give the couples

$$M_1H(1 + \phi H) = A_1,$$
$$M_2H(1 + \phi H) = A_2.$$

We deduce from these last two experiments the ratio

$$\frac{M_1}{M_2} = \frac{A_1}{A_2} = m;$$

consequently

$$M_1H = A\frac{m}{1+m},$$

$$M_2H = A\frac{1}{1+m}.$$

1182. Comparative Deflectors.—For travelling instruments it is unnecessary to know the distance of the deflecting bar from

* Joule. *Proceedings of the Manchester Lit. and Phil. Society*, Vol. vi., p. 129. 1867.—*Scientific Papers*, Vol. i., p. 561.

the magnet deflected, provided that this distance is the same in all observations.

The oscillations of the deflecting bar give a quantity proportional to the directing couple, and we may write

$$MH = C'N^2.$$

The deflection observed by the method of sines, so that the bars are always in the same relative position, also give a quantity proportional to the ratio $\frac{M}{H}$:

$$\frac{M}{H} = C'' \sin \alpha;$$

therefore

$$H^2 = \frac{C'}{C''} \frac{N^2}{\sin \alpha}, \qquad \text{or} \qquad H = C \frac{N}{\sqrt{\sin \alpha}}.$$

The constant factor C is deduced from comparisons with absolute measurements made at a principal station. The further precaution is taken of determining the deflection α by the mean of observations when the deflecting bar is inverted, and of changing side in respect of the deflected needle, in order to eliminate the want of symmetry and of centring.

1183. Dr. Lloyd* applied this method to the dipping needle. Besides the needles A_1 and A_2 which serve for the ordinary observations, two special needles A_3 and A_4 are employed, the magnetisation of which is never reversed. The former A_3 is well balanced, the second A_4 is provided with a suitable counterpoise which puts the centre of gravity outside the axis.

The observation is first made with the last needle in the magnetic meridian. Let M be its magnetic moment, and I′ the apparent inclination; equation (17) gives

$$(26) \qquad pd \cos I' = TM \sin (I - I').$$

This needle is then mounted almost at right angles and in a position which is always the same in a limb which has microscopes or lenses; the needle A_3 is put in its place and its apparent inclination I_1 observed.

* *Admiralty Manual of Scientific Inquiry*, 4th Edition, p. 105. 1871.

The reversal of the frame gives an inclination I_2, corresponding to a deflection in the opposite direction, and the mean of the deflections is

$$\delta = \frac{I - I_1}{2} + \frac{I_2 - I}{2} = \frac{I_2 - I_1}{2}.$$

The mean magnetic field of the deflecting needle on the deflected needle is proportional to its moment—that is, C′M—and makes with the direction of the second needle a constant angle θ; the condition of equilibrium is

$$(27) \qquad C'M \sin\theta = T \sin\delta.$$

Comparing equations (26) and (27), we get

$$T^2 = C'\frac{pd \sin\theta}{\sin\delta}\,\frac{\cos I'}{\sin(I - I')} = \frac{C}{\sin\delta}\,\frac{\cos I'}{\sin(I - I')}.$$

We shall thus have the total force T, if the constant C has been determined by comparison with absolute measures.

1183. Variation Apparatus. — In permanent observatories apparatus are arranged which give variations of the magnetic elements, either by direct observation frequently repeated, or, better, by continuous registration. It will be sufficient if we indicate the general principles.

Gambey's compass for variations of declination consists of a magnetised bar 50 cm. in length suspended by silk threads, and with a divided scale at each end which is read with a microscope. The angular value of the division being determined, observation gives the variation of the declination dD, from a mark representing a known declination.

Long needles have the serious inconvenience that the oscillations are too slow. Rapid changes may thus escape observation, and considerable disturbances produce oscillations of great amplitude which slowly become extinguished. It is better to take short magnets and observe by the method of the mirror.

1184. Several methods may be used for variations of the horizontal component.

The magnet of a declinometer being supported by a wire without torsion, is permanently deflected by an auxiliary magnet M arranged on a fixed post in the vicinity, so that the mean field of this bar

on the declinometer is very nearly perpendicular to the magnetic meridian.

Suppose, generally, that the direction of the field F makes an angle α with the perpendicular to the meridian, and the angle θ with the direction of the deflected magnet. The condition of equilibrium is

$$H \cos(\theta + \alpha) = F \sin\theta.$$

When the horizontal component varies by dH and the declination by $dD = d\alpha$, these variations being very small, the corresponding deflection $d\theta$ of the declinometer satisfies equation

$$\frac{dH}{H} - \tan(\theta + \alpha)\,(d\theta + d\alpha) = \frac{dF}{F} + \cot\theta d\theta.$$

If the defect of adjustment α is very small and the angle θ is near 45°, as is usually the case, we may write

$$\frac{dH}{H} = \frac{dF}{F} + 2d\theta + dD,$$

dD being given by the variation apparatus.

The force F changes with the temperature and with the time. If F_0 is its value at a certain epoch and the temperature 0°, we may represent it at the temperature t, and at the time T, by an expression of the form

$$F = F_0(1 - at)(1 - bT) = F_0(1 - at - bT);$$

it follows that

$$\frac{dH}{H} = -at - bT + 2d\theta + dD. \tag{28}$$

The greatest difficulty is to know the coefficient b of gradual enfeeblement. It is necessary often to compare the indications of the apparatus with the results given by the direct determinations of H.

1185. Another method consists in supporting a bar by a bifilar suspension, with a torsion such that the bar is almost in the transverse position.

Let θ be the angle of the bar with the direction it would take if it were not magnetised, and therefore the torsion of the bifilar,

and let α be the angle of this bar with the perpendicular to the magnetic meridian; the condition of equilibrium is

$$\mathrm{HM}\cos\alpha = \mathrm{C}\sin\theta,$$

and the simultaneous variations of the elements give

$$\frac{d\mathrm{H}}{\mathrm{H}} + \frac{d\mathrm{M}}{\mathrm{M}} - \tan\alpha\, d\alpha = \frac{d\mathrm{C}}{\mathrm{C}} + \cot\theta\, d\theta.$$

If c is the coefficient of variation with the temperature of the bifilar couple, observing that α is very small, we may write

$$(29) \qquad \begin{aligned} \frac{d\mathrm{H}}{\mathrm{H}} &= \cot\theta\, d\theta + ct + at + b\mathrm{T}, \\ &= \cot\theta\, d\theta + (a+c)t + b\mathrm{T}. \end{aligned}$$

This arrangement has the advantage that the variations of declination do not interfere to any sensible extent. The coefficient $(a+c)$ is determined as a whole by the observations themselves, and the value of $\cot\theta$ is given directly in the installation of the bifilar.

1186. To obtain variations of the vertical component, the changes of direction of a magnetised bar are observed which rests by a knife-edge on an agate support, like the beam of a balance, and which is adjusted so as to keep in equilibrium when nearly vertical. The axis of rotation may be perpendicular or parallel to the magnetic meridian.

Let us consider the general case. Let Z and H′ be the vertical and horizontal components of the projection of the terrestrial field on a plane perpendicular to the axis of rotation, I′ the apparent inclination in this plane, θ the angle which the magnetic axis of the bar makes with the horizon, β the angle of the plane perpendicular to the axis with the perpendicular d let fall from the centre of gravity on this axis of rotation, $\mathrm{Q} = pd$ the product of the weight of the bar by the distance d. The equation of equilibrium is

$$\mathrm{ZM}\cos\theta = \mathrm{Q}\sin(\beta+\theta) + \mathrm{H'M}\sin\theta.$$

As the angle θ is very small, the variations give, apart from quantities which may be neglected,

$$\mathrm{M}d\mathrm{Z} + \mathrm{Z}d\mathrm{M} = \mathrm{Q}\cos\beta\, d\theta + \sin\beta\, d\mathrm{Q} + \mathrm{H'M}\, d\theta.$$

Observing that $\tan I' = \frac{Z}{H'}$, and dividing the several terms of this equation by the former, we get, to the same degree of approximation,

$$\frac{dZ}{Z} + \frac{dM}{M} = (\cot\beta + \cot I')\,d\theta + \frac{dQ}{Q},$$

or, if q is the coefficient of variation of the product Q with the temperature,

$$(30) \qquad \frac{dZ}{Z} = (\cot\beta + \cot I')\,d\theta + (a+q)\,t + bT.$$

The principal factor $(\cot\beta + \cot I')$, by which we should multiply the angular variation observed $d\theta$, will be determined by experiment.

If the axis of rotation is perpendicular to the meridian, $I' = I$; if it is parallel to the meridian, $\cot I' = 0$. In this latter case, the formula is more simple; but the action of the earth tends to turn the knife-edge on the planes, and the course of the instrument may be less regular.

In the general case, the couple which tends to bring the bar to its position of equilibrium, when once it is deflected by a small angle δ, is

$$[Q\cos(\beta+\theta) + H'M\cos\theta + ZM\sin\theta]\,\delta,$$

an expression which, when the angle θ is very small, may be reduced to

$$(Q\cos\beta + H'M)\,\delta.$$

If then we make the balance oscillate in the meridian, and then in a plane perpendicular to this meridian, the corresponding numbers of oscillations n and n' give

$$\frac{n^2}{n'^2} = \frac{Q\cos\beta + HM}{Q\cos\beta} = 1 + \frac{HM}{Q\cos\beta}.$$

As we have sensibly

$$ZM = Q\sin\beta,$$

it follows that

$$\cot\beta = \frac{H}{Z}\,\frac{n'^2}{n^2 - n'^2} = \frac{n'^2}{n^2 - n'^2}\cot I.$$

The number N of oscillations of the bar oscillating about a vertical axis will give, in like manner,

$$\frac{N^2}{n'^2}=\frac{HM}{Q\cos\beta};$$

from this follows, as verification,

$$N^2=n'^2-n^2.$$

1187. The best method consists in determining the coefficients of the three instruments of variation by a direct comparison. Apart from the effects of temperature, the variations of the two components are given by expressions such as

$$\frac{dH}{H}=A d\theta, \quad \text{or} \quad \frac{dZ}{Z}=B d\theta.$$

An auxiliary bar M′ is placed successively at the same distance from the three apparatus of variation, in such a manner that its action F is perpendicular to the component H for the declinometer, parallel to H for the bifilar, and vertical for the balance. If the deflector is, for instance, in Gauss' second position in each experiment, the corresponding deflections δ, δ', and δ'' of the three apparatus, supposed to be very small, give

$$F=H\delta=AH\delta'=BZ\delta'';$$

the values of the coefficients are

$$A=\frac{\delta}{\delta'},$$

$$B=\frac{H}{Z}\frac{\delta}{\delta''}=\frac{\delta}{\delta''}\cot I.$$

1188. Variations of inclination may be directly observed on a dip-needle, supported by a knife-edge, and carefully adjusted; but it is generally preferable to calculate the variations of the two principal components.

From the equation

$$Z=H\tan I$$

is deduced

$$(31) \qquad \frac{2}{\sin 2I}\,dI = \frac{dZ}{Z} - \frac{dH}{H}.$$

The values of dI will be given to within a factor by the difference of the relative variations Z and H.

In the same way, for the total force T, equation

$$T^2 = Z^2 + H^2$$

gives

$$(32) \qquad \frac{dT}{T} = \frac{ZdZ + HdH}{T^2} = \frac{dZ}{Z}\sin^2 I + \frac{dH}{H}\cos^2 I.$$

CHAPTER II.

CONSTANT OF MAGNETISATION.

1189. Magnetic Moments.—Oscillations and Deflections. The oscillations of a bar magnet (1140) have already been utilised for estimating the directive couple MH in a uniform magnetic field of intensity H; the moment M will be determined in absolute value if we know the intensity of the field.

If we merely wish to find the ratio of the two magnetic moments M and M′, two successive experiments in the same field give

$$\frac{M}{M'}=\frac{n^2}{n'^2}\frac{K}{K'},$$

when n and n' are the numbers of oscillations in the same time, K and K′ the corresponding moments of inertia.

If the bars have a geometrical shape, so that their radii of gyration may be calculated, and if the weight of the apparatus of suspension may be neglected, then, if P and P′ are the weights of the bars, ρ and ρ' their radii of gyration,

$$\frac{M}{M'}=\frac{n^2}{n'^2}\frac{P}{P'}\frac{\rho^2}{\rho'^2}.$$

The moment of inertia of the system may be eliminated by taking care that it is the same in both cases. The two magnets are suspended, for instance, to the same oscillating system, so that they are parallel, first in one direction and then in the other. The ratio of the squares of the number of oscillations gives the ratio of the sum of their moments $M+M'$ to their difference $M-M'$; from this is deduced the ratio of the two moments M and M′.

We only obtain in this way, in strictness, the ratio of the apparent magnetic moments. In the latter arrangement, in particular, the magnets should be placed on the mounting at such a distance that their reciprocal action may be neglected.

When the bars M and M′, arranged on the common frame, make an angle θ with each other, the resulting magnet R sets in the plane of the magnetic meridian; and if α is the angle which the bar makes with the meridian, we have

$$\frac{R}{\sin\theta}=\frac{M}{\sin(\theta-\alpha)}=\frac{M'}{\sin\alpha}.$$

The value of the directive couple of the system is

$$RH=MH\frac{\sin\theta}{\sin(\theta-\alpha)}=M'H\frac{\sin\theta}{\sin\alpha}.$$

The ratio of the magnetic moments M and M′ is defined by the angles θ and α, for we have

$$\frac{M'}{M}=\frac{\sin\alpha}{\sin(\theta-\alpha)};$$

if the bars are at right angles, we have simply

$$\frac{M'}{M}=\tan\alpha.$$

This method, which was used by Bouty,* enables us to compare rapidly the magnetic moments of the bars.

1190. Methods of Torsion.—The methods of torsion also give the director-couple of a field on a magnet. They even present this special advantage—that, if the deflection is near 90°, the director-couple only depends on the rigid magnetism.

In merely comparing two magnetic moments, and working in the same field, the angles ω and θ, ω' and θ', relative to the magnets M and M′, give, according to the mode of suspension,

$$\frac{M'}{M}=\frac{\omega'-\theta'}{\omega-\theta}\frac{\sin\theta}{\sin\theta'},$$

$$\frac{M'}{M}=\frac{C'}{C}\frac{\sin(\omega'-\theta')}{\sin(\omega-\theta)}\frac{\sin\theta}{\sin\theta'}.$$

With a bifilar suspension, the ratio of the coefficients C and C′ is equal to the ratio of the corresponding weights P and P′ of the total system.

* Bouty. *Annales de l'École Norm.* [2], Vol. IV., p. 9. 1875.

1191. Use of the Balance.—The director-couple of a magnet may be counterpoised by a weight, by means of the ordinary balance. If a bar magnet is fixed vertically to the arm of a balance made of brass, so as to avoid any perturbing action,* and which oscillates in the magnetic meridian, the couple MH tends to incline the arm on one side, and the direction of its action changes when the magnet is reversed end-wise. The balance is counterpoised, in the two cases, by the weights P_1 and P_2. The semi-length of the arm being l, and the weights being expressed in grammes, the product $(P_1 - P_2)gl$ is equivalent to twice the director-couple, and we have

$$(1) \qquad 2MH = (P_1 - P_2)gl.$$

This is a kind of double weighing, in which the induced magnetisation has no appreciable influence, because its action is in the same direction in both cases; the difference of the induced magnetisations alone comes into play.

When the magnet is placed horizontally in the direction of the arm, two observations are made in the same way, turning the magnet end for end, or by turning the support of the balance through 180°. The difference of the tares Q_1 and Q_2 counterbalances twice the couple due to the vertical component, and we have

$$(2) \qquad 2MZ = (Q_1 - Q_2)gl.$$

Dividing equation (1) by equation (2), we shall have the ratio of the components H and Z, and therefore the value of the inclination (1167).

1192. Von Helmholtz† has pointed out another mode of utilising the balance. A vertical bar is suspended from one pan, and on the other a horizontal bar is directed towards the centre of the first. Let M_1 be the magnetic moment of the horizontal bar and $2L_1$ the distance of the poles, M_2 and $2L_2$ the same quantities for the vertical bar, R the distance of the centres, D the distance of the centre of the bar M_1 from the pole of the bar M_2, ω the angle between the directions D and R.

* A. Töpler. *Wiedemann's Annalen*, Vol. xxi., p. 158. 1884.

† Von Helmholtz. *Ber. der Akad. der Wissens. zu Berlin*, Vol. xvi. 1883.

The principal term of the expression for the vertical component of the action of the magnet M_1 on the pole $+m_2$ of the bar M_2 is (1146)

$$Ym_2 = 3\frac{M_1 m_2}{D^3}\cos\omega\sin\omega = 3\frac{M_1 m_2 L_2}{D^4}\frac{R}{D}.$$

The value of the vertical action on the pole $-m_2$ is the same; for the signs of m and of Y must be changed simultaneously. If $2m_2L_2$ is replaced by M_2, and the distance D by $\sqrt{R^2+L_2^2}$, we see that $3\frac{M_1M_2}{R^4}$ is the principal term of the resultant action.

The action of the magnet M_2 on the magnet M_1 is equal and opposite, but it takes place on the other side, and tends to incline the beam in the same direction.

The action again changes sign when one of the bars is inverted. If $P_{1\cdot2}$ is the difference of the tares necessary for restoring equilibrium in the two cases, we may write

$$P_{1\cdot2}\,l = 12\frac{M_1M_2}{R^4}(1+p),$$

and we should determine the correction p by working at two different distances. All defects of symmetry are eliminated by reversing the magnets and interchanging the pans. Here again only the differences of the magnetisations by the earth comes into play.

With three different bars, experiment will give the products M_1M_2, M_1M_3, and M_2M_3. Apart from the term of correction, we get the equations

$$\frac{M_1}{M_2} = \frac{P_{1\cdot3}}{P_{2\cdot3}}, \qquad \frac{M_1}{M_3} = \frac{P_{1\cdot2}}{P_{2\cdot3}}, \qquad \frac{M_2}{M_3} = \frac{P_{1\cdot2}}{P_{1\cdot3}},$$

which also give the moment of each bar; for we have

$$M_1^2 = \frac{(M_1M_2)(M_1M_3)}{M_2M_3} = \frac{R^4 l}{12}\frac{P_{1\cdot2}P_{1\cdot3}}{P_{2\cdot3}}.$$

1193. Measurement of the Moment by the Field.—In this experiment of Von Helmholtz the magnetic moment of a bar is determined by the measurement of its field at a certain distance.

This field may be evaluated by any other method—for instance, by oscillations or deflections.

If the magnet is placed so that its field on a magnet is parallel to the terrestrial field, the oscillations of the needle (1140) are determined in the terrestrial field alone, then in a field formed by the sum or difference of the terrestrial field and of the field of the magnet.

We may also work by deflection, by making the magnet act on a declinometer placed in a transverse position by a bifilar suspension (1142), or by a combination of oscillations and deflections.

We shall thus determine either the ratio of the field of a magnet to the terrestrial field or the ratio of the fields of two magnets. Allowing for the distance, and the term of correction if there is one, we shall deduce from it either the ratio of the magnetic moment M_1 to the terrestrial field or the ratio of the two magnetic moments M_1 and M_2 by the ratio of the corresponding fields F_1 and F_2.

If the field of the magnet is perpendicular to the magnetic meridian, it is made in like manner to act on a needle situated just in the meridian, and the deflection is measured as in Gauss' method (1174). We may also observe the oscillations of a declinometer situate in a transverse position, and subject first to the resultant field of the earth and the bifilar, then to the sum or the difference of the actions of this field and of a magnet.

1194. METHOD OF INDUCTION.—Suppose that a magnetised body of any given form is surrounded by a closed circuit in the plane perpendicular to the x axis. As the component of the magnetic induction perpendicular to this plane is X_1, the flow of induction across the circuit is (324)

$$Q = \int\int X_1 dydz = -\int\int \frac{\partial V}{\partial x} dydz + 4\pi \int\int A dydz,$$

the integral being extended to the entire face S of the circuit.

If the magnetisation is suddenly suppressed, the circuit will be the seat of an induced discharge which will give the value of Q. The discharge will be the same if the circuit is carried to a great distance from the magnet, or in a position for which the flow across it is null.

When the circuit closely encloses the surface of the magnet, the discharge gives the internal flow of induction in the corresponding section.

With a cylindrical magnet the value of Q for different points will enable us to construct what Gaugain has called the *curve of demagnetisation* (417). When the circuit is displaced between two positions x_1 and x_2, the induced discharge measures the variation $Q_1 - Q_2$ of the internal flow of induction from one section to another—that is to say, the flow of force which emerges from the bar (324) by the lateral surface between the two corresponding sections.

1195. Let us suppose now that the circuit S forms part of a cylindrical coil which has n_1 spires for unit length. The flow of induction across the spires comprised within a length dx is

$$dQ = -n_1 dx \int\int \frac{\partial V}{\partial x} dydz + 4\pi n_1 dx \int\int A dydz.$$

The ends of the coil being in the planes x_1 and x_2, the total flow of induction which traverses it, is the integral of this expression between the limits x_1 and x_2. If V_1 and V_2 are the potentials in the bounding planes and dv an element of volume of the magnet, we may write

$$Q = n_1 \int\int V_1 dydz - n \int\int V_2 dydz + 4\pi n_1 \int_1^2 A dv.$$

If the extremities of the coil are so distant on either side of the magnet that the potential is sensibly null, we get simply

$$Q = 4\pi n_1 \int A dv = 4\pi n_1 M_x,$$

M_x denoting the projection on the x axis of the magnetic moment of the body.

If the magnet is removed, the discharge induced in the circuit of the coil will enable us to determine the moment M_x.

More generally, we know (338) that the potential energy of a magnet M in a uniform field F with which it makes the angle θ is $-MF \cos \theta$. If a frame of any shape whatever, traversed by a current I, produces a uniform field GI, the potential energy of the magnet placed in this field will be $-MGI \cos \theta$, and the work WI necessary to bring the magnet in another direction θ' will be

$$WI = MGI (\cos \theta - \cos \theta').$$

But the product of the induced discharge by the resistance R of a circuit (515) is equal to the work W which corresponds to unit current. We have then

$$Rq = MG(\cos\theta - \cos\theta').$$

If the circuit is connected only with a ballistic galvanometer and the magnet is suddenly carried to a great distance, the corresponding discharge q will give the product MG cos θ—that is to say, the product of the constant G of the frame by the component $M_x = M\cos\theta$ of the magnetic moment parallel to the field.

If the magnet were first parallel to the field, a rotation of 90° would give the product MG, and a reversal end for end the value 2MG.

We may use in this way either a long cylindrical coil or a spherical coil (497), or any given system of frames with a uniform field (750 and 751).

Measuring by this method the components M_x, M_y, and M_z of the magnetic moment of the body in respect of three rectangular axes, we shall arrive at the resultant moment

$$M^2 = M_x^2 + M_y^2 + M_z^2,$$

and the angles α, β, γ of the direction of this moment with the axes will be defined by the equations

$$\frac{\cos\alpha}{M_x} = \frac{\cos\beta}{M_y} = \frac{\cos\gamma}{M_z} = \frac{1}{M}.$$

1196. Intensity of Magnetisation.—The quotient of the magnetic moment of a body by its volume gives the mean intensity of magnetisation.

From the experiments of Gauss,* for instance, the magnetic moment of the earth in C.G.S. units is

$$0{\cdot}33092\, R^3 = 8{\cdot}55 . 10^{25}.$$

In other words, the mean value of the horizontal component of the terrestrial field at the magnetic equator being about 0·33092,

* Gauss. *Allgem. Theorie des Erdmagn.* *Œuvres*, Vol. v., p. 164.

the intensity of magnetisation of the earth, regarded as a sphere uniformly magnetised, is equal to the fraction $\frac{3}{4\pi}$ of its action at the equator (355), which gives

$$I_a = \frac{3}{4\pi} 0{\cdot}33092 = 0{\cdot}079.$$

Gauss found in like manner that the magnetic moment of a steel bar weighing a pound was 10087·7, his numbers being reduced to C.G.S. units. Assuming that the density of the steel is 7·8, and that the pound in question is 453·6 grammes, the mean intensity of magnetisation will be 174, or 2200 times the terrestrial magnetisation.

Magnets such as are used in laboratories or for observations on terrestrial magnetism, have often a mean magnetisation of 200 to 400 units. The maximum magnetisation, however, which a magnet can acquire depends not only on the nature of the steel and its mode of tempering, but also on the dimensions of the bar. It is greater as the bar has the form of a longer and thinner cylinder.

Magnetisation, in fact, is especially localised in the superficial layers, either because the methods of magnetisation used do not readily act on the internal parts or because the effects of the tempering have not themselves penetrated to a great depth. Further, the reaction of a magnet on itself, or the demagnetising force (407), is less as the bar is longer in the direction of magnetisation.

By working on very thin rods, Kohlrausch* has observed that the magnetic moment of steel may attain 100 units per gramme, which would give a strength of magnetisation of 780, or nearly 10,000 times that of the earth.

1197. Method of Detachment.—The intensity of magnetisation may, in certain cases, be determined directly, without having recourse to the measurement of a magnetic moment.

Let us assume that, in a cylinder uniformly magnetised, a section S has been cut through at right angles to the axis, and that the surfaces have been brought in contact. The magnetic density on the surfaces is equal to the intensity of magnetisation I_a (322); the action for unit surface is equal to $2\pi I_a^2$, as for electrified surfaces (41); and the total attraction of the two surfaces is equal

* Kohlrausch. *Leitfaden der Prakt. Physik,* Fourth Edition, p. 174. 1880.

to $2\pi I_a^2 S$. By determining the weight P necessary to tear them asunder, or to prevent them from touching, we have

$$2\pi I_a^2 S = gP, \qquad \text{or} \qquad I_a = \sqrt{\frac{gP}{2\pi S}}.$$

If the surface S made an angle θ with the direction of the magnetisation, we should have

$$gP = 2\pi I_a^2 S \sin\theta.$$

1198. METHOD OF INDUCTION. — The flow of induction Q (**1194**), which traverses a circuit surrounding a magnet, may be written

$$Q = FS + 4\pi AS',$$

denoting by F the mean force on the surface S of the circuit, and by A the mean component of magnetisation in the corresponding section S′ of the magnet. The value of Q would be obtained either by the discharge due to the sudden suppression, or the establishment of the magnetisation, or by that produced when the circuit is transferred to a great distance, or to a position for which the flow of induction which traverses it is null.

The magnetising force F is due solely to the free magnetism of the magnet, or to the free ends of the magnetic filaments which compose it. The value of the term FS is a minimum when the circuit tightly encloses the magnet. If this is in the shape of a very elongated cylinder, and the circuit is placed near the median region, the force F is virtually null, and we have

$$Q = 4\pi AS'.$$

This equation is strictly rigorous even for an annular magnet forming a closed solenoid (**371**).

1199. The sudden changes of magnetism necessary for the application of this method can only be easily produced by currents.

Suppose, for example, that a cylindrical bar of section S′, and of great length in comparison with its diameter, is placed in the uniform field produced by a current I; for example, the field $4\pi n_1 I$ of a cylindrical coil (**495**) which is very long compared with the bar, or more generally in the field GI of a frame of uniform field. If the action of the ends of the bar itself in the central section of the

bar may be neglected, the intensity of the magnetisation of the bar being I_a, the value of the flow of induction across a closed circuit S, which surrounds the central section, is

$$Q_1 = GIS + 4\pi I_a S'.$$

Determining this flow by the induced discharge, and then repeating the same experiment with the coil alone, which will give $Q_2 = GIS$, we get, by difference,

$$Q_1 - Q_2 = 4\pi I_a S'.$$

If the bar forms a ring, and is surrounded by a ring-shaped coil (754), the internal field is sensibly uniform when the section of the bar is small in comparison with the diameter of the ring; experiment still gives in all cases the mean magnetisation.

With rings there are only variations of magnetisation, and the experiment says nothing as to the residual magnetism of the system, and therefore on its total magnetisation.

1200. Induced Magnetisation. — The quantities which we should endeavour to determine experimentally are, the coefficient of magnetisation k, which Sir W. Thompson has called its *magnetic susceptibility* (383), and the coefficient of induction $\mu = 1 + 4\pi k$, or the *magnetic permeability*.

These qualities vary with the magnetising force, and the problem can only be completely solved in particular cases.

The magnetising force results from the primitive field and from the induced magnetisation (385), and consequently it varies from one point to another. As the real state of a magnet cannot be deduced from its external actions, we must confine ourselves to the case in which, the field being uniform, the induced magnetism itself gives either a uniform field, or one which may be neglected.

In a uniform field of intensity ϕ, if the field of the induced magnetism is also uniform, the magnetisation I_a of the body is connected generally with the intensity of the field by an expression of the form

$$I_a = \frac{k}{1 - kC}\phi.$$

The value of the constant C is equal to $\frac{4}{3}\pi$ for a sphere, or for an ellipsoid, to one of the coefficients $-L$, $-M$, $-N$ the corresponding axis of which is parallel to the field (388). We may add

the case of a cylinder of great length, for which $C = 0$ or $C = -2\pi$, according as its axis is parallel or perpendicular to the field. Lastly, we have $C = 0$, for a ring surrounded by an annular coil.

The forms which present most advantages in practice are then: a sphere, an elongated ellipsoid of revolution, which may be compared with an ellipsoid of the same cross section and the same length, or which may better be considered infinitely long; and lastly, for a circular ring.

Experiment shows that the coefficient k, for a body of a given kind—iron, for instance—depends not only on its actual condition, its purity, temperature, temper, but also on the successive stages through which it has passed—that is to say, on its previous magnetic history.

Any modification in the magnetic condition of a body, even if apparently transient, should be considered as having produced a permanent alteration in its constitution, and this alteration can only disappear if the body is raised to a red heat. On the other hand, such an operation modifies the chemical or physical condition, so that nothing is more difficult than to obtain specimens of the same substance which may be considered as being perfectly identical.

The uniform fields which are ordinarily used are—the internal field of a coil, which is long in comparison with the dimensions of the body; the field of a system of frames, conveniently arranged; the field produced by a system of magnets; or simply the terrestrial field.

The preceding remark shows, further, that the manner in which the body is brought into the field is not a matter of indifference. If the body is placed in a coil, and the current is suddenly made or broken, we may have effects of induction which, at a given moment, give a value quite different from its permanent value, especially when the coil contains a magnetic core; in a field which is thus variable the body undergoes a sort of shock, which modifies the final condition.

Thus, when a bar of soft iron is magnetised in a coil, and is slowly removed, its residual magnetism may be almost twice that which would result from the sudden suppression of the current.

The method of induced discharges can thus only be employed with certain precautions. If we introduce the body into the coil when the current is already made, it passes through a variable field before attaining the uniform part. The best plan would be to introduce the body in the coil, and restrict ourselves to making and breaking the current in a slow and progressive manner.

1201. The field of a coil being proportional, by a known factor G, which depends on the form of the circuit, to the intensity of the current, the problem resolves itself into determining the intensity of magnetisation as a function of that of the current. We measure, in that case, either the intensity of magnetisation of the body by detachment or by induction, or its magnetic moment by a method of deflection.

In this latter case, if the body is placed in the field of a coil, the magnetic moment of the current itself should be determined, and deducted from the result obtained for the system of the coil and the magnetised body. It is then better to eliminate the action of the magnetising coil on the declinometer, compensating it by that of another coil placed on the opposite side, at a suitable distance, and which is traversed by the same current.

In like manner, if we wish to compare bodies of the same dimensions as to their magnetic properties—for instance, steel bars of different kinds, submitted to different operations of tempering or of annealing—it is advantageous to work by a method of reduction to zero. A typical bar being arranged in a fixed post on one side of a declinometer, the bar to be examined is placed on the opposite side, and its distance modified until the needle is restored to its original position. The ratio of the magnetic moments of two bars is equal, within a correction, to the cube of the inverse ratio of the distances for which they counterbalance the typical bar.

1202. With ring-shaped coils the magnetising force is not constant throughout the entire extent of the section S of the coil, and this is also the case with the coefficient k; but the variations of this factor may generally be neglected, and we may consider the internal field to have a constant intensity $F'_m I$ (**754**), the mean value F'_m referring to the mean section S' of the ring. In these conditions the intensity of magnetisation is equal to $kF'_m I$, and the corresponding flow of induction to $4\pi kF'_m S'$.

The total flow of induction across a closed circuit surrounding the ring p times is then

$$Q = pI\,(F'_m S + 4\pi kF'_m S');$$

and, if the wire is close to the ring,

$$Q = pI\,(1 + 4\pi k)\,F_m S = \mu p F_m SI;$$

the determination of Q by an induced discharge will give the value of μ.

As the method of rings only gives variations of the magnetic state, we only know the temporary magnetism by induced discharges which correspond to the making or breaking of the principal current, and the residual magnetism is determined by the difference of discharges in the making and breaking of the current.

The ring is thus in each experiment in a state which is imperfectly known. We should limit ourselves to only employing increasing currents, and the apparatus is out of use when the strongest currents have passed.

1203. PROPERTIES OF IRON.—The first numerical results on the magnetisation of soft iron by currents are due to Lenz and Jacobi.* They used the method of induction with cylindrical bars, and concluded from their experiments that the magnetisation is proportional to the intensity of the current. About the same time Joule† demonstrated the capital fact of the existence of a maximum. He worked by detachment, ascertaining what weight was necessary to separate two cylinders placed end to end and magnetised by a current, or two segments of a ring along a plane parallel to the axis. Joule observed that the weight commences to vary proportionally with the square of the intensity, that is to say, the magnetising force, in conformity with the law of Lenz and Jacobi, but then much less rapidly, and tends lastly towards a maximum. In the following table, taken from his memoir, the first four numbers relate to his own experiments; the three following are deduced from the previous experiments of Nesbit, of Henry, and of Sturgeon:

Section of Bars in Square Centimetres.	Maximum Weight in Grammes per Square Centimetre.	Maximum Magnetisation I_s.	Field $4\pi I_s$.
64·5	19,478	1742	21,880
1·263	17,570	1656	20,810
0·281	19,337	1740	21,860
0·0087	11,391	1334	16,750
29·02	22,291	1862	23,390
25·40	13,360	1442	18,120
1·26	17,931	1671	20,990

The intensity of magnetisation will then be represented as a function of the intensity of the current or of the magnetising force

* LENZ and JACOBI. *Pogg. Ann.*, Vol. XLVIII., p. 225. 1839.

† JOULE. *Annals of Electricity*, Vol. IV., p. 474, 1839; Vol. V., p. 187, 1840. —*Scientific Papers*, Vol. I., pp. 15–38.

by a curve (Fig. 244), composed of a right line OA and another part concave towards the axis of the abscissæ having as asymptote a horizontal line CD which would correspond to an intensity of magnetisation of 1700 or 1800.

Numbers of the same order have been obtained by different observers for the maximum of magnetisation:

Von Waltenhofen*	1670
Stefan†	1400
Fromme‡	1731

The maximum of magnetisation is a direct consequence of the hypothesis of Ampère, which consists in regarding particular currents as pre-existing in soft iron, and the act of magnetisation as a simple direction of the primitive currents.§

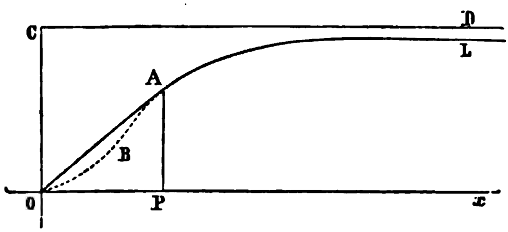


Fig. 244.

This hypothesis is confirmed by an ingenious experiment of Beetz.‖ A silver wire, which is coated with varnish except along a very fine line parallel to the axis, is taken as negative electrode in a bath of iron in a magnetic field parallel to its length. The thin film of iron obtained in these conditions has a very considerable permanent magnetisation, which is greater as the wire is finer, and therefore the reaction of the contiguous particles is less felt.

* Von WALTENHOFEN. *Pogg. Ann.*, Vol. CXXVII., p. 518. 1869.

† STEFAN. *Wiener Berichte* [2], Vol. LXIX., p. 200. 1874.

‡ FROMME. *Wied. Ann.*, Vol. XIII., p. 695. 1881.

§ AMPÈRE. *Réponse à Van Beck.* 1822.—*Collect. de Mém. de la Société Française de Phys.*, Vol. II., p. 214.

‖ BEETZ. *Pogg. Ann.*, Vol. CXI., p. 107. 1860.

Some experiments of Joule,* although limited to permanent magnetisation, show that for very small forces the magnetisation increases more rapidly than the magnetising force.

This result has been neatly demonstrated by the experiments of Professor G. Wiedemann† on the magnetisation of cylindrical bars by currents, by using the methods of deflection and determining separately the total magnetisation and the residual magnetisation, from which the temporary magnetisation is deduced by difference. It follows that the curve of the intensity of magnetisation (Fig. 244) has no rectilinear part (OA): it begins to turn its convexity towards the axis of abscissæ, has a point of inflection at B, and then tends towards a maximum. The curve of total and that of permanent magnetisation have the same features. The intensity of magnetisation begins then to increase more rapidly than the magnetising force.

1204. Variation of Magnetisation with the Magnetising Force.—Several methods may be used to express the results of experiments. The method which most directly answers to practical requirements consists in expressing the relation between the magnetic moment, or the intensity of magnetisation, and the magnetising force by a curve or by a formula.

Lamont‡ obtained a tolerably rational expression by assuming that the increase dm of the magnetic moment, produced by an increase $d\phi$ of the magnetising force, is at each instant proportional to the excess of the maximum moment M over the actual value m—that is to say, to the total increase of magnetisation which the bar is still capable of receiving. If we put

$$\frac{dm}{d\phi} = a\,(\mathrm{M} - m), \qquad \text{or} \qquad \frac{dm}{\mathrm{M} - m} = a\,d\phi,$$

it follows that

$$\mathrm{M} - m = \mathrm{A}e^{-a\phi};$$

as $\mathrm{M} = \mathrm{A}$ for $\phi = 0$, we get

$$\frac{m}{\mathrm{M}} = 1 - e^{-a\phi}.$$

The curve which gives the ratio $\frac{m}{\mathrm{M}}$, as a function of the magnetising force ϕ, starts from the origin, where the angular coefficient

* Joule. *Phil. Trans. for 1856*, p. 287.

† Wiedemann. *Galvanismus*, 1st Edition, Vol. II., p. 297.

‡ Lamont. *Handbuch des Magnetismus*, p. 407. 1867.

of the tangent is a, and it is an asymptote to the right line $y = 1$. We get thus at this general property that the magnetic moment m is first proportional to the magnetising force and tends towards a maximum. The intensity of magnetisation is proportional to m, and may be represented by $b(1 - e^{-a\phi})$; it follows that

$$k = \frac{1 - e^{-a\phi}}{b\phi}.$$

The initial value of the coefficient of magnetisation k will be equal to the ratio $\frac{a}{b}$, and will diminish to zero.

Lamont's formula represents very exactly the phenomena observed by Joule, and generally those which correspond to magnetising forces so great that we have exceeded the first elements of the curve of inverse curvature.

Instead of using an experimental formula, which has the inconvenience of greatly complicating calculations in practice, the magnetic moment may be represented, as Fröhlich has shown,* by the branch of a hyperbola under the condition of having the same tangent at the origin and the same asymptote as Lamont's exponential curve. We shall have then

$$\frac{m}{M} = \frac{a\phi}{1 + a\phi}, \qquad \text{or} \qquad k = \frac{a\phi}{b\phi(1 + a\phi)}.$$

Müller† found that for a soft iron bar of diameter d, the magnetic moment M corresponding to a field ϕ is pretty exactly represented by an empirical expression of the form

$$M = Cd^2 \text{ arc tan } \frac{\phi}{Ad^{\frac{3}{2}}},$$

in which A and C are constants; but this formula corresponds solely to the particular case of a bar magnetised by a much shorter coil placed at the centre. It has not then a general character, but it also indicates a maximum of magnetisation; and the narrower the bar, the more rapidly is this maximum attained.

* FRÖHLICH. *Electrotechnixte Zeitschrift*, Vol. II., p. 134. 1881.

† MÜLLER. *Pogg. Ann.*, Vol. LXXIX., p. 337, 1850; Vol. LXXXII., p. 181, 1851.

Weber* made a series of experiments with cylindrical bars and by the method of deflections which have been calculated by Kirchhoff.† For magnetising forces which first increase and then decrease, they give the following values for the coefficient of magnetisation k:

ϕ	k	ϕ	k
29·6	25·0	248·4	5·6
30·1	23·5	197·5	6·7
82·3	13·5	158·3	8·1
118·4	10·2	129·7	9·5
151·2	8·4	96·7	12·0
208·0	6·4	61·2	16·9
239·7	5·7	29·2	25·0.

The values of k, or of the magnetisation $k\phi$, are a little greater when the magnetising forces diminish; but the results agree in both cases either with the formula of Lamont or with that of Fröhlich.

Thalen‡ determined by a method of induction and under the mere action of the earth the magnetisation of cylindrical iron rods 2 cm. to 4 cm. in diameter and 40 cm. in length, which he placed in a vertical position and then suddenly inverted. With various specimens of annealed Swedish iron, the coefficient k varied from 27 to 45.

Analogous experiments made by Riecke,§ with ellipsoids of various eccentricities, and magnetising forces of 0·032 to 0·072, gave values of 13·5 to 25·4 for the coefficient k.

Quintus Icilius‖ used two ellipsoids the length of which was 101 or 165 times the diameter. He worked by the method of deflection for rather large magnetising forces, and by induction for feeble forces. The two series of observations were connected by an experiment made in the same conditions and simultaneously by the two methods.

* WEBER. *Electrisische Maasbestimmungen, Diamagnetismus.—Abhand. der König. Götting. Gesellschaft*, Vol. I., p. 170. 1852.

† KIRCHHOFF. *Crell's Journal*, Vol. XLVIII. 1853.—*Gesamm. Abhandlungen*, p. 193.

‡ R. THALEN. *Nova Acta Reg. Soc. Upsal*, 3rd Series, Vol. IV. 1863.

§ RIECKE. *Pogg. Ann.*, Vol. CXLI., p. 543. 1870.

‖ QUINTUS ICILIUS. *Pogg. Ann.*, Vol. CXXI., p. 125. 1864.

These results, calculated by M. Stoletow,* lead to values for the coefficient of magnetisation which first increase and attain a maximum of 110 or 120, corresponding to magnetising forces of 4 to 5 units.

Stoletow himself used a ring of rectangular section; he found values increasing from 21·5 to 174·2; the maximum takes place for a field equal to 3·21. The curve I (Fig. 245) represents the

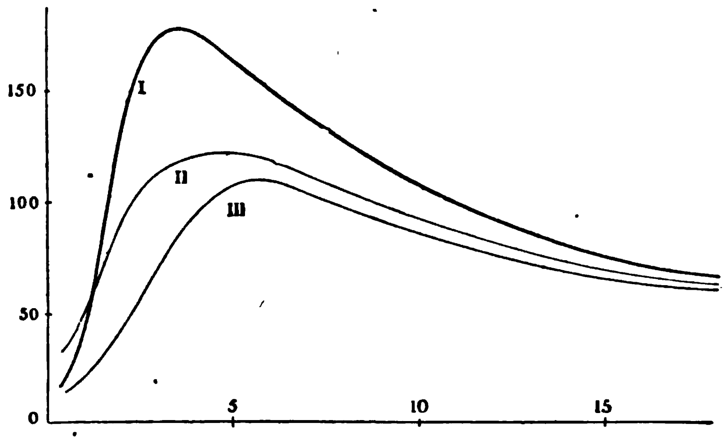


Fig. 245.

coefficient k as a function of the field according to Stoletow; the curves II and III refer to the experiments of Quintus Icilius.

Professor Rowland† has investigated, by the same method, several kinds of iron and steel in the form of rings with a circular section. Curves I, II, III (Fig. 246) represent the total magnetisation, the temporary magnetisation, and the permanent magnetisation for good Norwegian iron. They show that the increase of magnetisation is at first very rapid for small forces; that the magnetisation is at first simply temporary (and this effect is still more marked for steel than for iron); lastly, that the maximum is attained more rapidly for the permanent than for the total magnetisation.

1205. Instead of representing the intensity of magnetisation, or the coefficient k, as a function of the magnetising force, Rowland uses a mode of representation which seems far preferable;

* STOLETOW. *Pogg. Ann.*, Vol. CXLVI., p. 442. 1872.

† ROWLAND. *Phil. Mag.* [4], Vol. XLVI., p. 140. 1878.

for it enables us to obtain for all degrees of magnetisation very regular finite curves, by the aid of which we may determine with more precision the value of the maximum. It consists in taking as abscissa the intensity of magnetisation I_a, or the magnetic induction $F_1 = \phi + 4\pi I_a = \phi(1 + 4\pi k) = \mu\phi$, and as ordinates the magnetic permeability μ, or the quantity $\lambda = 4\pi\mu$.

The curves thus obtained (Fig. 247) have the form of inclined parabolas, and are well represented by the formula

$$\lambda = A \sin \frac{F_1 + a\lambda + b}{c},$$

in which a, b, and c are constants depending on the metal observed, and A is the maximum value of λ.

The figure refers to a series of observations on wrought iron of good quality. The constant A is equal to 30860, which gives

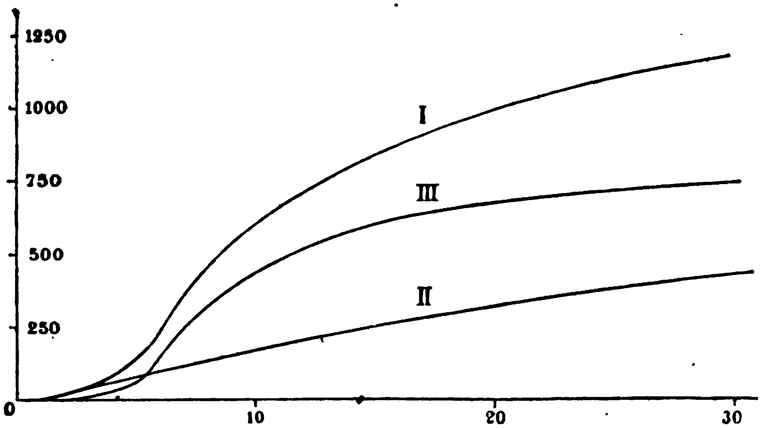


Fig. 246.

$\mu = 2456$, or $k = 195$, for $F_1 = 5968$ or $= 2\cdot43$. The point where the curve meets the x axis, corresponds to the maximum of magnetisation, which will be 1400 for $F_1 = 17500$.

The Norwegian iron already mentioned gave $A = 57820$, or $\mu = 4602$, and $k = 366$, for $F_1 = 5380$ or $\phi = 1\cdot169$.

1206. It is worthy of notice that these high values of the coefficient of magnetisation were obtained by the method of induction applied to ring-shaped pieces of soft iron, while the direct measurement of magnetic moments by deflections does not give

numbers of the same order. Such differences may be partially due to defects of the method of observation, but they are principally due to the demagnetising action of cylindrical rods. This action only becomes negligable provided the ratio of the length of the cylinder to its diameter is extremely great; it is recognised if the two methods are simultaneously used.

Iron rods magnetised by a long cylindrical coil are at the same time surrounded in the centre by a short coil of p spires, which communicates with a ballistic galvanometer. The magnetic moment M is determined by the deflection of a declinometer, and

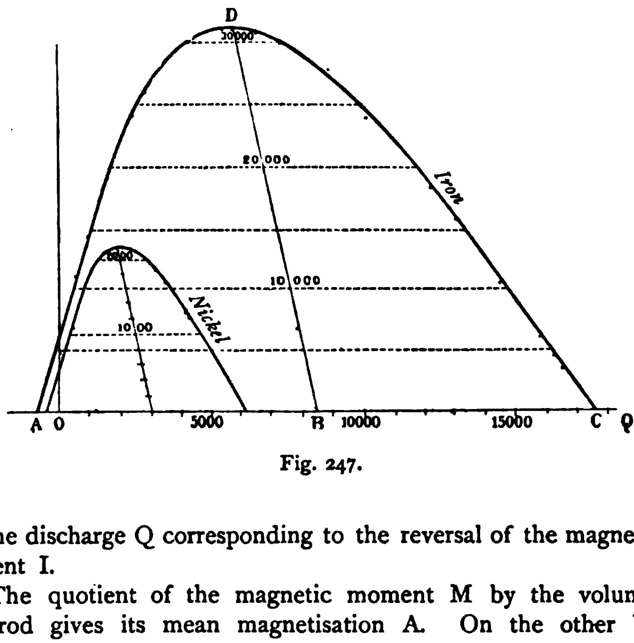


Fig. 247.

by the discharge Q corresponding to the reversal of the magnetising current I.

The quotient of the magnetic moment M by the volume of the rod gives its mean magnetisation A. On the other hand, if A_1 is the magnetisation of the median section, F_1 the magnetising action, and S the section of the rod, the discharge Q gives the product

$$2ps\,(4\pi A_1 - F_1) = 8p\pi S\left(A_1 - \frac{F_1}{4\pi}\right) = 8p\pi SA'.$$

The ratios of the quantities A and A′ to the magnetising force $4\pi n_1 I$, give two coefficients f and f' which have not the same meaning, but should be equal to each other and to the coefficient

of magnetisation k, if the ratio of the diameter of the rod to its length is so small that the magnetisation may be considered uniform.

Experiment shows at first that in all circumstances $f' > f$. These coefficients, while very different for short cylinders, become gradually nearer, and approach unity as the rod is longer. In the second place, the field in which the maximum of f or of f' is produced is stronger the shorter is the rod.

With iron wires of the same kind, when the length is from 40 to 600 times the diameter, the maximum value of the coefficients f or f' varied from 25 to 190, or from 40 to 220, while the corresponding fields diminished from 20 C.G.S. units to 3 units. The same wire used as rings gave for the coefficient k a maximum of about 200 with a field of 3 units. The agreement of these results may be considered as sufficient if we observe how difficult it is to obtain perfect identity of the specimens on which we work.

We may then investigate the magnetic properties of iron when formed into cylindrical rods, provided the length is at least 500 times the diameter. We have this advantage, that we can know at any moment the magnetic condition of the metal, and that it is easy to demagnetise it so as to subject it to fresh tests.

1207. Nickel and Cobalt.—With nickel and cobalt, Rowland observed the same phenomena as for iron except as regards the intensity. The curves of permeability as a function of the intensity of magnetisation have the same aspect (Fig. 247). At the ordinary temperature, the maximum of magnetisation is about 500 for nickel and 800 for cobalt. The maximum permeability is about 10 times as great for soft cobalt as for tempered cobalt.

Berson,* finally, has found that rolled nickel is more coercitive than fused nickel: the permanent magnetisation is then superior to the temporary magnetisation.

1208. Influence of Temperature.—The experiments of Professor Rowland, confirmed by those of Trowbridge and McRae,† have shown that the permeability given is virtually independent of the temperature between 0° and 280°. Berson verified this property as far as 330°. For higher temperatures the permeability should diminish, for the metal is no longer magnetic at a cherry-red heat.

* Berson. *Ann. de Chim. et de Phys.* [6], Vol. viii., p. 432. 1886.

† J. Trowbridge and McRae. *Proceedings of the American Academy of Arts and Sciences, May 13, 1885.*

Tempering has a much greater importance than any other physical property. While the maximum of μ is 4600 for iron heated to redness and then cooled slowly, this value is only 2250 for tempered metal, and rises to 6260 when it is reheated after tempering.

For nickel and cobalt, Rowland* found the curious result that at a temperature of 220° the curve of permeability becomes much closer, with a higher maximum; in other words, the variations of

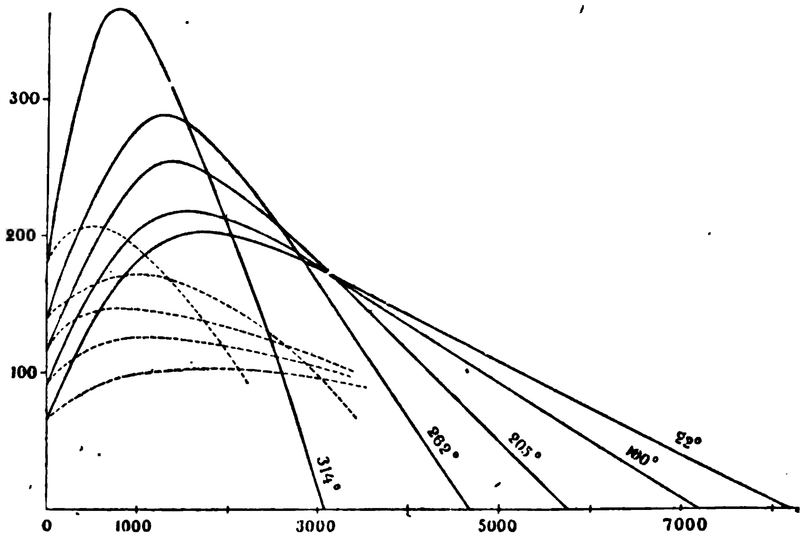


Fig. 248.

permeability are more rapid and the maximum intensity of magnetisation diminished.

The experiments on nickel have been completed and extended by Mr. Perkins.† The curves of Fig. 248 represent the results for several different temperatures. The dotted curves hold for temporary magnetism; they all start from the same origin as the corresponding curves of the total magnetisation—in other words, that, as for iron, the magnetism developed by weak magnetising forces is purely temporary.

* ROWLAND. *Phil. Mag.* [4], Vol. XLVIII., p. 321. 1874.

† PERKINS. *American Journal of Science*, Vol. XXX., p. 218. 1885.

According to M. Berson, the temporary magnetisation of nickel increases from 0° to 250° or 260°, then rapidly diminishes, especially from 280°, and disappears towards 340°.

The magnetic moment of a nickel rod, magnetised in the cold, becomes gradually weaker as the temperature rises, and is null towards 330°, as for steel; but it has this curious feature, that if we magnetise it at a high temperature of 200° to 290°, the residual magnetisation increases first during cooling, then slightly diminishes, always remaining at the ordinary temperature higher than what it was at the temperature of magnetisation.

For cobalt, the temporary and residual magnetisations increase steadily with the temperature between 25° and 325°. Contrary to what is the case for nickel, the residual magnetic moment of cobalt always diminishes, when we get away from the temperature of magnetisation, in one direction or the other.

These results agree with the older experiments, in which Faraday* had observed that, at the temperature at which olive oil begins to decompose, the magnetic properties of nickel have scarcely diminished, while, on the contrary, those of cobalt have increased.

1209. Steel and Cast-Iron.—The results for steel are less concordant, the complexity of the phenomena being greater. When the steel is magnetised for the first time, the composition of the metal, the degree and the duration of the temper, the temperature, and the duration of the annealing, the shape and dimensions of the bar, etc., must be taken into account. We may add also, that in bars which are somewhat large, the material is necessarily heterogeneous, as the effects of tempering are not equally felt in the various layers. Lastly, after several successive magnetisations, in the same or in opposite directions, the residual magnetism may have a very irregular distribution, and may even be formed of several successive layers, with magnetisations in opposite directions.

We owe to MM. Barus and Strouhal† numerous experiments on steel bars, always of small diameter, so as to obtain a more homogeneous material.

The different kinds of iron, soft iron, steel, cast-iron, differ greatly in conducting power; the specific resistance at 0°, which is equal to about 10 microhms for soft iron, rises to almost 100 for hard cast-iron, and varies from 47 to 15 for steel, according to the degree of

* Faraday. *Experimental Researches*, xxx., § 3424. 1855.

† Barus and Strouhal. *Bulletin of the United States Geological Survey*, No. 14, p. 1. 1885.

annealing. The coefficient of variation of the resistance with the temperature is connected with the resistance itself, by a law which seems to be the same for different kinds of iron, as is seen by the curve of Fig. 249, in which the specific resistances have been taken as abscissæ, and the variation for 1° at zero as the ordinate, the unit being the microhm.

The thermoelectric power of steel varies with the resistance, and proportionally so, according to these same physicists. The numbers are sensibly constant for steel of the same origin; they only vary within narrow limits, but in the same direction, from one specimen

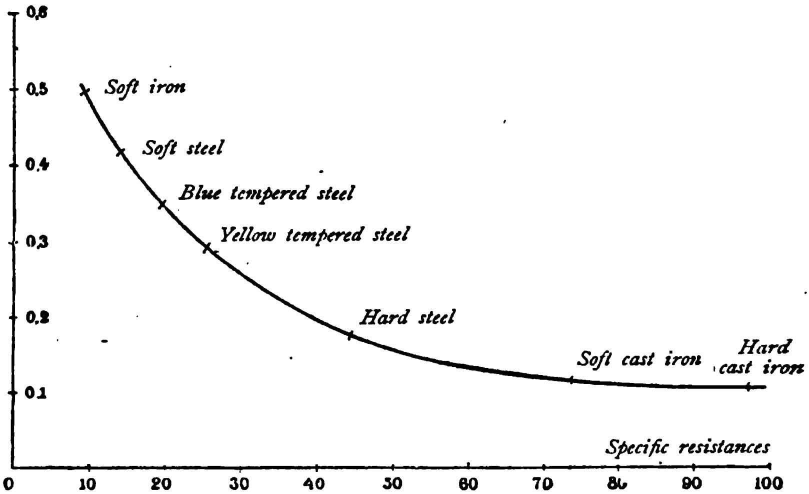


Fig. 249.

to another. It appeared natural, therefore, to define the condition of the steel, either by its specific resistance or by its thermoelectric power.

We find, in this way, that the effect of annealing is felt markedly even for temperatures below 100°—a circumstance which must be taken into account. For a given temperature, the effect of annealing increases with the time, and tends towards a limit.

In order to investigate the influence of the condition of the steel on the permanent magnetism, experiments have been made with very thin steel bars of the kind called *silver steel.* They were raised to a red heat by an electrical current, and then suddenly cooled by a current of water; they were then magnetised to saturation by placing

them in a long coil traversed by a current which was slowly raised to a fixed value, and which was then made to decrease in the same way, and the permanent magnetisation was determined. The bar was then re-annealed at successive temperatures for the time specified in the following table, then magnetised afresh in the same manner and in the same direction.

	Specific Resistance in Microhms.
Bar, hard temper	47·2
Reheated 1 hour, in steam at 100°	42·0
„ 3 hours „ „	39·5
„ 6 „ „ „	38·2
„ 10 „ „ „	37·4
„ 20 minutes, in aniline vapour at 185° ...	31·5
„ 1 hour „ „ „ ...	29·7
„ 3 hours „ „ „ ...	27·9
„ 7 „ „ „ „ ...	26·2
„ 13 „ „ „ „ ...	24·8
„ 10 „ in melted tin, at 240°	24·0
„ 10 „ in melted lead, at 330°	20·0
„ 1 „ in melted zinc, at 420°	17·5
„ at a red heat (soft steel)	15·7

The curves of Fig. 250 represent the magnetic moment for unit weight, or the specific moment—that is to say, the quotient of the permanent magnetisation by the density which corresponds to the different degrees of reheating or of resistance, for bars of the same diameter, 0.15 cm., and lengths respectively equal to 10, 20, 30, 40, or 50 times the diameter.

It will be seen that the reheating begins by increasing the specific moment up to a maximum which is obtained for a lower temperature of annealing, the longer is the bar. The moment diminishes then very rapidly, and again falls to a very low value for steel, which is heated to redness, or quite soft. With long bars, tempered at the blue, a specific moment may be obtained, exceeding 100 G.C.S. units—in other words, an intensity of magnetisation equal to 780, thus reaching about half the maximum magnetisation which soft iron acquires.

The amount of the magnetic moment is not the only quality to be considered in a magnet; this moment must be also as constant as possible, and undergo no permanent alteration from changes of temperature, shocks, vibrations, and time. Steel may acquire a

very high unstable magnetism—a kind of magnetic supersaturation, which does not stand mechanical actions, such as vibrations or shocks. It is sometimes sufficient to let a steel bar fall from a height of a metre, to destroy a considerable quantity—even half—of its magnetism, and its magnetisation is only permanent after a certain number of shocks.

On the other hand, experiment shows that if we reheat a bar at a given temperature for a given time, at any intervals, remagnetising it each time, it ultimately becomes insensitive, as regards its permanent moment, to any temperature below that of heating. According to

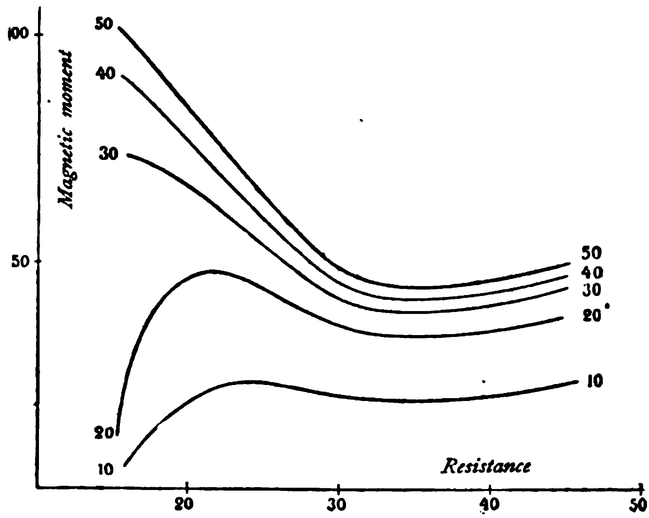


Fig. 250.

Barus and Strouhal, magnets which best satisfy the conditions of permanence are obtained by heating them for 20 to 30 hours, at several separate times, to a temperature of 100°—in steam, for instance—magnetising to saturation, subjecting them anew to the action of steam, and then finally magnetising. Magnets possess then the limiting resistance corresponding to 100°, and are no longer permanently modified by lower temperatures. The specific moment thus obtained may reach 45 or 50, corresponding to an intensity of magnetisation of about 300. In these conditions, the temporary variations due to changes of temperature are also very feeble.

M. Berson observed, in this way, curious effects of temperature. With tempered steel, which has been reheated to a higher temperature than will be used in the experiments, the temporary and the total magnetisation increase with the temperature between 0 and 340°, while the residual magnetism steadily decreases. The variations in temperature during the action of the magnetising force have a marked influence. For a bar magnetised at 290°, for instance, the three kinds of magnetisation—temporary, residual, and total—are considerably smaller than if the magnetising force has been applied while the temperature rises from 240° to 290°.

Singular cases of unstable equilibrium are met with. A steel bar, tempered and not reheated, magnetised at 240°, and suddenly cooled immediately after magnetisation, retains a residual magnetisation considerably higher than that which would be acquired in the cold, under the influence of the same magnetising force. But the bar is, so to say, supersaturated; for this magnetisation disappears under the influence of shocks and agitations far more rapidly than that which would be produced in ordinary conditions.

1210. Mechanical Actions. — This influence of magnetic actions on magnetisation is a general fact, and one long known. If a bar of soft iron be placed parallel to the terrestrial field, the temporary magnetism which it acquires in this situation becomes permanent when the bar is struck at one end by a hammer. When struck again transversely to the field, all the magnetism is destroyed. Besides the influence on the coercive force, we may say that any shock or vibration aids the effect of the actual magnetising forces as if they facilitated the arrangement of the molecules by making them more mobile. Thus during magnetisation the shocks contribute to increase the magnetic moment of the bar. After magnetisation, shocks promote demagnetising actions, and may, as we have seen, cause a steel bar to lose great part of its magnetism.

Joule* has proved that an iron bar lengthens during magnetisation; when the action of the field is suppressed the bar shortens, without however reverting to its original length. The elongation produced by magnetisation is not so great if the wire is strongly stretched; the wire shortens even, if it is subject to a tension near that which produces a rupture. Cobalt

* Joule. *Annals of Electricity*, Vol. viii., p. 219. 1842.—*Scientific Papers*, pp. 46, 205.—*Phil. Mag.* [3], Vol. xxx., pp. 76, 225. 1847.

behaves like iron, but the reverse is the case for nickel: magnetisation shortens the bar. According to Professor Barrett,* the same magnetising force, in analogous conditions, produces an elongation of $\frac{1}{260,000}$ for iron, of $\frac{1}{425,000}$ for cobalt, and a shortening of $\frac{1}{130,000}$ for nickel.

In like manner Sir W. Thomson† found that, in general, traction increases the magnetisation of an iron bar and diminishes that of a nickel bar. On increasing gradually the intensity of the field in which the bar is placed, we find that the increase of moment due to traction goes on increasing to a maximum, then decreases and becomes null for a certain value of the field; beyond this critical value traction diminishes magnetisation. There appears also for nickel to be a critical value of the field, for which the effect of traction on magnetisation changes the sign; but this limit is higher than for iron, and has not been attained.

The deformations which accompany magnetisation explain the sounds produced by iron and steel when these metals are submitted to a rapid series of magnetisations and of demagnetisations. The sounds due to this cause may possibly play an important part in the telephone.

A bar of soft iron becomes heated when submitted to a series of successive magnetisations and demagnetisations, or to magnetisations in opposite directions. Joule‡ even used this phenomenon for one of his determinations of the mechanical equivalent of heat. But it is impossible to separate the heat developed by induced currents in the mass of iron from that which might correspond to the work of magnetisation or of demagnetisation. An analogous question, and which seems quite as difficult to solve, is the retardation which a bar of soft iron presents when placed in a field which periodically varies, and owing to which the maximum of magnetisation is at a greater or less interval after the period of the maximum of the field. It is difficult to say whether this

* BARRETT. *Nature*, Vol. XXVI. 1882.

† Sir W. THOMSON. *Proceedings of the Roy. Soc.*, Vol. XXIII., pp. 445, 473, 1875; Vol. XXXII., p. 442, 1878.—*Phil. Trans. Roy. Soc. Lond.* [2], Vol. CLXVI., p. 693. 1877.

‡ JOULE. *Phil. Mag.* [3], Vol. XXIII., pp. 260, 347, 405. 1843.—*Scientific Papers*, Vol. I., p. 120.

retardation is due solely to induced currents, or if magnetisation does really take an appreciable time to establish itself. MM. Bichat and Blondlot have observed* that by passing the oscillating discharge of a Leyden jar through a coil surrounding a piece of flint glass, the oscillations of the plane of polarization coincide to within 0·00003 of a second with the oscillations of the needle; even if this retardation exists in the case of diamagnetism, it does not attain this value.

1211. DISTRIBUTION OF MAGNETISM IN MAGNETS.—We know (**315**) that the internal action of a magnet is equivalent to that of a fictive layer formed of two equal and opposite masses distributed on its surface, or more generally on a closed surface S which surrounds it.† We may then propose to determine either the fictive density of the magnetism on this surface, and the problem is then definite, or the real distribution of the internal magnetisations, if we make a hypothesis as to the mode of distribution.

We have again seen (**498**) that the external action of a magnet is equivalent to that of a system of superficial currents—that is to say, of magnetic shells. We may then replace the magnet by a body of the same form with a solenoidal or with a lamellar magnetisation.

The methods of Coulomb, by the oscillations of a magnetic needle near a magnet (**414**) or by a torsion balance (**415**), give the normal component of the force, if we take care that the magnetism of the needles is not altered by the field and does not react on the magnet itself.

The use of soft iron (**416**), by oscillations or by the test-contact, enables us to make experiments near the surface; but it does not give more certain results, because we must assume further that the coefficient of magnetisation is independent of the magnetising force, which is far from being the case.

A more correct method would consist in applying to the surface S a small circuit of surface S′, which, by analogy with the proof plane, may be called *proof ring*, and to measure the charge induced when it is carried to a great distance. We should then have the total flow of force in the surface S′, and consequently the mean value of the normal component. We may in like manner place the circuit S′ in any direction in the field of the magnet,

* BICHAT and BLONDLOT. *Comptes rendus*, Vol. XCIV., p. 1590. 1882.

† POISSON. *Ann. de Chim. et de Phys.* [2], Vol. XXV., p. 127. 1824.

and determine, for instance, the components in three rectangular directions, which would give the true force at each point.

Finally, when the magnet is one of revolution, the magnetic images produced by iron filings in a meridian plane give very sensibly the direction of the lines of force, and we could deduce from them the relative magnitudes of the forces. Suppose that, at a point P at a distance r from the axis of revolution, the force is F, and the perpendicular distance of two adjacent lines of force is l; the section of the tube of force determined by these two lines, and by the corresponding lines of a meridian plane which makes the angle θ with the first is $l \times r\theta$, and the corresponding flow is $Flr\theta$. For another section $l'r'\theta$ of the same tube, where the force is F′, the value of the flow of force is the same, which gives

$$Flr = F'l'r', \qquad \text{or} \qquad \frac{F}{F'} = \frac{l'r'}{lr}.$$

1212.—Determination of the Fictive Layer.—We have often considered the curve of normal components as representing the law of the distribution of magnetism; but the reasonings by which Coulomb attempted to justify this mode of view are quite insufficient. There is really no simple relation between (**418**) the real distribution, nor even between the distribution of the fictive layer, and the external field of the magnets. It may not be without use to show this in greater detail by a few instances.

Let us again take the case of a uniform linear magnet, which is equivalent to two isolated masses $\pm m$ situate at a distance 2L; the potential at a point P (**1152**) at the distances r and r' from the masses $+m$ and $-m$, is

$$V = m\left(\frac{1}{r} - \frac{1}{r'}\right).$$

If y be the distance of the point P from the right line 2L, ω and ω' the angles of the radii vectores r and r' with this same right line measured in the direction of the magnetisation, the component of the force parallel to y is

$$-\frac{\partial V}{\partial y} = my\left(\frac{1}{r^3} - \frac{1}{r'^3}\right) = \frac{m}{y^2}(\sin^3\omega - \sin^3\omega').$$

This would be the expression for the component perpendicular to the surface of a cylinder of radius y concentric with the magnet. If this radius is very small compared with the distance 2L of the

two masses, and if we consider the points situate in the neighbourhood of one of them m, the action of the other mass may be neglected.

If z is the perpendicular force, and x the *abscissa* of the point P, measured on the axis of the magnet, starting from the mass $+m$ as origin, we may write

$$x = \frac{m}{y^2}\sin^3\omega = \frac{my}{(y^2+x^2)^{\frac{3}{2}}}.$$

The curve corresponding to this equation, in which y is considered as a constant, presents exactly the same aspect as those which have been given by Coulomb to represent the distribution of magnetism in cylindrical magnets (419), if we put aside the ordinate relative to the end of the magnet, knowing that this extreme value was the result of an estimate.

We shall compare, for instance, the values calculated on the hypothesis of a uniform linear magnetisation with an experiment which has been given in detail by Coulomb,* and which he indicates as giving in the most complete manner the distribution of magnetic densities on a thin steel rod 2 lines in diameter.

The small magnet used for the oscillations was 6 lines in length, and was at a distance of 8 lines from the rod investigated. The distance from the centre of the magnet to the axis of the rod was therefore one inch, and Coulomb's numbers represent actions produced at points situate at different distances from the ends. We may then either make $y=1$ inch in the preceding formula, and calculate the constant my by one of Coulomb's numbers, or determine at what distance y the magnet should be placed to satisfy two of the experimental numbers. We find in this way

	Values of z		
		calculated assuming	
Distance from the End of the Magnet x.	From Coulomb's Experiments.	$y=1$ inch.	$y=2{\cdot}096$ inches.
Inches.			
0·0	165	536·6	122·4
1·0	90	189·7	122·4
2·0	48	189·7	46·4
3·0	23	17·0	46·4
4·5	9	5·5	9·2
6·0	6	2·4	4·4

* COULOMB. *Mém. de l'Acad. des Sciences, 1879.* 3rd Mem., 8th Exp.—*Collection de Mém. de la Société de Physique*, Vol. I., p. 293.

This experiment is certainly in contradiction with the hypothesis of a uniform magnetisation, and we may say that there was magnetism outside the ends. But if, just as well, the same results had been obtained at twice the distance, Coulomb's reasoning would not be sensibly modified; while the real magnetisation would be equivalent to masses $\pm m$, situate at the centres of the two bases.

In the case of a short cylinder, uniformly magnetised, the same reasoning, applied to different magnetic filaments, would give the perpendicular component on the lateral faces—or rather, on the surface of a concentric cylinder of somewhat greater diameter. We could then apply the formulæ which enable us to calculate the action of a uniform circular layer (767), or the external action of a cylindrical coil (768).

1213. Let us again suppose that the surface S which surrounds the magnet is an infinite plane or a spherical surface. The fictive layer on this surface is the algebraic sum of those corresponding to different internal masses; it is the problem of electrical images (148). For a mass m, situate in a sphere of radius R, at a distance L from the centre, the value of the density σ_1 of the fictive layer at a point P, at the distance r from the acting mass, putting $k^2 = \frac{R}{L}$, is

$$\sigma_1 = \frac{2a}{4\pi k^2} \frac{m}{r^2}.$$

The numerator $2a$, which represents the distance of this mass from its external image, is

$$2a = R\frac{k^4 - 1}{k^2} = Rk^2 \frac{k^4 - 1}{k^4} = Rk^2 \frac{R^2 - L^2}{R^2};$$

it follows that

$$(3) \qquad \sigma_1 = \frac{R^2 - L^2}{4\pi R} \frac{m}{r^2}.$$

If ω is the angle which the radius of the point P makes with the right line L, the perpendicular component of the force is

$$N_1 = \frac{m}{r^3}(R - L\cos\omega).$$

When the magnet reduces to two masses $\pm m$, on either side of the centre of the sphere, at the distance $2L$, the distance from the point to the mass $-m$ being r', the density of the fictive layer is

$$(4) \qquad \sigma = \frac{R^2 - L^2}{4\pi R} m \left(\frac{1}{r^3} - \frac{1}{r'^3} \right).$$

If this value of σ be developed as a function of the quantity

$$z = \frac{2LR}{R^2 + L^2} \cos \omega,$$

we find

$$\sigma = \frac{3}{4\pi} \frac{(R^2 - L^2)m}{R(R^2 + L^2)^{\frac{3}{2}}} \left[1 + \frac{5.7}{4.6} z^2 + \frac{5.7.9.11}{4.6.8.10} z^4 + \ldots \right].$$

This expression for the density is very different from that of the perpendicular component of the action

$$N = m \left(\frac{R - L \cos \omega}{r^3} - \frac{R + L \cos \omega}{r'^3} \right),$$

the expansion of which, as a function of the same value of z, has been given above.

For a magnet of any constitution comprised within the sphere, the superficial layer will be determined by the superposition of the layers σ_1 relative to all the masses, and, if the magnetisation is symmetrical in respect of the centre, by the superposition of the layers σ relative to the magnetic filaments into which it can be decomposed.

1214. The general problem of the distribution of the fictive layer on the surface S, when we know the internal distribution of the magnetism, is the same as that which consists in calculating the electric layer whose actions are equivalent to those of masses comprised within a surface; it presents great difficulties, and the inverse problem of determining the fictive layer by external actions is of the same kind.

We observe, in the first place, that the field of a magnet outside a closed surface S, which surrounds it, is completely defined when we know the potential and its perpendicular components at all points of this surface.

If, in Green's formula (34), the letters U and V are interchanged, and if the two equations are subtracted, we get

$$(5) \qquad \int U\Delta V dv - \int V\Delta U dv = \int \left(U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n} \right) dS,$$

the second expression being extended to the entire surface S, and the first to the volume which it contains.

We shall apply this equation to the volume of the magnet, assuming first that V represents its potential, and ρ the magnetic density at a point M. Consider an external point P, at the distance r from the point M: we shall represent by U the potential $\frac{1}{r}$ at M of a mass equal to unity at the point P; we shall have then, within the limits of the integral,

$$\Delta U = 0, \qquad \Delta V = -4\pi\rho.$$

The former member of the equation (5) reduces to

$$\int U\Delta V dv = -4\pi \int \frac{\rho dv}{r} = -4\pi V_p,$$

V_p denoting the potential at the point P; it follows that

$$(6) \qquad 4\pi V_p = \int V \frac{\partial \frac{1}{r}}{\partial n} dS - \int \frac{1}{r} \frac{\partial V}{\partial n} dS.$$

The value of V_p only depends then on the potential V, and on its differential perpendicular to the exterior of the surface S.

It is sufficient if we know, to within a constant, the potential on the surface, for, by calculating the external potential, this constant would be determined by the condition that the potential is null at a great distance from the magnet.

There is, moreover, a relation between the potential V on the surface and its perpendicular differential. Let us, in fact, consider a point in the interior, and, the function U being defined in the same way, apply equation (5) to the volume bounded by the surface S and

by the surface S′ of a sphere of radius r', which is very small, having its centre at the point in question. Equation (6) is exact on condition that the integral of the second member is extended to the surfaces S and S′.

In this case, the second member contains implicitly the term $4\pi V_p$; for the surface S′ we have $dn = -dr'$, $dS' = r'^2 d\omega$, $d\omega$ being the angle at the centre which corresponds to the surface dS', and

$$\int V \frac{\partial \frac{1}{r}}{\partial n} dS' - \int \frac{1}{r} \frac{\partial V}{\partial n} dS' = r' \int \frac{\partial V}{\partial r'} d\omega + \int V d\omega.$$

When the radius r' becomes null, the first term of the second member tends towards zero, and the second towards $4\pi V_p$. Equation (6) reduces then to

$$(7) \qquad \int V \frac{\partial \frac{1}{r}}{\partial n} dS = \int \frac{1}{r} \frac{\partial V}{\partial n} dS.$$

This condition being independent of the position of the point P, it follows that the two quantities V and $\frac{\partial V}{\partial n}$ are a function of the other; the external field of the masses comprised within a surface S is then defined if we know the potential on the surface, or the perpendicular differential of the potential—that is to say, the perpendicular component of the force on an indefinitely near point.

If S is the surface of a sphere of radius R, and we take the point P′ at the centre, we have

$$\int -\frac{1}{r} \frac{\partial V}{\partial n} dS = \frac{1}{R} \int -\frac{\partial V}{\partial n} dS = \frac{Q}{R} = \frac{4\pi M}{R},$$

Q denoting the total flow of force from the sphere, and M the sum of the acting masses which it contains; further

$$\int V \frac{\partial \frac{1}{r}}{\partial n} dS = -\frac{1}{R^2} \int V dS.$$

It follows that

$$\int V dS = 4\pi RM.$$

With a magnet, $M = 0$; in this case the mean value of the potential on the surface is also zero.

1215. If we have determined by any method the external potential of the magnet, and the internal potential of the fictive layer, by the aid of the potential on the surface, the density at each point (498) will be given by the equation

$$(8) \qquad 4\pi\sigma + \frac{\partial V}{\partial n} + \frac{\partial V}{\partial n'} = 0,$$

the external and internal perpendiculars being counted from the surface.

The determination of these potentials reduces to problems of electrical influence.

Suppose that the function U represents the potential of unit mass at a point P, and of the layer which this mass, if it were electrical, would produce by induction on the surface S, which is supposed to be a conductor, and in connection with the earth, and let u be the density of this layer on the element dS.

We shall apply equation (5) to the volume bounded by the surface S and the surface S′ of a sphere of infinitely small radius, having the point P as centre.

For the surface S′, the value of U tends to become equal to $\frac{1}{r'}$, and, when r' tends towards zero,

$$\int U \frac{\partial V}{\partial n} dS' = -\int \frac{1}{r'} \frac{\partial V}{\partial r'} r'^2 d\omega = -r' \int \frac{\partial V}{\partial r'} d\omega = 0,$$

$$\int V \frac{\partial U}{\partial n} dS' = -V_p \int \frac{\partial \frac{1}{r'}}{\partial r'} r'^2 d\omega = 4\pi V_p.$$

The value of the second member of the equation (5) for the surface S′, reduces then to $-4\pi V_p$.

The terms of the second member relative to the surface S are null, for we have $U=0$ and $\frac{\partial U}{\partial n}=0$; this differential being counted towards the interior or towards the exterior of the surface, according as the point P is itself on the outside or inside.

If the point P is outside, $\Delta V=0$ within the limits of the integral, except perhaps on the surface S, but in that case $U=0$; we have then

$$\int U\Delta V dv=0.$$

On the other hand, $\Delta U=0$ in the same conditions, but we must allow for the layer on the surface S, when V is not null. The former member of the equation then becomes

$$-\int V\Delta U dv=4\pi\int V u dS.$$

It follows that

$$V_p=-\int V u dS.$$

The external potential of the magnet is then expressed solely as a function of the potential on the surface.

If the point P is within the surface S, we shall replace the magnet by the fictive surface layer. The same reasoning will then apply without any modification, and the expression for the potential is the same.

In both cases the density u will be defined by the aid of potential U by the condition

$$4\pi u=-\frac{\partial U}{\partial n}.$$

If a be the perpendicular distance of the point P to the surface, the density u on the element dS is a function of a, and we have

$$\frac{\partial V_p}{\partial a}=-\int V\frac{\partial u}{\partial a}dS.$$

The values of this expression, inside and outside, for $a=0$, will give the intensity σ by equation (8).

The investigation of the external field by one of the methods previously indicated, will give us the potential on the surface to within a constant, starting from any given equipotential surface V_1, and taking the integral $\int F dn$, which is equal to $V_1 - V$ along a line of force to the surface S. In the case of a bar magnet, for instance, two level surfaces V_1 and V_2 will be chosen which surround the extremities; the difference of potentials $V_1 - V_2$, and, starting from one or the other, the potential over the whole surface will be determined.

1216. Poles of Magnets.—The poles of magnets, or the centres of gravity of the positive and negative masses which they contain (295), may be determined if we know their magnetic structure, or simply the distribution of the surface fictive layer.

In the case of cylinders of small diameter, the centre of gravity of the surface bounded by the curve of lateral distribution will give the position of the pole, provided always that the elementary surface near the end comprises also the magnetism which is found on the terminal faces.

From the experiments of Coulomb (419) the distance x_1 of the pole from the end will be $\frac{1}{6}$ of the length for short magnets with a linear distribution, or $\frac{25d}{3}$ for magnets whose length is 50 times the diameter of d.

If the curve of distribution were replaced by a parabola tangential to the direction of the magnet at the distance l from the ends, we shall have $x_1 = \frac{l}{4}$.

Biot's law, finally, $\lambda = a\mu^x$, where the constant μ is smaller than unity, gives $x_1 = -\frac{1}{l.\mu}$. For bars magnetised to saturation, the constant a, according to Jamin's experiments,* would only depend on the nature of the steel, and the constant μ on its tempering or annealing.

These various methods all fail precisely from the manner in which the law of distribution is determined.

1217. Measurement of magnetic moments gives the best results, but it is still necessary to have recourse to some hypotheses.

* Jamin. *Comptes rendus*, Vol. LXXX., p. 1553. 1875.

Coulomb found that for similar bars, magnetised to saturation, the magnetic moments are proportional to the cube of homologous dimensions; the mean is then constant, and the distribution of magnetism should be independent of the dimensions.

For needles 2 lines in diameter the length of which increases to 5 inches, or 30 times the diameter, the magnetic moment is nearly proportional to the square of the length.

When the ratio of the length to the diameter is greater than 30, the increase of the magnetic moment is nearly proportional to the increase of length, although a little more rapid; if m and x_1 are two constants and $2l$ is the length,

$$\text{(9)} \qquad M = 2m(l - x_1).$$

In this case, the mean magnetisation, being proportional to the ratio $1 - \frac{x_1}{l}$, will increase with the length.

If we consider the factor m as representing the mass of each pole, the distance x_1 of the pole from the end will be constant; but the restriction introduced into the exactitude of this experimental relation by Coulomb himself, shows that the mass of the poles increases with the length; and this should be the case, for with the same distribution at the ends, the demagnetising force diminishes as the length of the magnet increases.

The same is the case, according to M. Bouty,* for the temporary magnetisation of steel needles of various lengths in the uniform field of a coil. The distance from the pole to the ends, calculated by equation (9), is seen to be sensibly independent of the length of the needle and of the magnetising force; this latter result, however, does not agree with an ingenious experiment by which Professor Rowland† has observed that the distribution of the induced magnetism changes with the magnitude of the magnetising force.

Green's formula (422)

$$M = 2m\left[l - \frac{1}{\beta}\frac{e^{\beta l} - e^{-\beta l}}{e^{\beta l} + e^{-\beta l}}\right]$$

gives, for the distance of the pole from the end, a constant quantity

* Bouty. *Journal de Physique*, Vol. IV., p. 367. 1875.

† Rowland. *Phil. Mag.* (4) Vol. XLVI, p. 142. 1873.

$\frac{1}{\beta}$ in the case of long magnets; it also represents satisfactorily the experiments of M. Bouty on the magnetic moment of a needle magnetised to saturation—at any rate so long as the length exceeds 10 times the diameter, but it does not suit shorter bars.

If, finally, a bar already magnetised be temporarily magnetised, the total magnetisation* will not be represented by Green's formula, but by the sum of two similar formulæ with different constants. The rigid and the temporary magnetism would be almost independent of each other, the poles of the latter being much nearer the ends, and its distribution comparable with that of long magnets.

Comparison of the magnetic moments of bars of the same sections and of different lengths gives thus a closer value for the position of the poles, although the hypothesis of constant masses which it implies is less exact the shorter are the bars.

1218. Kohlrausch† endeavoured to determine the magnetic length 2L of a magnet by the second term of the series which expresses the action of the magnet as a function of increasing powers of the inverse distance (**1153**). Thus, for points in the axis

$$F = \frac{2M}{R^3}\left[1 + \frac{2L^2}{R^2}\right],$$

and by experiments made at two different distances, we may deduce the constant 2L.

Working with very various magnetising forces on solid or hollow cylindrical bars, tempered or annealed, the length of which was from 10 to 30 times the diameter, or even with rectangular bars, the dimensions of which were $44 \times 2{\cdot}3 \times 1$, the ratio of the constant 2L to the length of the magnet varies from 0·81 to 0·86; we may then consider this fraction as being in all cases sensibly equal to 0·83 or five-sixths.

Nevertheless, apart from the fact that the experiment is of no great accuracy, it does not give us the real distance of the poles. Suppose, for instance, the case of a linear and symmetrical magnet, whose linear density is λ at a distance x from the centre. The mass for length dx being λdx, the magnetic moment of the two

* BOUTY. *Journal de Physique*, Vol. v., p. 346. 1876.

† KOHLRAUSCH. *Wied. Ann.* Vol. XXII., p. 411. 1884.

symmetrical masses will be $2\lambda x dx$; if, then, $2l$ is the length of the magnet, we shall have

$$ML^2 = 2mL^3 = 2\int_0^l \lambda x^3 dx,$$

$$L^3 \int_0^l \lambda dx = \int_0^l \lambda x^3 dx.$$

This equation will give the constant L, but the distance a from the middle of the magnet to the centre of gravity of each layer is defined, on the contrary, by the condition

$$a\int_0^l \lambda dx = \int_0^l \lambda x dx;$$

and, whatever is the mode of distribution, outside the case of two masses we have always $L > a$.

For a linear distribution, for instance, in which the density λ is proportional to x, the ratio of the values of a and L to the semi-length of the magnet will be

$$\frac{a}{l} = \frac{2}{3} = 0{\cdot}667,$$

$$\frac{L}{l} = \sqrt[3]{\frac{2}{5}} = 0{\cdot}737.$$

These few examples will be sufficient to show the difficulties presented by the problem of the distribution of magnetism and the position of the poles.

1219. Bodies which are Feebly Magnetic or Diamagnetic.—After iron, nickel, and cobalt, the most powerfully magnetic bodies are the oxides and the salts of iron, but in a far lower degree; the magnetisation is still more feeble in diamagnetic substances. The experiments present then special difficulties arising on the one hand from the smallness of the effect to be measured, and on the other from the sources of error due to the presence of the smallest traces of iron in the bodies in question.

When diamagnetic phenomena were discovered, Faraday ascribed them to a polarity the inverse of that which a magnetic body would give; he gave up this explanation when he had found that all the facts

were sufficiently accounted for on the assumption that magnetic bodies move towards points where the force is maximum, and diamagnetic towards the points of minimum force. The opinion of physicists was at first divided between these two views of Faraday, but this divergence ceased from the time when Sir W. Thomson showed that these two explanations merge into each other (395).

We shall describe two experiments by Tyndall* which show clearly the inverse polarity of bismuth.

A bismuth bar is suspended horizontally in the axis of a magnetising coil A, so that its ends are opposite poles of the contrary kind of two electromagnets. The deflection of the bar is the inverse of that of an iron bar; the current of the coil A develops then in

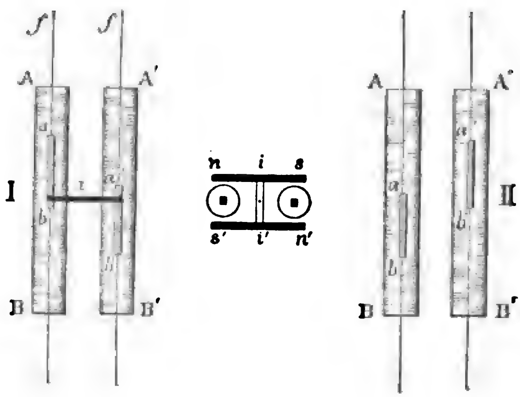


Fig. 251.

the bismuth a pole N at its right, and a pole S at its left. The deflection changes its direction with the direction of the current in the coil A, which proves that the magnetisation of bismuth is due mainly to the field of this coil, and that the influence of the electromagnets is only secondary.

The principle of the second experiment is due to Weber.† The apparatus consists of two long vertical coils AB, A'B' (Fig. 251), placed side by side. By an endless wire passing over two pulleys, two bismuth bars *ab*, *a'b'* can be displaced in opposite directions along the axis of each coil. Two magnetised needles *ns*, and *n's'*, in

* TYNDALL. *Phil. Trans.* p. 24. 1855.

† WEBER. *Electrodyn. Maasbestim.*, *Abh. der Königl. Säch. Gesellschaft*, Vol. I., p. 483. 1852.

the same horizontal plane, and forming an almost astatic system, comprise between them the two coils.

The astatic system, which has a small directive force, is placed in the plane of symmetry perpendicular to the axis of the coils, and should not be acted upon by them; this absolute neutrality is difficult to obtain, but the residual effect is counterbalanced by a small auxiliary coil. The needles are contained in a small copper box, which acts as damper.

The current is passed in contrary directions in the two coils, and by the mirror method the position of equilibrium is observed when the two bars occupy the position (I), and then the position (II); it is observed that the deflection with bismuth is opposite that which would be obtained with a magnetic body.

We may, with Weber, increase the deflections by the method of multiplication, by giving the two bars a backward and forward motion in harmony with the oscillations of the needle.

The coils being so long that the field may be considered uniform within the limits of the swing, there are no other currents than those due to the displacement of the bars in respect of the needles. Experiment shows that this effect is almost null even with copper bars.

1220. Experimental Methods.—In order to determine the magnetic susceptibility k, of feebly magnetic or diamagnetic bodies, recourse is had to the action of a non-uniform field. The variation of energy of a small volume u, at a point where the intensity of the field is ϕ, is equal to $-\frac{uk}{2}d\phi^2$, and the effort with which it tends to move along a path ds is expressed by $-\frac{uk}{2}\frac{\partial\phi^2}{\partial s}$ (394).

The differential $\frac{\partial\phi^2}{\partial s}$ only depends on the form of the field; but it is unnecessary to get it by experiment, for it disappears in the ratios. We might, moreover, determine it by an examination of various points of the field by Verdet's method.

If the field is produced by a current, the value of $\frac{\partial\phi^2}{\partial s}$ is the product of the square of the intensity I of a current, by a factor depending on the shape of the coil. In a series of successive experiments, the current should be constant, or else its variations be allowed for.

When the magnetic field produced by electromagnets is used, a fresh difficulty is met with. The action is no longer in a simple ratio to the intensity of the exciting current, and the very form of the field may also be considerably modified; hence in this case the current must be kept unchanged.

1221. When the field has two symmetrical planes, the oscillations of a small needle (404) enable us to determine the coefficient k. We have then

$$k = \frac{\pi^2}{A+B} \frac{\rho}{t^2},$$

ρ denoting the density of the substance, A and B constants defined by the shape of the field. In this case, the duration t of oscillations is independent of the length of the needles, so that for comparative experiments it is unnecessary to use needles of the same length, or to know the form of the field.

If the needle is so long that variations of ϕ^2 in the space occupied by the needle cannot be expressed by the two first terms of the expansion of a series (184), we may still put

$$k = A' \frac{\rho}{t^2},$$

but the coefficient A′ is a function of the shape of the field and of the length of the needle. In this case we should restrict ourselves, for comparison, to needles of the same dimensions.

1222. The torsion balance gives a method frequently employed, which does not require the field to be symmetrical. At one end of the needle is placed a small specimen of the body to be examined—a sphere, for instance. When the magnetic field is caused to act, the sphere tends towards points where the force is maximum or minimum, according to the sign of k, and it is restored to its original position by the torsion A of the wire. If l is the length of the arm of the lever, ds the element of the circle described by the sphere, and $d\theta$ the corresponding angle, the couple produced on the sphere by the action of the field is

$$-l\frac{uk}{2} \cdot \frac{\partial \phi^2}{\partial s} = -\frac{uk}{2} \frac{\partial \phi^2}{\partial \theta}.$$

If m is the magnetic moment of the sphere, and C the coefficient of the wire, the equation of equilibrium is

$$CA = -\frac{uk}{2}\frac{\partial\phi^2}{\partial\theta} = -uk\phi\frac{\partial\phi}{\partial\theta} = -m\frac{\partial\phi}{\partial\theta}.$$

If, for instance, the field is obtained by means of Faraday's electromagnet, provided with armatures symmetrical in reference to a common axis, the needle will be arranged so that the sphere can move along this line; the original position, moreover, being outside the centre. The centre, in short, would represent a position of unstable equilibrium for diamagnetic bodies.

It is even advantageous to place on the electromagnets armatures which are symmetrical in respect of the centre, but dissymmetrical in reference to the common axis, and to use a long needle, the middle of which is in the centre of symmetry.* As the direction of equilibrium of the needle makes an angle θ with the axis, the couple due to the action of the field on an element of volume du is $-k\phi\frac{\partial\phi}{\partial\theta}du$; the resultant couple gives, as condition of equilibrium,

$$CA = -k\int\phi\frac{\partial\phi}{\partial\theta}du.$$

The integral by which we should multiply the coefficient k will be constant for needles of the same form placed in the same field, provided the position of equilibrium is not modified.

M. Becquerel also used the ordinary balance, and this method is especially convenient for the case of gases. A glass globe, placed below the armature of an electromagnet, is suspended to the beam of a balance. The weight necessary to restore equilibrium, when the electromagnets are excited, gives a measure of the action of the field on the globe, and the gas which it contains. The globe being at first empty, and then filled with a gas at known pressure, the difference of the two weighings gives the action of the gas. Becquerel found that air is magnetic, and that this magnetism is due to the oxygen alone.

1223. As Faraday has shown,† experiments made in air only give *apparent* values of the coefficient of magnetisation—that is

* E. BECQUEREL. *Ann. de Chim. et de Phys.* [3], Vol. LXXXII., p. 68. 1851.

† FARADAY. *Experimental Researches*, § 2362. 1845.

to say, the excess of the value for the coefficient for a vacuum over the coefficient for air, also in respect of vacuum. Becquerel used this relation for a great number of liquids and of gases.

Let k_1 be the apparent coefficient of a substance in air, k its value in vacuum, and k' the coefficient of air; we have

$$k_1 = k - k'.$$

Without altering the rest of the experiment, the air is replaced by another medium (water, for instance), the coefficient of which is k''; the new value k_2 gives, in like manner,

$$k_2 = k - k''.$$

The coefficient of water in respect of air is then

$$k_1 - k_2 = k'' - k'.$$

1224. One arrangement used by Faraday, and then by Weber,* consists in placing a bar of the substance to be investigated (bismuth, for instance) in what is the uniform field of a long cylindrical coil. Between the bar and the magnetising coil is interposed a second coil, in which the wire is wound in opposite directions on the two halves. The two ends of this coil are connected with a delicate galvanometer. When the bar is displaced in either direction along the axis, its magnetic state is unchanged; but the two poles develop in each half of the coil induced currents which add themselves. By combining this motion of the bar with that of the needle we may work by multiplication. By giving to the bar a rapid backward and forward motion, and rectifying the currents by a commutator, we may obtain a permanent deflection. The effect produced is then compared with that which a bar of soft iron would give. By this method Weber found that for the same weight the moment of bismuth is 456,000 times as small as that of iron.

The difference in the inductive effects of two concentric coils, according as a magnetic core is or is not introduced into the axis, renders it possible to determine the coefficient of magnetisation of the core; the direct method would give no result with bismuth, but it becomes applicable if a differential method is made of it.

* WEBER. *Electrodyn. Maasbestim.*, *loc cit.*

Let there be two systems of identical coils, each composed of an inducing coil and of a concentric induced coil.* The inducing coils are traversed by the same current, and the two induced coils are connected by a delicate galvanometer, so that their currents have no action on the needle. By introducing the bismuth bar in one of these coils, the inductive effects are diminished; the galvanometer measures then the difference of two induced currents, so that the coefficient k' may be determined.

It must be observed that with a conducting core without magnetic action a diminution of the same kind would be observed, owing to the induction currents developed in the mass; but it can be seen that a copper bar produces much greater diminution than a bismuth bar. Nevertheless there is a source of error which must be taken into account in the case of bodies which are very feebly magnetic or diamagnetic, if they are not bad conductors; they tend to give too high a value for diamagnetic bodies.

1225. This same method has been applied to perchloride of iron. A vessel filled with the solution is taken as core. M. Silow† uses a long tube placed vertically, in which is coiled the magnetising spiral; the induced coil, which is much shorter, is placed equidistant from the two ends. M. Bergmann‡ places the liquid between two concentric coils of the same lenght. The coefficient of mutual induction M (771) should then be multiplied by $1 + 4\pi k$. The same physicist also used annular coils. Silow has shown that solution of ferric chloride obeys the same law as powerfully magnetic metals; the coefficient begins to increase rapidly with the magnetising force, passes through a maximum, and then slowly decreases. The maximum is with a magnetising force of 0·4 C.G.S. units, and amounts to 179.10^{-6}.

It is generally assumed that, for feebly magnetic or diamagnetic bodies, the value of k is a constant. It must, however, be remarked that intensity of magnetisation has been compared, not to the intensity of the field itself, but to the intensity of the current, which cannot be considered equivalent in the case of electromagnets. We have seen that in any case this law does not hold for perchloride of iron in solution—at any rate for small forces. For greater forces the coefficient seems to be tolerably constant.

* Töpler. *Pogg. Ann.*, Vol. cliv., p. 60. 1875.

† Silow. *Wied. Ann.*, xi., p. 324. 1880.

‡ Bergmann. *Beiblatter der Physik*, Vol. iii., p. 812. 1879.

Experiments do not seem to indicate a maximum for bismuth. The following are the principal results which have been obtained:—

Weber*	$16{\cdot}4.10^{-6}$
Töpler and Ettingshausen†	$15{\cdot}1.10^{-6}$
Christie‡	$14{\cdot}6.10^{-6}$
Ettingshausen§	$14{\cdot}5.10^{-6}$

The first three numbers were determined by comparison with iron, the last, by absolute direct measures by the aid of a method of torsion.

1226. ANISOTROPIC BODIES.—The magnetic properties of substances of fibrous or crystalline structure were discovered by Plücker.|| The general case of a non-homogeneous body situated in a field which is not uniform gives rise to very complicated phenomena, because each element of volume is attracted towards maxima or minima regions of force, according as it is magnetic or diamagnetic, at the same time that the axis of greatest magnetisation tends to set in a direction parallel or perpendicular to the field (397). The action reduces to a couple when the field is uniform.

Let us consider a homogeneous body of volume u placed in a uniform field ϕ, the direction of which makes angles with the principal axes of magnetisation whose cosines are λ, λ', and λ'' (392). If the principal coefficients k, k', and k'' are very small, the resultant magnetisation is independent of the shape of the body. The magnetic moments m, m', and m'', parallel to the axis, are respectively

$$m = u\phi k\lambda, \qquad m' = u\phi k'\lambda', \qquad m'' = u\phi k''\lambda'',$$

and the couples relative to the three axes,

$$\begin{aligned} D &= m'\phi\lambda'' - m''\phi\lambda' = u\phi^2(k' - k'')\lambda'\lambda'', \\ D' &= m''\phi\lambda - m\phi\lambda'' = u\phi^2(k'' - k)\lambda''\lambda, \\ D'' &= m\phi\lambda' - m'\phi\lambda = u\phi^2(k - k')\lambda\lambda'. \end{aligned}$$

* W. WEBER. *Electrodynam. Maasbestim.—Diamagnetismus*, p. 523.

† TÖPLER and ETTINGSHAUSEN, *Pogg. Ann.*, Vol. CLX., p. 1. 1877.

‡ CHRISTIE. *Pogg. Ann.*, Vol. CXXXIII., p. 589. 1858.

§ ETTINGSHAUSEN. *Wied. Ann.*, Vol. XVII., p. 272. 1882.

|| PLÜCKER. *Pogg. Ann.*, Vol. LXXII., p. 315. 1847.

If the principal coefficients are constant without being very small, the body should have the shape of a sphere. The magnetisation parallel to the component $\phi\lambda$ of the field is then

$$\frac{k}{1+\frac{4}{3}\pi k}\phi\lambda = \frac{3}{4\pi}\frac{\mu-1}{\mu+2}\phi\lambda = \frac{3h}{4\pi}\phi\lambda,$$

and the coefficient's $\frac{3\lambda}{4\pi}$ for each of the principal axes will play the same part as the coefficients k.

Supported by this observation, all the properties we are about to establish, on the hypothesis of very small coefficients, might be applied to anisotropic spheres, the coefficients of magnetisation of which are of any order of magnitude.

1227. When the structure of a body is symmetrical in respect of a right line, the same symmetry is again met with in the magnetic properties, and two of the coefficients (for instance k and k') are equal.

In this case, if the body is movable about its axis of symmetry, the couple D for this axis is null, and the body is in neutral equilibrium in all directions.

If the body is movable about a perpendicular to the axis of symmetry, the component H of the field perpendicular to the axis is alone operative. If δ is the angle of this force with the axis of symmetry, the expression for the couple D″ becomes

$$D'' = u\mathrm{H}^2(k-k')\sin\delta\cos\delta;$$

equilibrium takes place when the axis of symmetry is parallel or perpendicular to the effective component H, according as $k>k'$ or $k<k'$.

This couple may be measured by a method of torsion (for instance by a bifilar C) which has been turned through an angle ω from the direction of equilibrium; we have then

$$k-k' = \frac{2C\sin(\omega-\delta)}{u\mathrm{H}^2\sin 2\delta}.$$

If the body is suspended by a wire without torsion, and oscillates freely, the directive couple for very small deflections is

$u\,H^2(k-k')$. If Δ is the density of the body, ρ its radius of gyration, and n the number of oscillations per second, we have

$$k-k'=\frac{\pi^2\Delta\rho^2}{H^2}n^2.$$

We only obtain the difference of the coefficients k and k'; another experiment would be necessary to determine one of them in absolute value.

Bismuth* has the character of symmetry in question. Its crystalline shape is that of a rhombohedron of 87° 40′ with a cleavage perpendicular to the axis. It is diamagnetic and uniaxial. Moreover, the direction of the axis puts it in stable equilibrium parallel to the field; we have then, in numerical strength, $k<k'$.

Iceland spar* often seems magnetic, but it only possesses this property when it contains traces of carbonate of iron. In the pure state it is diamagnetic, and, contrary to what is the case for bismuth, the axis of crystallisation is in equilibrium perpendicularly to the field—that is when $k>k'$.

When the three coefficients k, k', and k'' are unequal, their difference in pairs may be determined by means of the couples D, D′, and D″ in respect of the same direction along which the principal axes have been successively placed.

The number of oscillations n, n', and n'', for instance, observed in the three cases, will give, assuming $k>k'>k''$,

$$\frac{k'-k''}{n^2}=\frac{k-k''}{n'^2}=\frac{k-k'}{n''^2}=\frac{\pi^2\Delta\rho^2}{H^2}.$$

The equation of condition which follows from this has been verified by Plücker with *formiate of copper*, which crystallizes in the klinorhombic system.

1228. Ellipsoid of Induction.—The magnetic properties of anisotropic bodies may be represented by an ellipsoid* analogous to the ellipsoid of polarization, by which Fresnell represented the double refraction of light; we thus find correlated properties in the two orders of phenomena.

* Faraday. *Experimental Researches*, XXII. Series. 1848.

If we call an *ellipsoid of induction* the ellipsoid referred to the three principal axes, and the equation of which is

$$kx^2 + k'y^2 + k''z^2 = 1,$$

the inverse squares of the axes of the ellipsoid being respectively equal to k, k', and k''.

If the body turns about the x axis, the couple of rotation is proportional to $k' - k''$—that is to say, to the difference of the inverse squares of the principal sections of the ellipsoid of induction, by the plane perpendicular to the axis of rotation.

This is a general property. If the body rotates about any given direction, the couple is proportional to the difference of the inverse squares of the axes of the ellipse obtained by cutting the ellipsoid of induction by a diametral plane perpendicular to the axis of rotation. In particular, if this axis is perpendicular to one of the cyclical planes of the ellipsoid, the curve of intersection is circular and the equilibrium neutral.

The corresponding directions are situate in the plane xz of the extreme magnetisations, and satisfy the conditions

$$(k - k')x^2 = (k' - k'')z^2;$$

the angle A, which they make with the least axis of magnetisation, is given by equation

$$\tan^2 A = \frac{x^2}{z^2} = \frac{k' - k''}{k - k'}.$$

The coefficients k, k', and k'' play then the same part as the squares of the velocities of propagation of plane waves in double refracting media with two axes, and all the theorems established in optics have their analogues in the phenomena of magnetisation.

We may, by analogy, in crystallised bodies distinguish between magnetic *uniaxial* and *biaxial* crystals.

In like manner magnetic uniaxial bodies will be considered as positive or negative, according as the coefficient of magnetisation in the direction of the axis of symmetry is of greater or less value than in the perpendicular direction. From what has been said above, bismuth is negative and Iceland spar positive.

Magnetic biaxial crystals, finally, are positive or negative according as the angle A of the magnetic axes with the direction of

least magnetisation is less or greater than 45°; that is, according as

$$k' - k'' \lesseqgtr k - k', \qquad \text{or} \qquad 2k' \lesseqgtr k + k''.$$

1229. The direction which a crystal takes in a uniform field only depends on the difference of the coefficients; we may then add a constant value to each without modifying the phenomena. From this it follows that the conditions of equilibrium are not changed when the body is immersed in any given medium, magnetic or diamagnetic.*

If there were bodies (397) whose coefficients k, k', and k'', were not all of the same sign, the body being magnetic in certain directions and diamagnetic in others, the properties might still be represented by an ellipsoid of induction which would be obtained by adding a constant to all the coefficients.

This indifference of the medium does not exist for a non-uniform field, for an element of volume is attracted towards points of maximum or minimum force, according as the coefficient of apparent magnetisation is positive or negative. Faraday observed, for instance, that a crystal of ferrid-cyanide of potassium, varnished on the surface so as to prevent its being dissolved, is constrained to move in the direction of increasing forces when it is immersed in water, and, on the contrary, moves towards decreasing forces when it is immersed in a concentrated solution of ferrous sulphate. But in a liquid formed of 15 volumes of the concentrated solution, to which 6 volumes of water have been added, the crystal is magnetic or diamagnetic according as its axis of symmetry is parallel or perpendicular to the lines of force—that is to say, that in the first case it moves towards increasing, and in the second towards decreasing, forces.

Tyndall and Knoblauch† have observed that we can obtain analogous figures with bodies in powder compressed in a definite direction, or of systems of superposed layers. The direction of greatest density always tends to set parallel to the lines of force, and perpendicularly to those lines for diamagnetic bodies. This experiment is made by a paste of gum, in which is incorporated powdered bismuth, or grains of carbonate of iron.

* FARADAY. *Experimental Researches*, Series, XXII. and XXX. 1855.

† TYNDALL and KNOBLAUCH. *Phil. Mag.*, Vol. XXXVI., p. 178. 1850.

By pressing a series of sheets of paper, which are then cut in the form of a cylinder, the axis is magnetic with emery paper and diamagnetic with paper covered with powdered bismuth.

In like manner a bismuth cube, which has been compressed, sets in such a manner that the line of pressure is axial.

These experiments seem to show that partially, at least, the phenomena of crystalline magnetism are due to inequalities of pressure, resulting from molecular structure, rather than to the molecules themselves.

1230. INFLUENCE OF SHAPE ON ISOTROPIC BODIES.—The preceding considerations only apply when the coefficients of magnetisation have considerable values, for we must allow for induced magnetisation (**358**), which depends on the shape of the bodies; but isotropic bodies of any shape give analogous results in certain particular cases.

Let us suppose an ellipsoid of an isotropic substance placed in a uniform field, and that the coefficient of magnetism is uniform. The magnetic moments parallel to the three axes are respectively proportional (**388**) to

$$\frac{k}{1+k\mathrm{L}}, \qquad \frac{k}{1+k\mathrm{M}}, \qquad \text{and} \qquad \frac{k}{1+k\mathrm{N}}.$$

These three factors, which we shall denote as f, f', and f'', may be considered as coefficients of mean magnetisation; it is seen that the ellipsoid behaves as a crystallised body, having for principal coefficients of magnetisation the same quantities f, f', and f'', and all the preceding theorems are applicable. In particular, the couples for the three axes are respectively proportional to the differences $f-f'$, $f'-f''$, and $f-f''$.

This is also the case for a parallelopipedon of dimensions $2a$, $2b$, and $2c$; the couple of rotation about a line parallel to the edges only depends on the difference of the mean coefficient of magnetism parallel to the two systems of edges perpendicular to the first.

This is the case of a magnet which oscillates in the terrestrial field. If f and f' are the mean coefficients of longitudinal and transverse magnetism, M the rigid magnetic moment of the magnet, and u its volume, the directing couple is

$$\mathrm{MH} + u\mathrm{H}^2(f-f') = \mathrm{MH}\left[1 + \frac{u(f-f')}{\mathrm{M}}\mathrm{H}\right].$$

The factor ϕ, which has been used previously (1140) to represent the influence of the magnetisation induced by the earth in the oscillations, is then proportional to the difference $f-f'$, and not merely to the magnetisation parallel to the length of the magnet.

If the bar turns about a line parallel to the edges c, the components of the induced magnetic moment are $u\mathrm{H}f\lambda$ and $u\mathrm{H}f'\lambda'$. It may be decomposed into two others, one m parallel to the greatest length a, the other m' parallel to the field; m and m' satisfy the conditions

$$m + m'\lambda = u\mathrm{H}f\lambda,$$
$$m'\lambda' = u\mathrm{H}f'\lambda',$$

which give

$$m' = u\mathrm{H}f',$$
$$m = u\mathrm{H}(f-f')\lambda.$$

It will be seen that the magnetic moment m' parallel to the field is a constant, and that the moment m is sensibly constant when the deflections are very small; the first does not give any couple, and the second may be considered as adding to the rigid magnetisation M (1140).

Generally, whatever be the form of an isotropic body placed in a uniform field, there are always three rectangular directions for which the magnetisation is parallel to the field, and may be defined by the mean coefficients f, f', and f''. These determine the same ellipsoid of induction

$$fx^2 + f'y^2 + f''z^2 = 1,$$

as if it were the case of an anisotropic body defined by the coefficients of magnetisation f, f', and f''.

1231. MARINER'S COMPASS.—The term *compass* is usually applied to the magnetic apparatus used in navigation. It consists of a needle or system of needles rotating on a point; the needle supports a card, on which is a line of sight corresponding to the magnetic axis and a series of circular divisions. The line of sight does not usually coincide with the magnetic meridian; the iron and steel of which the ship is constructed, or which forms part of its equipment, produce a deviation which it is necessary to correct or to compensate. The correction of the compass is based on a theory which was propounded by Poisson.*

* POISSON. *Mem. de l'Institut.*, Vol. v., p. 521. 1824.

The masses of steel or hard iron, which have been magnetised during the construction, behave like magnets; this magnetism is sometimes called *sub-permanent*, for its value, which depends on the position of the ship during building, diminishes slowly at sea, and only becomes stationary after the vessel has made a certain number of voyages. Soft iron, on the other hand, becomes magnetised by the earth; the temporary magnetisation thus produced varies with the direction of the ship and its geographical position. There are thus two kinds of disturbing actions—one set constant, the other varying with the direction of the ship.

Let us first assume that the ship is upright. The *head* of the ship is the plane of symmetry from stern to bow—that is to say, from back to front. Let

ζ be the azimuth of the ship's head with the magnetic meridian, or the *magnetic course*, this angle being counted towards the east;

ζ' the azimuth of the head with the direction of the compass, or the *compass course;*

$\delta = \zeta - \zeta'$ the *deviation* of the compass.

The angle V which the compass makes with the geographical meridian, or the apparent declination, is sometimes called the *variation*. If Δ is the real declination, we have

$$V = \Delta + \delta.$$

We shall assume that both the permanent and sub-permanent magnetism of the ship are independent of the temperature; the variations which it might undergo from this may be neglected in comparison with other sources of error.

It will be assumed that for masses of soft iron the induced magnetisation is proportional to the magnetising force, from which it will follow that magnetisations produced by different causes will superpose themselves; these conditions are sufficiently realised with actions of the same order as that of the earth.

We shall finally suppose that the compass needle is very small compared with its distance from the nearest masses of iron or steel, and therefore that it moves in a sensibly uniform field. It is sufficient, therefore, to calculate the disturbance of the terrestrial field at the centre of the compass.

1232. Let us consider three rectangular co-ordinates, one x along the ship's head, y from larboard to starboard—that is, towards the right—and z downwards towards the keel. This latter axis is vertical, and the two former are horizontal when the ship is on an even keel.

The permanent magnetism, and the induced magnetisation which it gives rise to in masses of soft iron, produce on the compass a force which is constant in magnitude and direction in reference to the ship; let P, Q, and R be the projections on the three axes. When the distribution of the magnets and of the soft iron is symmetrical, the value of Q is very small, and the horizontal component $F_1 = \sqrt{P^2 + Q^2}$ is sensibly parallel to the plane of symmetry if the compass itself is situated in this plane. But the component Q has appreciable values whenever the head of the vessel on the building slips was not in the magnetic meridian; the force F_1 makes then with the head an angle α which is called the *starboard angle.*

If this were the only action, then if H is the horizontal component of the earth, H_1 the resultant of the forces H and F_1, and δ_1 the deviation or the angle of the resultant H_1 with the magnetic meridian,

$$\frac{\sin(\zeta+\alpha)}{H_1} = \frac{\sin\delta_1}{F_1} = \frac{\sin(\zeta+\alpha-\delta_1)}{H}.$$

The deviation δ_1 is null when the head is in the azimuth $-\alpha$ or $\pi-\alpha$, and it changes sign on passing from one side to the other of this direction; this is a *semicircular* deviation. If the ratio of the forces F_1 and H is very small, we may write approximately

$$(10) \qquad \sin\delta_1 = \frac{F_1}{H}\sin(\zeta+\alpha) = \frac{P}{H}\sin\zeta + \frac{Q}{H}\cos\zeta.$$

The deviation is sensibly in the inverse ratio of H; the sign is the same all the world over, and the same value in all points of a magnetic parallel.

1233. For the terrestrial field, we may first of all observe that the vertical component Z produces a magnetisation independent of the direction of the ship, and that its action on the compass gives a component situate in the plane of symmetry. The corresponding deflection is then *semicircular* also; it becomes zero at the equator, and changes sign on passing from one hemisphere to the other.

In order to calculate the effect of the horizontal component H, we shall replace it by its two projections

$$X = H\cos\zeta, \qquad Y = -H\sin\zeta.$$

The magnetisation produced by the former gives an action the horizontal component of which, F_2, is in the plane of symmetry. If we put

$$F_2 = aH\cos\zeta,$$

and let H_2 be the resultant of the forces H and F_2, the deviation δ_2 due to the horizontal component will be defined by equations

$$\frac{\sin\zeta}{H_2} = \frac{\sin\delta_2}{F_2} = \frac{\sin(\zeta-\delta_2)}{H},$$

which give sensibly

(11) $$\sin\delta_2 = \frac{a}{2}\sin 2\zeta.$$

This deflection is null when the head is directed towards one of the four cardinal magnetic points, and its signs are different in two adjacent quadrants; this is a *quadrantal* deviation.

The projection $Y = -H\sin\zeta$ gives, in like manner, a horizontal action

$$F'_2 = -eH\sin\zeta.$$

The corresponding deflection δ'_2 is sensibly determined by the equation

(12) $$\sin\delta'_2 = -\frac{e}{2}\sin 2\zeta;$$

this is still a quadrantal deviation.

1234. When the disturbing actions are weak, it may be assumed that the deflections produced by each of them simply add themselves, and the total deflection may be represented by an expression of the form

(13) $$\sin\delta = (A) + B\sin\zeta + C\cos\zeta + D\sin 2\zeta + (E\cos 2\zeta).$$

The two terms in brackets are usually very small, since they simply arise from want of symmetry in the distribution of masses of iron or steel, or the disymmetric position of the compass on board.

We may, lastly, to the same degree of approximation, replace, in the second member, the real azimuth of the head, which is unknown, by the apparent azimuth ζ'; the deflection is then expressed by the formula

$$(14)\quad \sin\delta = A + B\sin\zeta' + C\cos\zeta' + D\sin 2\zeta' + E\cos 2\zeta',$$

the coefficients of which will have to be determined experimentally. The first term A is the mean value of the deviation; the two following represent a semicircular deviation, and the two latter a quadrantal deviation.

1235. There is still to be considered the action produced by the obliquity of the vessel, or the *heeling error*. Let i be the inclination of the ship on the starboard, or towards the right. The components of the terrestrial action should be replaced by

$$Y_i = Y\cos i + Z\sin i,$$
$$Z_i = Z\cos i - Y\sin i,$$

and the component X does not vary. In this case each of the coefficients A, C, and D contains a term sensibly proportional to the inclination.

1236. In order to make the calculation more completely, we may observe that, when the ship is on an even keel, and assuming the superposition of the different magnetisations, the components X′, Y′, and Z′ of the resultant field may be represented by the expressions

$$(15)\quad \begin{aligned} X' &= X + P + aX + (bY) + cZ, \\ Y' &= Y + Q + (dX) + eY + (fZ), \\ Z' &= Z + R + gX + (hY) + kZ, \end{aligned}$$

in which the parameters a, b, c, d.... refer to the induced magnetisation of the earth. We have put in brackets the terms which are very small, for they destroy themselves in the conditions of symmetry of the ship and of the compass.

If H′ is the horizontal component of the resultant field, I the magnetic inclination, we have

$$X = H\cos\zeta, \qquad Y = -H\sin\zeta, \qquad Z = H\tan I.$$
$$X' = H'\cos\zeta', \qquad Y' = -H'\sin\zeta'.$$

Putting again

$$P = pH, \qquad Q = qH, \qquad R = rZ;$$
$$H' = h'H, \qquad Z' = z'Z.$$

Substituting these values in the preceding expressions, we obtain

$$h' \cos \zeta' = (p + c \tan I) + (1 + a) \cos \zeta - b \sin \zeta,$$
$$-h' \sin \zeta' = (q + f \tan I) - (1 + e) \sin \zeta + d \cos \zeta,$$
$$z' = (r + 1 + k) + g \cot I \cos \zeta - h \cot I \sin \zeta.$$

These equations give the components of the resultant field as a function of the terrestrial field of the permanent magnetism, of the parameters, and of the ship's head.

In order to get the deviation itself, the two first equations are added after having multiplied them respectively by $\sin \zeta$ and $\cos \zeta$, the second is then subtracted from the first after having multiplied them respectively by $\cos \zeta$ and $\sin \zeta$, so that the former member contains the sine and cosine of the angle $\zeta - \zeta' = \delta$. We thus find, after a few simple reductions,

$$h' \sin \delta = \frac{d-b}{2} + (p + c \tan I) \sin \zeta + (q + f \tan I) \cos \zeta$$
$$+ \frac{a-e}{2} \sin 2\zeta + \frac{b+d}{2} \cos 2\zeta,$$
$$h' \cos \delta = 1 + \frac{a+e}{2} + (p + c \tan I) \cos \zeta - (q + f \tan I) \sin \zeta$$
$$+ \frac{a-e}{2} \cos 2\zeta - \frac{b+d}{2} \sin 2\zeta.$$

Dividing these two equations by the quantity

$$\lambda = 1 + \frac{a+e}{2},$$

and then the first by the second, we get

$$\frac{\sin \delta}{\cos \delta} = \frac{A + B \sin \zeta + C \cos \zeta + D \sin 2\zeta + E \cos 2\zeta}{1 + B \cos \zeta - C \sin \zeta + D \cos 2\zeta - E \sin 2\zeta}.$$

where A, B, C, D, and E are coefficients, the meaning of which follows from the course of the calculation.

Removing the denominator and replacing the angle $\zeta - \delta$ by ζ', we have finally

$$(16)\ \sin\delta = A\cos\delta + B\sin\zeta' + C\cos\zeta' + D\sin(2\zeta' + \delta) + E\cos(2\zeta' + \delta).$$

It will be seen that, by definition, if we refer to the original equations, the coefficients A and E are very small. Disregarding δ in the terms of correction, we find again the expression (14).

We may also replace $\cos\delta$ in equation (16) by unity, which gives

$$\sin\delta = \frac{A + B\sin\zeta' + C\cos\zeta' + D\sin 2\zeta' + E\cos 2\zeta'}{1 - D\cos 2\zeta' + E\sin 2\zeta'},$$

the term $E\sin 2\zeta'$ in the denominator being negligable.

Without insisting on the methods used for determining, the values of the coefficients, we may add that this mode of correction gives excellent results when the deflection does not exceed 20°; but it becomes very difficult to apply when the deviations are considerable, as is often the case for ships which contain large masses of iron and steel.

1237. The action of the ship on the compass is really equivalent to that of a magnet, which would produce the components P, Q, R, and of a mass of soft iron placed near the compass in a determinate direction and at a convenient distance, provided that the action of the compass itself on this mass of soft iron produces no induced magnetisation capable of producing a distinct perturbation by its reaction on the compass.

It is therefore possible to compensate exactly the action of the vessel, by placing in a fixed position near the compass a magnet the components of whose field are $-P$, $-Q$, and $-R$, and a mass of soft iron which counterbalances the magnetisation of the mass of soft iron by the earth; the deflection will be neutralised. It is, however, difficult to arrange the compensation in this manner by methodical trials, and in practice it is better to use several magnets by which the various terms of the deflection are separately neutralised.

This mode of correction is due particularly to Sir George Airy.* It may first of all be observed that the vertical components and the

* G. Airy. *Phil. Trans.* 1856.

parameters g, h, k, do not affect the deviation of a ship on an even keel; we may then annul the components P and Q and the other parameters. Longitudinal and transverse magnets are arranged on the deck, which separately compensate P and Q, and then bars of soft iron, or boxes of chains, are added, which compensate the magnetisation induced by the earth.

Ordinary compasses have long and highly magnetised needles, so that it is necessary to keep the correcting instruments at such a distance that they produce on the needle an almost uniform field, and thus avoid the reaction of the induced magnetism. We are thus led to use very powerful magnets and great masses of iron.

1238. In order to approximate more closely to the theory, Sir W. Thomson uses, on the contrary, a series of small needles connected by silk threads to a very light card of paper or of mica, so that the whole does not weigh more than 30 grammes. Although these needles are moreover very slightly magnetised, the moment of inertia of the system is so small that the compass rapidly acquires its position of rest, without having recourse to artificial means as in compasses with liquid. The smallness of the needles and their small magnetic moment render it possible to bring the correcting parts much nearer; the magnets are placed in the compass box itself, and the soft iron is formed of two symmetrical spheres on the binnacle, at the two ends of a horizontal diameter passing through the needle.

The component Q is compensated by a transverse magnet, and the component P by a pair of longitudinal magnets symmetrically placed under the compass in respect of the vertical. The *semicircular* error is thus eliminated.

The quadrantal error is got rid of by two spheres placed at a convenient distance, and at right angles to the plane of symmetry of the ship in the usual case, or in an oblique direction if the coefficient E is not null—that is to say, if the quadrantal error itself is oblique. This latter circumstance occurs when the compass is not in the plane of symmetry, or that the distribution of iron on board is not symmetrical. In this case there is a constant error A independent of the direction of the ship, and which may be determined once for all, if it is not preferred to correct it by a simple rotation of the binnacle.

A vertical magnet, lastly, corrects the *heeling error*.

The correction is easy when the ship can be swung in sight of land, or with a clear sky, for the head can then be put in different directions, which separately neutralise the several terms.

The correction of the *quadrantal* deviation by masses of iron is easy, and when once made it is correct for all magnetic latitudes.

This is not the case for the *semicircular* deflection corrected with magnets. Experiment made in one and the same place does not allow us to separate the subpermanent magnetism in the coefficients B and C, from the magnetism produced by the vertical component of the earth. As this latter varies with the latitude, the correction for a given place is inexact away from this place.

Sir W. Thomson remedies this defect by correcting the semicircular deflections both with magnets and with a vertical soft iron rod called *Flinder's bar;* at starting an approximate correction is made, and it is modified during the voyage by varying the ratio of the bar and the magnets. When the value of this ratio has been found by trial, the correction holds good for all ultimate displacements.

An analogous remark would apply to the correction of the heeling error, but no account is taken of it in practice.

When the correction is complete, the compass is only under the action of the terrestrial field. By means of a *deflecting* magnet, put on the lid of the compass in a given position, it is then ascertained that the deflection produced when the deflector is at right angles with the needle is independent of the direction of the vessel.

The heeling error is itself corrected if $Z' = Z$; that is to say, if the apparent inclination determined on board is equal to the magnetic inclination of the place.

These two methods of verification give a means of re-establishing the correction even at sea, and assuming that the state of the sky prevents a determination of the geographical meridian by astronomical observations. It is sufficient for this purpose that the deviation produced by the deflector is independent of the head, and that the inclination measured on board is equal to the real inclination which is known by the approximate position of the vessel.

PART IV.

COMPLEMENT.

CHAPTER I.

INDUSTRIAL APPLICATIONS.

1239. GENERAL CHARACTER.—The numerous applications to which electricity gives rise may be classed in different groups.

The shocks produced by electric sparks were the starting point for what is now a very extended branch of science, electrophysiology, which comprises the effects of electricity on living beings, and the investigation of the electromotive forces of which their organs are the seat.

In another series of applications use is made of the rapidity with which electrical phenomena are propagated in a conductor. The work of currents is sometimes directly utilised to produce a mechanical effect at a distance, as in bell work and direct telegraphs; but most frequently the electricity only serves to produce a signal, or by a sort of detachment sets to work other mechanical forces, such as the action of a weight, a spring, or of a local battery; such are most of the telegraphs, registering instruments, microphones, etc. In all cases the work, and the expense of the electrical energy utilised, are of no great importance; all the progress to be realised in this kind of apparatus lies in the rapidity of the signals and the certainty with which the various parts work.

We shall restrict ourselves to those applications in which it is especially proposed to utilise the electrical energy itself, as in galvanoplastics, lighting, or the transmission of force. In these different cases, we are concerned with the amount of work utilised and its economical efficiency.

1240. Electrostatic Machines.—In apparatus for producing statical electricity, as in frictional or multiplying machines (**194**), the motion produces a continuous transport of electricity between the two conductors which join the collectors or the poles of the machine. If the poles are connected by a conductor, the action of the machine maintains a continuous current in the conductor, and the difference of potential of the poles is feeble. If the intermediate conductor is connected by a spark discharger, a series of discharges is produced, and the difference of potential of the poles increases periodically and disappears at each spark. In all cases the efficiency of the machine is the quantity of electricity which flows in unit time.

The *striking distance*, which increases with the difference of potential, is only limited by the dimensions of the parts and the distance of the collectors.

The *quantity of electricity* produced is almost independent of the striking distance, so long as this is small compared with the greatest striking distance—that is to say, a few millimetres in ordinary machines—but it then diminishes rather rapidly. Other things being equal, the quantity is proportional to the velocity of rotation, or, more exactly, to the extent of surface of the plates or cylinders which are utilised by the comb in unit time. The ratio of the quantity to the surface utilised appears greater for reaction machines than for frictional machines, and varies notably with the nature of the glass.

It is interesting to estimate the quantity of work transformed into electricity in an electrostatic machine. A Holtz machine with two rotating plates 60 cm. in diameter* was used to charge a battery, the electrostatic capacity of which was 225 metres or 22,500 (C.G.S.) —that is to say, in practical units (**613**),

$$\frac{225.10^2}{3^2.10^{20}} = 0{\cdot}025.10^{-15}, \qquad \text{or} \qquad 0{\cdot}025 \text{ microfarads.}$$

With seven turns of the machine, the charge of the battery was sufficient to produce a spark of 0·1 cm. in a discharger the balls of which were 2·2 cm. in diameter, which corresponds to a difference of potential (**822**) of 5,490 volts.

* Mascart. *Traité d'Électricité Statique*, Vol. II., p. 324.

The charge of the battery in coulombs was therefore

$$0{\cdot}025.10^{-6}.5490 = 137{\cdot}25.10^{-6},$$

which is about 0·00002 coulombs for one turn.

If the machine made 10 turns in a second, the current will be 0·0002 ampère; this is almost the result obtained by connecting the poles by a galvanometer, the wires of which are suitably insulated.

In the preceding case, the work utilised, U, is equivalent to the discharge of the battery, which gives in kilogrammetres

$$U = \frac{137{\cdot}25.10^{-6}.5490}{2 \times 9{\cdot}81} = 0{\cdot}0384 \text{ kgm.}$$

The work utilised for each turn is then 0·0055 kgm., and it would be twice as much if the difference of potential was constant. For a velocity of 10 turns, the work per second will be 0·11 kgm.; it would be about $\frac{1}{682}$ of a horse-power.

The electrical work, however, which corresponds to the working of the machine increases greatly with the striking distance. In order to form some idea of this, we will suppose that the product is constant, and that the poles are separated, so as to keep unchanged the difference of potential which gives the maximum striking distance; this was 22 cm., corresponding to about 133,000 volts. In this case the work per second, and for the same velocity of 10 turns, will be $\frac{133{,}000}{5490} = 24{\cdot}68$ times as much; it will represent then $\frac{1}{27{\cdot}6}$ of a horse-power.

These considerations show that the energy utilised by these electrostatic machines is really very small. It is observed that, with reaction machines, the work of keeping up their motion increases when they are primed; but the greater part of the work expended is absorbed in friction and in losses of electricity.

1241. On Voltaic Batteries. — In an ordinary battery, the energy is withdrawn from the work of chemical action. Let us suppose that the battery consists of N identical couples, associated in q parallel series of n elements each; let e be the electromotive force and r the resistance of each element; the electromotive force of the battery is $E = ne$, and its internal resistance $R_0 = \frac{nr}{q}$.

If the intensity of the current is I, the work done by the chemical action in each unit of time is

$$W = EI,$$

and the external utilisable work

$$U = EI - R_0 I^2 = EI\left(1 - \frac{R_0 I}{E}\right) = R_0 I\left(\frac{E}{R_0} - I\right);$$

the efficiency u is then

$$u = \frac{U}{W} = 1 - \frac{R_0 I}{E}.$$

If $I_0 = \frac{E}{R_0}$ is the intensity of the current corresponding to an external work null, the expressions for U and u become

$$U = R_0 I (I_0 - I),$$
$$u = 1 - \frac{I}{I_0} = \frac{I_0 - I}{I_0}.$$

The external work U first increases, and then diminishes, when the current diminishes, starting from its maximum value I_0 to zero; the efficiency, on the contrary, always increases in the same conditions, and tends to unity, as the useful work tends towards zero.

The sum of the two factors I and $I_0 - I$ being constant, the external work attains its maximum value U_m for the condition $I_0 = 2I$. The efficiency is then equal to 0·50, and we have

$$U_m = \frac{R_0 I_0^2}{4} = \frac{E^2}{4R_0} = nq\frac{e^2}{4r} = N\frac{e^2}{4r}.$$

The maximum work which can be utilised per second is then simply proportional to the number of couples, whatever is their arrangement, and the value of the maximum work of each couple, or its mechanical power, is $\frac{e^2}{4r}$. This number is characteristic of the couple in question. The value of e only depends on the nature of the bodies which constitute the couples; that of r depends, on the contrary, on the shape of the elements, and on their dimensions.

In the ordinary couples, the electromotive force e rarely attains 2 volts. The resistance r varies, on the contrary, within wide limits; the smallest values are obtained with Grove's or Bunsen's elements, and accordingly they are used when the battery has to do much work.

The mechanical power which corresponds to unit of electrical work per second is now often called a *watt;* according to this definition, a watt is equal to $\frac{1}{9{\cdot}81} \times \frac{1}{75}$, or about $\frac{1}{736}$ of a horse-power. A Bunsen's element, for instance, for which $e = 1{\cdot}8$ volt, and $r = 0{\cdot}01$ ohm, would have a mechanical power of 81 watts, or $\frac{1}{9}$ of a horse-power.

1242. Chemical Expenditure.—Let p be the chemical equivalent of one of the bodies which plays a part in the action—that is to say, the weight of this body, which corresponds to an ampère per second, or a coulomb; for a total current of intensity I, the weight of this body, which comes into play in each couple in unit time, is $p\frac{\mathrm{I}}{q}$. The weight P for the entire pile, for the time t, is then

$$\mathrm{P} = \mathrm{N}p\frac{\mathrm{I}}{q}t = np\mathrm{I}t,$$

and the energy corresponding to this chemical action, expressed in kilogrammetres, is

$$\mathrm{W} = \frac{\mathrm{EI}t}{9{\cdot}81} = \frac{e}{9{\cdot}81}\frac{\mathrm{P}}{p}.$$

The chemical energy, as might be foreseen, is independent of the time of the operation, of the number, and arrangement of the

elements; it only depends on the electromotive force of each couple, and on the number $\frac{P}{p}$ of equivalents concerned.

The work of a horse-power during an hour, or the horse-power-hour, being $75 \times 36,000 = 270,000$ kgm., the number of chemical equivalents corresponding to a horse-power is

$$\frac{P}{p} = \frac{9\cdot81}{e} 270,000 = \frac{2\cdot65}{e} 10^6,$$

which for hydrogen, with an electrochemical equivalent of $0\cdot1035\cdot10^{-4}$, is equal to

$$\frac{2\cdot65 \times 10\cdot35}{e} = \frac{27\cdot42 \text{ gr.}}{e}.$$

This value for hydrogen would enable us to calculate the corresponding weights of different bodies, provided the reactions were well known; but it frequently happens that the effects are rather complicated. This, for instance, is the case with nitric acid, which serves as depolarizer in Bunsen's element. The passage of a coulomb destroys then an equivalent, or a third of an equivalent, according as the acid is transformed into hyponitric acid, or into bioxide of nitrogen; we may then take as mean half an equivalent. We thus find, for two elements of frequent use:—

Expenditure for a Horse-Power-Hour.

Daniell's Element ($e = 1$).

	Kgm.
Zinc ($Zn = 33$)	$= 0\cdot905$
Sulphuric Acid ($SO^3,HO = 49$)	$= 1\cdot344$
Sulphate of Copper ($CuO,SO^3 + 5HO = 124\cdot75$)	$= 3\cdot420$

Bunsen's Element ($e = 1\cdot8$).

	Kgm.
Zinc ($Zn = 33$)	$= 0\cdot503$
Sulphuric Acid ($SO^3,HO = 49$)	$= 0\cdot746$
Nitric Acid ($NO^5,4HO = 90$)	$= 0\cdot685$

The sulphate of zinc may be considered as a useless product in both cases. It is possible, on the other hand, in Daniell's cell, to reproduce the sulphate of copper from the copper thrown down,

so that the expenditure reduces itself to that of the corresponding sulphuric acid.

We may estimate the price of these various substances at 4½*d.* each for the zinc and nitric acid, and 1*d.* for the sulphuric acid; but the cost of amalgamating the zinc, so as to protect it from waste, will double the price. We have then

COST OF THE HORSE-POWER-HOUR.

For Daniell's Cell	10*d.*
„ Bunsen's „	7½*d.*

It is important to observe that only a fraction of this energy can be utilised. For the conditions of maximum work, all the preceding numbers should be doubled.

The comparison of this expenditure with that of steam engines shows how illusory it would be to seek for any economy in the use of batteries as sources of energy. Moreover, the liquids are only useful when they have the proper concentration, or when they are not charged with other bodies in solution; and, lastly, local actions are always produced on the electrodes, which represent an unprofitable expenditure of energy. All these causes contribute to increase the cost of this source of electrical energy.

1243. AVAILABLE WORK.—The work required from the current consists in overcoming either a resistance R′, or an electromotive force E′, or both together; if the intensity is I, the useful work is $R'I^2$, $E'I$, or $R'I^2+E'I$, according to circumstances. Moreover, wires offering a resistance ρ are necessary for connecting the electromotor with the points where the work is done, so that the total unutilised resistance is

$$R=R_0+\rho=\frac{nr}{q}\left(1+\rho\frac{q}{n}\right).$$

The work U really utilised is then $U \doteq EI-RI^2$, and the expression for the efficiency

$$u=\frac{U}{W}=1-\frac{RI}{E}.$$

The maximum useful work, lastly, corresponding to a yield of 50 per cent., is

$$U = \frac{1}{4}\frac{E^2}{R} = N\frac{e^2}{4\left(r+\rho\frac{q}{n}\right)}.$$

It will be observed that the useful work

$$U = E'I + R'I^2 = (E' + IR')I = \left(R' + \frac{E'}{I}\right)I^2$$

is equivalent to that which would be absorbed either by an electromotive force $E_1 = E' + IR'$, or by a resistance $x = R' + \frac{E'}{I}$, and that the efficiency may be written

$$u = \frac{E_1}{E} = \frac{x}{R+x}.$$

For an efficiency of 50 per cent., the useful resistance x should be equal to the useless resistance R, or the useful electromotive force E_1 equal to half that of the battery.

Equations

$$W = EI = (R+x)I^2,$$
$$U = EI - RI^2 = xI^2 = E_1I,$$
$$u = \frac{U}{W} = \frac{E_1}{E} = \frac{x}{R+x},$$

contain several quantities which give rise to a great number of problems, when there are only three unknown quantities, and when any relations are established between them, provided these relations are compatible with the physical phenomenon.

We may add that the same considerations apply to an electromotor of any nature whatever, chemical or mechanical, which is characterised by a constant electromotive force E, and a constant resistance R_0.

1244. Choice of Conductors.—If S is the section of a conductor, σ its specific resistance, the resistance for unit length is

$\frac{\sigma}{S}$, and the corresponding loss of energy per second is $\frac{\sigma}{S}I^2$. Two conductors of different metals are therefore equivalent for the same current when the sections are proportional to the specific resistances. With iron and copper, for instance, these sections should be in the ratio of 6 : 1.

We ought not, however, to conclude that in all cases the best conducting metals and the greatest sections should be used, for the price of the metal and the cost of establishing the wire play an important part.

In the case of bare conductors the most economical conditions are furnished by a given strength of current for each metal. Let

P be the price of unit work in the conditions given by the chemical or mechanical electromotor,
n the number of seconds in a day (86400),
f the fraction of the time in which the current is utilised,
Q the price of unit volume of the metal,
t the rate at which the interest and the redemption of capital is estimated,
N the number of days in the year.

The energy lost in heating the conductor per day, and for unit length is equal to $fn\frac{\sigma}{S}I^2$, and the corresponding expense $Pfn\frac{\sigma}{S}I^2$.

On the other hand, the price of unit length of the conductor is QS, and the cost per day $QS\frac{t}{100N}$.

The total expense D per day and unit length is then

$$D = QS\frac{t}{100N} + Pfn\frac{\sigma}{S}I^2.$$

The section of the conductor which is adapted to reduce this expense to a minimum is defined by the condition that the differential of the second member is small—that is to say, that the two terms

are equal. In this case the cost of interest and of redemption represent a sum which is equal to the expense for the energy lost, and we have

$$I^2 = \frac{S^2}{100} \frac{Qt}{NnP} \frac{1}{f\sigma},$$

$$D^2 = 4I^2 \frac{ftn}{100N} PQ\sigma.$$

It will be seen that, for a total minimum expense, the density of the current $i = \frac{I}{S}$ only depends on the metal and the cost of the energy, and that the relative expense for the same current is independent of the section of the wire.

With two different metals, the prices of which per kilogramme are respectively k and k', the densities d and d', and the rates of redemption t and t', the ratio of the densities of the current i and i', and that of the corresponding expenditures D and D′ are

$$\frac{i}{i'} = \sqrt{\frac{Qt}{Q't'} \frac{\sigma'}{\sigma}} = \sqrt{\frac{Kdt}{K'd't'} \frac{\sigma'}{\sigma}},$$

$$\frac{D}{D'} = \sqrt{\frac{Qt\sigma}{Q't'\sigma'}} = \sqrt{\frac{Kdt\sigma}{K'd't'\sigma'}}.$$

It may be estimated that, for equal volumes, the price of copper of good quality is 10 times the cost of iron. If, in the preceding expressions,

$$\frac{Q}{Q'} = \frac{Kd}{K'd'} = 10,$$

and, assuming that the redemption is the same in both cases, we find

$$\frac{i}{i'} = 1{\cdot}299, \qquad \frac{D}{D'} = 7{\cdot}74.$$

The density of the current will therefore be a little greater with copper than with iron, but the expense is much greater. We may, however, remark that for the same current the sections of wires are inversely as the values of i and i'; the weight of iron

is then greater, which increases the cost of installation, and moreover the metal rapidly deteriorates, which necessitates a higher rate of redemption.

These considerations do not hold for wires covered with insulators; the cost of insulation increases in that case the value of the conductors, and the use of copper is then justified.

In order to determine the density of the current numerically, the cost of the energy must be estimated. The kilogrammetre is worth $9{\cdot}81 . 10^7$ units of work (612) C.G.S., and one horse-power can produce $75 \times 9{\cdot}81 . 10^7 = 74 . 10^8$ units of work per second.

If C is the price of a horse-power working continuously for a year, the price of unit work is

$$P = \frac{C}{Nn} \frac{1}{74 . 10^8},$$

which gives

$$NnP = \frac{C}{74 . 10^8}.$$

The most advantageous current then becomes

$$i = \frac{I}{S} = \frac{1}{10} \sqrt{\frac{Qt}{f\sigma} \frac{74 . 10^8}{C}} = 10^3 \sqrt{\frac{74 Qt}{Cf\sigma}}.$$

The specific resistance of copper is about 1600. Assume that the price of copper is $1{\cdot}75$ francs per kilogramme, or $0{\cdot}0156$ per cubic centimetre, and that the price for a horse-power-year is 250 francs. The electromotor being produced by mechanical means, let us suppose that the current is used uninterruptedly, and that the redemption is 10 per cent.; we shall then have $t = 10$, $f = 1$, which gives

$$i = \sqrt{\frac{74 \times 0{\cdot}156}{0{\cdot}25 \times 1{\cdot}60}} = 5{\cdot}37.$$

The current will be $5{\cdot}4$ C.G.S. units per square centimetre or 54 ampères—that is to say, $\cdot 54$ ampères per square millimetre.

We may add that these results only apply to the ordinary conditions; for the cost of installation, and especially the use of insulating materials, tends to raise the price Q of unit volume of the metal. Moreover, the factor f is less than unity when the

apparatus do not work continuously, which is the most frequent case—for instance, in lighting. In practice, much denser currents are used, attaining 3 to 4 ampères per square millimetre.

1245. LIGHTING BY INCANDESCENCE.—In incandescent lighting, the calorific energy of the current is utilised in raising to a very high temperature a filament of carbon placed in a vacuum.

The luminosity rapidly increases with the temperature, but in practice it is necessary to keep the heating considerably below the temperature at which the wire breaks or rapidly deteriorates.

When the temperature is in equilibrium, the energy of the current is compensated by radiation. The ratio of the quantity of light emitted by the lamp to the energy expended is a function of the temperature alone, and does not depend on the shape of the carbon threads, provided they have the same emissive power.

Experiment shows, for example, that good conditions of lustre and duration are obtained with incandescent lamps, equal to about 1·71 carcel, or 12·5 candles, when they are traversed by a current of 0·8 ampère with a difference of potential of 100 volts at the terminals.

The resistance of each lamp is then $\frac{100}{0 \cdot 8} = 125$ ohms, and the energy necessary for maintaining it is $100 \times 0 \cdot 8 = 80$ units of work per second, or 8·155 kgm.; the electrical energy equivalent to a horse-power will then be capable of producing an illumination of $\frac{75 \times 1 \cdot 71}{8 \cdot 155} = 15 \cdot 73$ or 16 carcel lamps. Experiments at the Electrical Exhibition in 1881 gave 12 to 22 carcels.

If the lighting were done by Bunsen's elements with a yield of 80 per cent., the price per electrical horse-power would be $\frac{0 \cdot 92}{0 \cdot 8} = 1 \cdot 15$ francs, and that of a carcel 0·072 francs.

The lamps are usually arranged as shunts on the circuit. Supposing there are m identical lamps; let r be the resistance of each, i the current which traverses it, E the electromotive force of the electromotor, R its resistance with the connections. The resistance of the whole of the lamps is $x = \frac{r}{m}$, and we have

$$E = (R + x) mi = (mR + r) i,$$

$$u = \frac{x}{R + x} = \frac{r}{mR + r}.$$

1246. For the same expenditure of energy incandescent lamps give far more light when the difference of potential at the terminals is increased, but they waste more rapidly. In conditions which are near those practically used, the luminosity is nearly proportional to the 6th power of the difference of potential V at the terminals, and the waste is proportional to the 25th power of V.*

Let H be the number of hours that a lamp lasts, L its luminous intensity in carcels, and W the energy expended per hour; if we assume that these different quantities are proportional to powers of V, we may put generally

$$L = AV^{\alpha}, \qquad H = BV^{-\beta}, \qquad W = CV^{\gamma}.$$

If P is the price of the lamp, and Q the price of unit of electrical work, taking into account the fuel or the consumption of products, according as we deal with mechanical electromotors, or batteries, together with the sinking fund for the installation, and the yield. The expense per carcel-hour is $p = \frac{P}{HL}$ on the part of the lamp, and $q = \frac{QW}{L}$ for the electromotor; the total expense

$$p + q = \frac{P}{AB} V^{\beta - \alpha} + \frac{QC}{A} V^{\gamma - \alpha}$$

is evidently a minimum for conditions which make the differential of the second member in respect of V null—that is to say, when we have

$$(\beta - \alpha) p = (\alpha - \gamma) q.$$

The ratio of the expense of the lamps to the total expense is then

$$\frac{p}{p+q} = \frac{\alpha - \gamma}{\beta - \gamma}.$$

The exponent γ will be equal to 2, if the resistance of the carbon were constant; it is really greater, and we shall have a higher

* FLEMING. *Phil. Mag.* [5], Vol. XIX., p. 368. 1885.

limit if we make $\gamma=2$. On this supposition, and with the above numbers, we get

$$\frac{p}{p+q}=\frac{6-2}{25-2}=0{\cdot}174.$$

It is remarkable that this ratio is independent of the nature and price of the electromotor, the luminosity of the lamps forming the compensation.

1247. GALVANOPLASTICS.—The useful work in the galvanic deposition of metals consists solely of an electromotive force to be overcome. In practice this electromotive force is extremely feeble when the positive electrode is formed of the metal to be deposited; there is then as much dissolved on the one side as is deposited on the other, and the solutions retain their strength. The useful effect $u=\frac{E_1}{E}$ increases as the electromotive force of the battery diminishes. In order to obtain an important effect, without diminishing the efficiency too much, the internal resistance of the battery and the resistance of the connections must be diminished as much as possible.

1248. ELECTRICAL ARC.—Between the carbons which produce the electrical arc there is a fall of potential varying from 40 to 70 volts, according to the strength of the current. M. Edlund* has shown, and all experiments have since confirmed, that the greatest part of this fall is due to an inverse electromotive force which may be estimated at 30 volts; the rest is due to the resistance of the interposed layer of gas, a resistance which varies with the length of the arc and its temperature, and is usually from 0·5 to 1·5 ohms.

The brightness of the lamps varies greatly with the density of the current and with the length of the arc, which may be almost null, or may amount to several millimetres—that is to say, it depends ultimately on the extent of the incandescent surfaces and their temperature.†

With a continuous current, the maintenance of a bright arc light, of 100 carcels, requires a current of about 15 ampères, and

* EDLUND. *Ann. de Chim. et de Phys.* [4], Vol. XIII., p. 450; Vol. XIV., p. 493; Vol. XV., p. 479. 1868.

† ALLARD, LE BLANC, JOUBERT, POTIER, and TRESCA. *Comptes rendus*, Vol. XCV., pp. 747, 806. 1882.

a fall of potential of 50 volts; the electrical work per second is

$$U = \frac{50 \times 15}{9 \cdot 81} = 76 \cdot 45 \text{ kgm.},$$

or 1·02 horse-power. The arc light is thus six times as powerful as the incandescent lamp, for the same expense.

1249. ACCUMULATORS.—When two electrodes, one of which is polarized, are connected by a conductor, the secondary current (**253**), or rather the discharge of the polarization, is usually rather weak, unless the polarization is kept up by an extraneous cause, as in Grove's gas battery, in which two platinum plates, which act as electrodes, are respectively immersed in oxygen and in hydrogen.

Nevertheless, with certain metals (**1081**) the capacity of polarisation may become considerable, because the reactions, instead of being confined to a thin surface layer, penetrate into the interior of the electrode, and thus put in play an important weight of matter.

M. G. Planté* has investigated most of the metals from this point of view, and has shown that a quantity of electricity, and therefore a considerable quantity of energy, may be stored in such a secondary battery. For some years these secondary couples have been called *accumulators* of electricity.

The best arrangement, according to M. Planté, is to immerse two plates of lead parallel and very close to each other in dilute sulphuric acid. The maximum electromotive force is higher than 2 volts. A remarkable property, also observed by M. Planté,† is that the capacity of polarization increases in proportion as the accumulator is formed—that is to say, according to the number of times it is charged and discharged, so that the oxidizing action of the primary current penetrates more deeply, the galvanic deposits thus formed being exceptionally porous. The formation of these couples is greatly accelerated by covering the lead plates, or filling the cavities they contain with a layer of red lead. This idea seems to have been due to M. Faure;‡ but in practice it is difficult to make the metal arising from the reduced oxide adhere to the lead plates.

M. Planté§ has observed that the electrochemical formation of the couples is accelerated by a rise of temperature, but he attains

* G. PLANTÉ. *Recherches sur l'Électricité.* Paris, 1879.

† G. PLANTÉ. *Comptes rendus*, Vol. LXXIV., p. 592. 1872.

‡ E. REYNIER. *Comptes rendus*, Vol. XCII., p. 951. 1881.

§ G. PLANTÉ. *Comptes rendus*, Vol. XCV., p. 418. 1882.

the same result more rapidly by first immersing the lead plates for a day or two in nitric acid diluted with its own bulk of water. With a small loss of weight, the metal experiences a kind of scraping or roughening, which doubtless increases its porosity and facilitates the ultimate penetration of the oxidation produced by the primary current.

In any case when a formed accumulator is polarized to excess, the electrode which becomes positive in the discharge supports a certain quantity of binoxide of lead PbO^2, and the negative electrode an equivalent quantity of metallic lead in the porous condition. The couple being closed, the positive electrode is reduced, the other becomes oxidised; the electromotive force of the secondary couple retains a constant value for the greater part of the discharge, and then diminishes slowly until the depolarization is complete.

In practice it is desirable to use only the first part of the operation, that which corresponds to a constant electromotive force; the accumulator is then charged anew, either by a battery or by any other electromotor. It is advantageous to group the accumulators abreast during the charge, so that only a small electromotive force has to be overcome, and to arrange them in series for the discharge.

1250. The investigation of the reactions which take place in the discharge of an accumulator is somewhat difficult. According to the thermochemical researches of M. Tscheltzow,* sulphate of lead simply is found on the two electrodes, according to the equation

$$Pb + PbO^2 + 2SO^3 + Aq = 2PbO.SO^3 + Aq.$$

The sum of the quantities of heat liberated in this reaction corresponds in fact to 1·93 volts; and sensibly represents the electromotive force of the accumulator.

As the equivalent of lead is 103·5, the weight of lead in the binoxide which corresponds to the passage of a coulomb (**1242**) is 1·0712 mgr., so that a kilogramme of lead, half of which comes into play on each electrode, may furnish

$$\frac{500,000}{1 \cdot 0712} = 467,000 \text{ coulombs.}$$

* TSCHELTZOW. *Comptes rendus*, Vol. C., p. 1458. 1885.

If the electromotive force is 2·1 volts, the disposable energy for a kilogramme of lead is

$$\frac{467,000 \times 2\cdot1}{9\cdot81} = 100,000 \text{ kgm.}$$

The work of a horse-power-hour requires therefore that the reaction takes place on 2·7 kilos. of lead.

It is useful to make the same calculation for the zinc used in batteries. The electrochemical equivalent of zinc being $33 \times 0\cdot01035$ or 0·3416 mgr., the solution of a kilo. of zinc produces

$$\frac{1,000,000}{0\cdot3416} = 2,927,000 \text{ coulombs.}$$

For an electromotive force of 1·8 volts, as in Bunsen's element, the disposable work is

$$\frac{2,927,000 \times 1\cdot8}{9\cdot81} = 537,000 \text{ kgm.}$$

For an equal weight, the zinc of Bunsen's batteries produces thus 5 times as much work as the lead of the accumulators.

The acids and vessels which are necessary in both cases, and the carbon of the Bunsen's element, must clearly be taken into account.

1251. The principal property of an accumulator, besides the retention of its charge, is the quantity of energy which it can furnish for a given weight.

From experiments made by a Commission of the Exhibition,* 35 Faure's accumulators, arranged in series, and each weighing 43·7 kilos., gave a useful effect of 619,600 coulombs, with an electromotive force of 2·1 volts per cell. The expenditure per kilo. is thus

$$\frac{619,600}{43\cdot7} = 14,179 \text{ coulombs.}$$

* ALLARD, JOUBERT, POTIER, and TRESCA. *Comptes rendus*, Vol. XCIV., p. 603. 1882.

and the disposable energy

$$\frac{619{,}600 \times 2\cdot 1}{43\cdot 7 \times 9\cdot 81} = 3035 \text{ kgm.}$$

Hence 89 kilos. are necessary for a horse-power-hour. The weight of lead which enters into the action is then a fraction of the total weight equal to

$$\frac{14{,}179}{467{,}000} = 0\cdot 03 \text{ nearly.}$$

If we assume that the lead is two-thirds of the weight of the couple, we conclude that the kilogramme of lead absorbs 21,269 coulombs.

Planté arrived at much higher numbers—at 36,000, and even 61,765 coulombs per kilogramme of lead, which corresponds to 0·08 or 0·14 of the total weight, or 30·65 kilos. of accumulator per horse-power-hour.

We have scarcely any practical data as to the work which can be produced by a Bunsen's battery. According to M. Duboscq, who has had much experience in this direction, it is possible with 100 elements to maintain a good arc for 15 hours, using 50 couples for the first third of the time, 50 others during the second third, and 100 couples in two series of 50 as a battery in the last third.

If the light is 80 carcels, the energy expended in the lamp is $76\cdot 45 \times 0\cdot 8 \times 15 \times 3600 = 3{,}300{,}000$ kilogrammetres. As each couple weighs 6 kilos., the useful energy for a kilogramme of battery is

$$\frac{3{,}300{,}000}{600} = 5500 \text{ kgm.}$$

The work of the battery would correspond to a horse-power-hour for 49 kilogrammes. The zinc dissolved, apart from local action, is a fraction of the whole weight equal to

$$\frac{5500}{537{,}000} = 0\cdot 01 \text{ nearly.}$$

1252. Another property of the accumulator is to restore the greatest possible quantity of the electrical energy which it has

received during the charge. The Commission found that the accumulators investigated, having absorbed 694,500 coulombs, gave out 619,600. The efficiency in electricity is then

$$\frac{619,600}{694,500} = 0{\cdot}89.$$

This ratio does not give the yield in energy, since during the charge the electricity enters the battery at a higher mean potential than that which corresponds to the discharge. Let E, I, R, be the electromotive force of the battery, the current, and the resistance during the charge; E′, I′, and R′ the same quantities during the discharge; the difference of potentials at the terminals is $E+IR$ in the first case, and $E'-I'R'$ in the second. The quantities of electricity set free being Q and Q′, the efficiency expressed is

$$u = \frac{Q'(E'-I'R')}{Q(E+IR)}.$$

It is always useful in practice to use a current for the charge which is weaker than the discharge—that is to say, to make $I < I'$; we thus obtain a better result.

In the experiments cited, the difference of potential at the terminals were in the ratio of 2 to 3; it follows that

$$u = \frac{Q'}{Q}\frac{2}{3} = 0{\cdot}89\frac{2}{3} = 0{\cdot}59.$$

1253. Induction Electromotors.—Shortly after Faraday's discovery, which was communicated to the Academy of Sciences on Dec. 26th, 1831, Pixii, under the direction of Ampère,* constructed a machine intended to obtain induced currents by the rotation of a magnet in face of an electromagnet. By the effect of rotation, the current produced in the wire of the electromagnet is of course alternating; but by a commutator fixed on the axis of rotation the currents in the external circuit are rectified.

This first machine of Pixii has been modified in many different ways. Clarke,† keeping the magnet stationary, caused the electromagnet to rotate; Page‡ surrounded the magnet by a conducting

* Ampère. *Ann. de Chim. et de Phys.* [2], Vol. LI., p. 71. 1832.

† Clarke. *Phil. Mag.* [3], Vol. IX., p. 262. 1836.

‡ Page. *Ann. of Electricity, Magn., and Chemistry*, Vol. III., p. 489. 1838-39.

wire, and rotated a piece of soft iron in front of it. The researches of Masson and Breguet* have shown the remarkable effects obtained by the induction of broken currents. The induction coil which they used has come, by successive improvements, into general use.

For some years these machines have emerged from the laboratories, and have taken an important part in industry. They may be divided into two groups, according as the current which they produce is sensibly *uniform*—that is to say, is always in the same direction with small variations of intensity, or *non-uniform*, as the current can change its direction, or undergo sudden changes.

1254. The machines which give a uniform current—or at any-rate a current of constant direction when their motion is kept up by external force—become, on the contrary, motors when excited by an external current.

We shall first suppose it to be the case of perfect machines—such as Faraday's disc, in which the current is perfectly uniform. In this case (**550** and **557**) the electromotive force of induction E, other things being equal, is proportional to the velocity—that is, to the number n of turns per second, and to a function $\phi(I)$ of the intensity of the current, the form of which depends on the mode of construction and on the play of the machine; we may write

$$E = n\phi(I).$$

If a machine is used as a motor by exciting it by a constant source, of electromotive force E_0, the intensity of the current in a circuit of resistance R is given by equation

$$IR = E_0 - E = E_0 - n\phi(I).$$

The work expended in unit time is $W = E_0 I$. If I_0 is the current $\frac{E_0}{R}$ which would correspond to the motor at rest, the work utilised may be written

$$U = EI = E_0 I - RI^2 = RI(I_0 - I),$$

and the efficiency

$$u = \frac{U}{W} = \frac{E}{E_0} = \frac{n\phi(I)}{E_0}.$$

* Masson and Breguet. *Ann de Chim. et de Phys.* [3], Vol. IV., p. 129. 1842.

As the useful work is a maximum, and the efficiency equal to 0·50, when $2I=I_0$, the corresponding velocity of the motor is

$$n=\frac{E_0}{2\phi\left(\frac{E_0}{2R}\right)}.$$

If the machine is left to itself, assuming that all sources of friction are suppressed, the velocity increases until the current is zero; the electromotive force of the machine is then equal to that of the source, and the efficiency equal to unity.

The maximum number of turns per second is given by the condition $N\phi(0)=E_0$, which corresponds to a finite or infinite velocity according to the form of the function $\phi(I)$.

1255. When the machine is used as electromotor the current can only be kept up provided the energy expended EI is higher than, or at least equal to, the energy which the heating of the circuit consumes, that is if we have

$$n\phi(I)>IR.$$

In order that the machine shall charge itself by being put in motion, we must suppose the circuit traversed by a very small current i; we have

$$n\phi(i)>iR.$$

If the external work is reduced to the heat disengaged in the circuit, equilibrium holds for a current I_0 defined by the equation

$$n\phi(I_0)=I_0R,$$

which generally gives a finite value for I_0. As the intensity of the current diminishes as the resistance increases for a given velocity, we see that the ratio $\frac{I}{\phi(I)}$ increases generally with the intensity.

1256. Just as with batteries, the available work U of an electromotor is only the excess of the energy expended over that

which corresponds to the heat disengaged in the resistance R of the machine alone. We have then

$$U = EI - RI^2 = I[n\phi(I) - IR],$$

and the maximum of useful work corresponds to the condition

$$\frac{dU}{dI} = 0, \qquad \text{or} \qquad n[\phi(I) + I\phi'(I)] = 2IR.$$

The current I which gives the maximum useful work is no longer in general half the current I_0, which the machine would produce if short circuited.

The efficiency for the maximum work is

$$u_m = 1 - \frac{IR}{n\phi(I)} = \frac{1}{2} - \frac{I\phi'(I)}{2\phi(I)}.$$

This efficiency is equal to 0·50 if the function $\phi(I)$ is a constant; it is lower or higher than 0·50 according as the derivative $\phi'(I)$ is positive or negative.

Let us imagine two machines traversed by the same current I, one of which serves as electromotor and the other as motor, the former absorbing the work $nI\phi(I)$, and the second producing the useful work $n_1 I\phi_1(I)$. The expression for the efficiency is

$$u = \frac{n_1\phi_1(I)}{n\phi(I)};$$

it is simply equal to the ratio of the velocities when the function $\phi(I)$ is the same in the two machines.

1257. Various Types of Machines.—All the properties of a machine with a strictly uniform current are then defined by the function $\phi(I)$, and the form of this function might serve to classify various types of machines. We shall observe that the product $I\phi(I)$ represents the work absorbed by the machine during a period.

1st. We shall apply the term *electrodynamic* to those machines in which the inductor and the induced body consist simply of wires, and traversed by the same current. The work W_1 is then proportional to the square of the intensity of the current, and we may write

$$W_1 = CI^2, \qquad \text{or} \qquad \phi I = CI.$$

2nd. In *magnetic* machines the induction is produced by the motion of a circuit in an invariable magnetic field, like that of a system of permanent magnets. The work W_1 is then proportional to the current, which gives

$$W_1 = AI, \qquad \phi I = A.$$

3rd. In *magneto-electrical* machines, the inductor and the induced body are coils containing soft iron or electromagnets, the two systems of wires being traversed by the same current, or by given fractions of the same current. The work W_1 comprises: 1st, a term of the form CI^2, due to the action of the two systems of wires; 2nd, a term C_1MI, due to the action of the magnetised iron of each of the systems on the wires of the other, and proportional to the magnetism M; 3rd, finally, a term C_2M^2, due to the action of the two systems of magnets.

If the currents are weak, the magnetism M of the armatures is sensibly proportional to the current, and we may write

$$W_1 = (C + C_1 + C_2)I^2, \qquad \phi(I) = (C + C_1 + C_2)I.$$

As the current increases, the intensity of the magnetisation of the iron tends towards a maximum; the products C_1M and C_2M^2 tend to become constants C′ and C″, which gives

$$W_1 = CI^2 + C'I + C'', \qquad \phi(I) = CI + C' + \frac{C''}{I}.$$

4th. We may, lastly, call those machines *mixed* in which the inductor is a constant magnetic field due to an external agency, and the induced body is an electromagnet.

The work W_1 is then made up of two terms $AI + A_1M$, which correspond to the action of the field on the wire and on the magnetised iron of the induced body.

For weak currents we have still

$$W_1 = (A + A_1)I, \qquad \phi(I) = A + A_1,$$

and, when the currents can produce the maximum magnetisation

$$W_1 = AI + A', \qquad \phi(I) = A + \frac{A'}{I}.$$

Electrodynamic machines have no practical interest since their electromotive force would only have an appreciable value for excessive velocities.

A machine on the Siemens' type might be considered as a magnetic one, as also the Thomson-Ferranti machine; but they are only used for alternate currents.

Most of the present machines are magneto-electrical. If one of the systems only contains iron, the coefficient C is null; this is the case with Froment's motor.

The older machines, finally, are *mixed*, if we disregard the fact that they naturally produce alternating currents, and that even with a commutator the current cannot be considered as sensibly uniform, Such are the machines of Pixii, of Clarke, of Nollet, of Méritens, etc. The Gramme machines are mixed, and have a sensibly uniform current, when the field is produced by magnets, or when the inducing electromagnets are excited by an extraneous current.

1258. Some particular cases give rise to remarkable properties.

1st. If the function ϕ (I) is a constant (magnetic machines, or mixed machines with weak current), the electromotive force is proportional to the velocity. For a given regime, the machine used as electromotor is exactly comparable with an ordinary battery.

2nd. If the function ϕ (I) is of the form CI (electrodynamic machines, or magneto-electrical machines with weak currents), and the machine is used as motor, the current is given by the equation

$$E_0 - nCI = IR, \qquad I = \frac{E_0}{R + nC}.$$

The machine only affects the expression for the current in the form of a resistance proportional to the velocity; the efficiency is

$$u = \frac{nCI}{E_0} = \frac{nC}{R + nC} = \frac{1}{1 + \dfrac{R}{nC}}.$$

Apart from friction, the velocity would continue to increase without limit.

If the machine is used as electromotor, it can only start provided

$$nCi > iR, \qquad \text{or} \qquad n > \frac{R}{C}.$$

For a smaller velocity the machine discharges itself, whatever be the current. For a velocity higher than this limit, the energy absorbed nCI^2 is greater than the calorific energy disengaged; the circuit becomes heated, whatever be the means of cooling, until the resistance, which varies with the temperature, satisfies the condition $R = nC$.

If the machine with its circuit is kept in a bath which equalises the temperatures, equilibrium is only possible for a certain temperature which depends on the velocity.

The intensity of the current which gives the temperature of equilibrium depends itself only on the calorific energy Q lost, and therefore on the method of cooling; we have

$$I^2 = \frac{Q}{R} = \frac{Q}{nC}.$$

3rd. In magneto-electrical machines, the term CI, which corresponds to the electrodynamic action, is generally very small

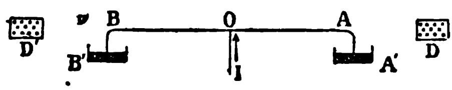


Fig. 252.

compared with the action of the magnetised pieces; the function ϕ (I) is first proportional to the intensity I for weak currents, and tends to become constant for very powerful currents

1259. Let us consider as an example a rectilinear conductor AB (Fig. 252) of length $2a$, movable about a vertical axis passing through its middle, and the bent ends of which dip in a ring-shaped bath of mercury A'B'; the mercury is connected with the point O by another conductor, so as to form a circuit of resistance R.

The only effective force is the vertical component of the terrestrial field. If the rotation is uniform, and makes n turns in a second, the flow of force cut by each branch in unit time is $n\pi a^2 Z$, or for the two $2n\pi a^2 Z$.

If the total current is I, or $\frac{I}{2}$ in each branch, the work necessary f r keeping up the motion is $\pi n a^2 ZI$, and the electromotive force

$$E = n\pi a^2 Z.$$

This is a machine which belongs to what we have called the magnetic type.

The value of Z at Paris being about 0·422 (C.G.S.), if $a = 1$ metre, we have

$$E = n\pi 10^4 \times 0{\cdot}422 = n1{\cdot}33.10^4 \text{(C.G.S.)},$$

$$E = \frac{n1{\cdot}33}{10^4} \text{ volts.}$$

To obtain a volt, the number of turns per second should be $n = \dfrac{10^4}{1{\cdot}33} = 7474.$

1260. The same apparatus may be transformed into an electrodynamic machine. Let us suppose that the circuit comprises a coil DD′, the mean plane of which passes through the conductor, and the axis of which coincides with the axis of rotation.

The mean action of the coil on the circle of radius a being F_m for unit current, the electromotive force for a current I, apart from the action of the earth, will be

$$E = n\pi a^2 F_m I.$$

Used as an electromotor, the machine can only produce a current provided $n > \dfrac{R}{\pi a^2 F_m}$.

If the coil contains N turns of mean radius A, and we take as approximate value the mean action $Fm = \dfrac{2N\pi}{A}$, we get $n > \dfrac{RA}{2N\pi^2 a^2}$.

The total length of the circuit is sensibly $2N\pi A$, and if S is the section of the wire, and σ its specific resistance, the resistance R is equal to $\dfrac{\sigma}{S} 2N\pi A$; it follows from this that $n > \dfrac{\sigma A^2}{S\pi a^2}$.

This condition only depends on the ratio of the squares of the radii A and a, and therefore on the ratio of the corresponding surfaces.

The specific resistance of copper is about 1600; if we use wires whose section is a square centimetre, and assume $A^2 = 2\pi a^2$, we get $n > 3200$.

1261. The preceding results are very approximately verified* for machines, like those of Gramme, where the current is sensibly uniform, either by measuring the electromotive forces, or by determining the efficiency. Several causes however contribute to complicate the effects.

1st. The current is never strictly uniform, and the variations in intensity require us to take into account the resulting extra currents.

2nd. It is probable that the soft iron in electromagnets is not instantaneously magnetised, and this retardation in the magnetisation gives rise to a loss of useful effect.

3rd. For the magnets themselves, changes are produced in the magnetisation in one direction or the other, and to unequal extents, according as the actions are repulsive or attractive, which again causes an analogous loss. The same is the case for electromagnets.

4th. Magnets and cores of magnets are finally the seat of induced currents, or what are called *Foucault's currents*, which heat them to the detriment of the usual work.

The influence of all these reactions may be represented, at any rate approximately, by adding to the circuit a *fictive* resistance which is sensibly proportional to the velocity of the machine.

1262. GRAMME'S MACHINE.—Pacinotti† devised the principle of this machine as long ago as 1860, but his invention remained unknown and without application. M. Gramme‡ subsequently rediscovered the same principle, and had the merit of applying it to the construction of true industrial machines; these machines have served as models for most of those used at the present day.

The principal part is a coil made up of a series of coils B (Fig. 253) surrounding a ring A of soft iron; the coils are connected separately to a series of conducting plates or *strips* insulated from each other, and arranged on the surface of a cylinder which acts as a commutator; each of the individual coils may be formed of several turns. The circuit of the coils is thus closed on itself, but the two halves are traversed by currents in opposite directions; two brushes C and C′ press on two strips which are diametrically opposite, and serve to transmit the current into an external circuit, and form the poles of the machine.

* MASCART and ANGOT. *Journal de Physique*, Vol. VII., pp. 79, 363. 1878.

† A. PACINOTTI. *Nuovo Cimento*, Vol. XIX., p. 378. 1864.

‡ GRAMME. *Comptes rendus*, Vol. LXXIII., p. 175. 1871.

When the ring turns in a magnetic field, we may propose to determine the electromotive force in each coil, the position of the brush which produces the maximum current, and the intensity of this current.

Let $4m$ be the total number of coils,
p the number of windings of each coil,
$P = 4mp$ the total number of windings,
n the number of turns in a second,
$\omega = 2n\pi$ the angular velocity,
ρ the resistance of one winding,
$r = p\rho$ the resistance of a coil,
l its coefficient of self-induction,
a the resistance of a ring-shaped coil,
R the total useless resistance, comprising the wires of the machine and the connections.

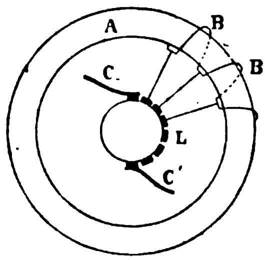


Fig. 253.

When the brushes merely rest on a strip, the resistance of the coil is $\frac{1}{2} 2mr = mr = P\frac{\rho}{4}$, and when each rests on two successive strips $\frac{1}{2}(2m-1)\,r = (P-2p)\,\frac{\rho}{4}$. We may, without appreciable error, replace this variable resistance by an intermediate constant value

$$a = mr - \frac{r}{4} = (P-p)\,\frac{\rho}{4}.$$

1263. We shall assume that the external field is symmetrical with respect to a plane CD (Fig. 254) passing through the axis of rotation; we shall also assume that it is symmetrical to within a

sign in respect of the transverse plane AB perpendicular to the first—that is to say, that the forces F on one side are directed towards the transverse plane, while the symmetrical forces F′ are directed from this plane. The ring of soft iron introduced into the field does not modify this kind of symmetry. The flow of force is absorbed on one side of the ring; an equal flow is emitted from the other, and a part of the total flow, the least possible, proceeds from one half of the ring to the other by the internal void space. The figure given by iron filings well illustrates this distribution.

It will be assumed that the magnetisation of the ring by the field is the same in motion as at rest—that is to say, there is no retardation in the magnetisation.

When a winding moves from A to M, it cuts a flow of force which is a function $f(\alpha)$ of the angle α which its plane makes

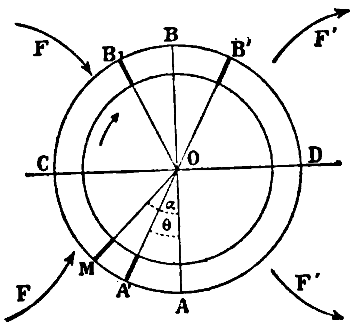


Fig. 254.

with the transverse plane. The flow of force cut by a coil, the mean plane of which has traversed this same angle, is

$$q = pf(\alpha).$$

For a displacement $d\alpha$, the electromotive force induced in the ring is

$$\frac{dq}{dt} = pf'(\alpha)\frac{d\alpha}{dt} = p\omega f'(\alpha).$$

If θ is the angle of lead of the brush—that is to say, the angle of the diameter A′B′ of the contacts with the transverse plane—the flow of force cut by the ring from A′ to B′ comprises

two symmetrical parts B_1B and BB', which give flows of force of opposite signs, and neutralise each other. There remains then for the useful effect only the angle $A'OB_1 = \pi - 2\theta$; the corresponding flow of force, which is equal to $p[f(\pi) - 2f(\theta)]$, tends to make the brush B′ positive, for instance, and A′ negative.

When the coil has passed the brush B′, it undergoes the same influence in the opposite direction in the second half of the ring, so that it tends to give the same signs respectively to the two brushes.

The flow of useful force cut by a coil during an entire rotation is

$$2p\left[f(\pi) - 2f(\theta)\right].$$

For the $4m$ coils, with the velocity indicated, the total flow of force cut in a second is

$$Q = 4mn2p\left[f(\pi) - 2f(\theta)\right] = 2Pn\left[f(\pi) - 2f(\theta)\right].$$

The current is not absolutely uniform, even if we suppose the external resistance constant, for the electromotive force in each half of the ring is given by the sum $p\omega\Sigma f'(\alpha)$ of the terms relative to each coil, and the resistance of the induced wire varies periodically. The period T of the variations of the current is the time necessary for one coil to replace the preceding one—that is to say $\frac{1}{4mn}$, and the external current may be represented by an expression of the form $I + A\sin 2\pi(4mnt + \beta)$ in which A and β are constants. The introduction of a telephone into the circuit, or into a shunt, will render evident these oscillations.

If, however, the coils are numerous, the undulations of the current are very small in comparison with its mean value I, and it may be regarded as sensibly uniform.

1264. As the current is $\frac{I}{2}$ in each half of the induced wire, the electromagnetic work per second is

$$W = nP\left[f(\pi) - 2f(\theta)\right]I = nE_1I.$$

The electromotive force is nE_1, and the factor E_1 represents the electromotive force for one turn per second.

The ring is really magnetised afresh by the current which surrounds it, so that the field is no longer symmetrical. It may, however, be assumed, as a first approximation, that this magnetisation is added to the original magnetisation. In this case, no change takes place in the electromotive force, for the sum of the corresponding flow of force, cut by a ring from A′ to B′, is null.

The energy lost comprises, first, that which arises from the heating of the circuit, or I^2R.

Moreover, every time a coil is short-circuited by the contact of the brush on two adjacent strips, the current is suppressed, and the intrinsic energy is lost; the loss is $\frac{l}{2}\frac{I^2}{4}$ at each brush, or $l\frac{I^2}{4}$ for a rotation, or $mnlI^2 = a_1I^2$ in a second for all the coils.*

In conclusion, the excess of energy expended over the energy lost, or the disposable energy U, is

$$U = nE_1I - (R + a_1)I^2.$$

The effects of self-induction may thus be represented by a fictive resistance proportional to the velocity.

1265. The best angle of lead is practically defined by the condition that no sparks are produced. But these sparks are dangerous, not when the brush meets a strip, but when it leaves it. Hence two successive strips, that which is in contact with the brush, and that which escapes, should be at the same potential, or that the corresponding coil should contain at this moment an electromotive force capable of producing the mean current $\frac{I}{2}$.

This electromotive force comprises a part of $p\omega f'(\theta)$ which is due to the field, and a term in the opposite direction proportional to the magnetisation of the ring to the velocity and to the number of turns, and which may be written $-\omega pM$.

* JOUBERT. *Comptes rendus*, Vol. XCVI., p. 641. 1883.

If the current has had the time to establish itself in the closed coil, the equation which defines the angle of lead is then

$$p\omega\left[f'(\theta) - \mathrm{M}\right] = r\frac{\mathrm{I}}{2} = p\rho\frac{\mathrm{I}}{2},$$

$$f'(\theta) = \mathrm{M} + \frac{\rho}{\omega}\frac{\mathrm{I}}{2}.$$

When the magnetisation M is proportional to the current, we may represent it by $\mathrm{M} = r'\frac{\mathrm{I}}{2}$, which gives

$$f'(\theta) = \frac{\mathrm{I}}{2\omega}(\rho + \omega r').$$

The magnetisation only comes then into the angle of lead as a fictive increase of the resistance of a winding, which would be proportional to the velocity.

1266. The useful work may be represented by a resistance x equivalent to the useful resistance and to the electromotive forces overcome. The current and the angle of lead are then given by the equations

$$\mathrm{I} = \frac{n\mathrm{P}\left[f(\pi) - 2f(\theta)\right]}{\mathrm{R} + a_1 + x},$$

$$f'(\theta) = \mathrm{M} + \frac{f(\pi) - 2f(\theta)}{4\pi}\frac{\mathrm{P}\rho}{\mathrm{R} + a_1 + x},$$

and, if the magnetism is proportional to the current,

$$f'(\theta) = \frac{f(\pi) - 2f(\theta)}{4\pi}\frac{\mathrm{P}(\rho f \omega r')}{\mathrm{R} + a_1 + x}.$$

It will also be observed that the coefficient l is proportional to the square of the number of turns of each coil; and may be represented by $4\lambda p^2$; we have then

$$a_1 = mnl = 4mnp^2\lambda = n\mathrm{P}p\lambda.$$

In order to diminish the influence of self-induction, it is advantageous as much as possible to form the coils of a single winding.

Making the same assumptions, the angle of lead for an infinite velocity would be

$$f'(\theta)=\frac{f(\pi)-2f(\theta)}{2}\frac{r'}{p\lambda}.$$

We should still have to bring in other losses of work, such as are due, for instance, to currents induced in the ring, to the action of the field on the movable ring, if there is a retardation in the magnetisation, to the reaction of this ring on the external field, etc. These various effects may be represented, at any rate approximately, by a fictive increase of resistance proportional to the velocity and comprised within the term $a_1=mnl$.

1267. If we represent the function $f'(\alpha)$ by the ordinates of a curve C (Fig. 255), in reference to the semi-circumference AB rectified, the flow of force $f(\pi)$ is represented by the total area ACB,

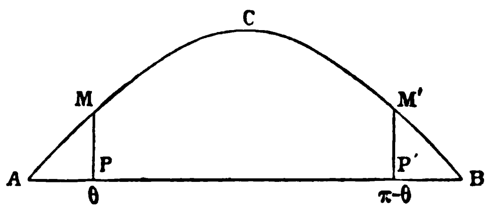


Fig. 255.

and the term $2f(\theta)$ by the sum of the areas AMP and M'P'B, so that the total electromotive force is proportional to the area PCP'. We may moreover determine the curve $f'(\alpha)$ by experiment.

1st. If a single coil is closed by a galvanometer, and this coil is suddenly displaced through a very small angle $\Delta\alpha$, the induced discharge gives the corresponding flow of force $pf'(\alpha)\Delta\alpha$,—that is to say, the ordinate $f'(\alpha)$ of the curve C. The operation being repeated for a series of equal displacements $\Delta\alpha$, from 0 to α, we can deduce $f(\alpha)$ by an integration or by the measurement of a surface.

2nd. If the coil passes rapidly from the angle α_1 to the angle α_2, the induced discharge gives $p[f(\alpha_2)-f(\alpha_1)]$; we may thus also determine $f(\pi-\theta)-f(\theta)=f(\pi)-2f(\theta)$.

3rd. Instead of insulating a coil, the brushes are connected with the ballistic galvanometer. If they are first in the transverse plane,

and the ring is turned through the angle of two successive coils, $\frac{2\pi}{4m}=\beta$, the effect is the same as if a single coil had turned, for each half of the ring, through the angle π; the induced discharge corresponds to the flow of force $pf(\pi)$.

4th. If the brushes make an angle β with the transverse plane, a rotation β corresponds to the flow of force $p[f(\pi)-2f(\beta)]$. We have thus the series of determinations

$$\begin{aligned} A_0 &= f(\pi), \\ A_1 &= f(\pi)-2f(\beta), \\ A_2 &= f(\pi)-2f(2\beta), \\ &\ldots\ldots\ldots\ldots\ldots\ldots; \end{aligned}$$

consequently

$$f(\beta)=\frac{A_0-A_1}{2},$$

$$f(2\beta)=\frac{A_0-A_2}{2}, \text{ etc.}$$

These successive values of the function, from $f(\beta)$ to $f(m\beta)$, determine sufficiently all intermediate values.

1268. Particular Case.—The ring is usually a kind of hollow cylinder, formed of iron wires, or by a thin strip coiled as a spiral, the successive layers of which are insulated. The induced currents are thus avoided which tend to form in the meridian section, and which, by heating the metal, would cause a considerable loss of energy.

If the original field is uniform, and of the intensity F, the ring becomes magnetised almost like an unlimited cylinder (388)—that is to say, uniformly—and the intensity of magnetisation is

$$I_a=\frac{k}{1+2\pi k}F,$$

or sensibly

$$I_a=\frac{F}{2\pi}.$$

If a is the radius of the cylinder, its potential at the distance r from the axis, at an external point, the abscissa of which is x, is expressed by (359)

$$V_2=I_a\frac{2\pi a^2}{r^2}x=F\frac{a^2}{r^2}x.$$

The potential of the original field being $V_0 - Fx$, the resultant potential is

$$V = V_1 + V_2 = V_0 - Fx\left(1 - \frac{a^2}{r^2}\right).$$

The component N of the field along the direction r is

$$N = -\frac{\partial V}{\partial r} = F\left[\left(1 - \frac{a^2}{r^2}\right)\frac{\partial x}{\partial r} - x\frac{2a^2}{r^3}\right],$$

which gives, for $r = a$—that is to say, at the surface of the cylinder,

$$N = 2F\frac{x}{r} = 2F \sin\alpha\,;$$

this is a value double the component, which would be due to the original field alone.

If L is the length of the cylinder,

$$\begin{aligned}
f'(\alpha) &= 2FLa \sin\alpha,\\
f(\alpha) &= \int_0^\alpha f'(\alpha)d\alpha = 2FLa(1 - \cos\alpha),\\
f(\pi) &= 4FLa,\\
f(\pi) - 2f(\theta) &= f(\pi)\cos\theta,\\
E &= nPf(\pi)\cos\theta,\\
I &= \frac{nPf(\pi)\cos\theta}{R + a_1 + x}.
\end{aligned}$$

The equation which defines the angle of lead (1265) becomes

$$\frac{f(\pi)}{2}\sin\theta = M + \frac{\rho}{\omega}\frac{I}{2} = M + \frac{f(\pi)\cos\theta}{4\pi}\,\frac{P\rho}{R + a_1 + x},$$

and, if the magnetisation M of the ring may be considered as proportional to the current,

$$\sin\theta = \frac{2}{f(\pi)}\left(r' + \frac{\rho}{\omega}\right)\frac{I}{2}.$$

For a constant velocity, the sine of the angle of lead is then proportional to the intensity of the current, and for the same current this angle diminishes as the velocity increases.

As a function of the external circuit, we have

$$\tan\theta = \frac{P}{2\pi}\frac{\rho + \omega r'}{R + a_1 + x},$$

and, for an infinite velocity,

$$\tan\theta = \frac{r'}{p\lambda}.$$

1269. DYNAMOS.—This is the name usually given in practice to magneto-electrical machines, like those of Gramme, in which the current itself is used either wholly or in part to excite the inductors. The current of the inductors being I, the function $\phi(I)$ which, with M. Marcel Deprez, we may call *characteristic function*, is at first null, for $I = 0$, to almost the remanent magnetisation; it then increases with the intensity, and tends in general towards a maximum with the magnetisation of the armatures.

For a machine in full work, the energy absorbed by each turn of the ring, $W_1 = I\phi(I)$, only depends on the strength of the current. The same work W_1 would be effected by a force applied at the end of a lever equal to unity, the numerical value of which was $\frac{W_1}{2\pi}$; this quotient $\frac{W_1}{2\pi}$ is the *motor couple* of the machine. W_1 is again the numerical value of the force which should be applied at the end of a radius of length $\frac{1}{2\pi}$, corresponding to a circumference equal to unity, in order to obtain the same work; this is what has led M. Marcel Deprez* to call this quantity the *static effort* of the machine.

In the expression

$$W_1 = I\phi(I) = PI\left[f(\pi) - 2f(\theta)\right],$$

* MARCEL DEPREZ. *Comptes rendus*, Vol. xcv., p. 778. 1882.

the factor within the bracket only depends on the magnetic field—that is to say, on the inductors; the two former depend on the ring, and on the intensity of the current.

The excitation of the inductors, if they are not permanent magnets, requires an expenditure of energy, which should be made as small as possible. Given the armatures of the inductors, the magnetisation is defined by the currents which surround them. In this case, with wires covered by an insulating layer proportional to their thickness and occupying a given volume, the energy expended in producing the same magnetic field is independent of the diameter of the wire, as well as of the intensity of the current, and only depends on the *density* of the current in the meridian section occupied by the wire; we ought then to seek for economy simply in the choice of the wire, the shape of the cores, and the pole pieces, and the mode of distribution of the currents.

Other things being equal, the energy expended in heating the inductors is proportional to the square of the intensity of the current, while the magnetisation increases more slowly, and tends towards a maximum. The ratio of the energy expended to the field produced increases then rapidly, and there is a degree of magnetisation which it is not advantageous to exceed in practice.

On the other hand, the field being determined, the motor couple is proportional to the product PI—that is to say, to the total current for unit section of the whole of the wires which surround the ring, or the density of the current. Then, again, if we neglect the effects of self-induction which show themselves at the passage of the brush, the thermal energy does not alter if this product remains the same.

With some allowance for the effects of mutual induction or of self-induction of the various parts of the circuit, we may say then, with M. Deprez, that the energy necessary to produce a given motor couple (or a static effort) is independent of the resistance of the wires which are coiled on the inductors and on the ring, provided the volume occupied respectively by these two systems of wires does not change.

1270. The characteristic function may be experimentally determined by various methods.

1st. An auxiliary current I is passed in the inductor, the brushes being left in the position which they are to occupy, and the ring is rotated; the electromotive force $n\phi(\mathrm{I})$ will be measured by opposition, either by an electrometer, or with a galvanometer of great resistance.

2nd. The previous determination of the function $f(a)$ will give

$$\phi(\mathrm{I}) = \mathrm{P}\left[f(\pi) - 2f(\theta)\right].$$

These two methods have the inconvenience of giving the electromotive force which corresponds to zero current in the ring, and leave aside the effect due to the magnetisation of the ring.

3rd. The machine being in motion and producing the current I, the ring is pressed by a sort of forked contact, consisting of two branches kept at a distance of two strips by an insulating plate, and which communicate with a galvanometer of high resistance g. The difference of potential ϵ_a of two successive strips is equal to the excess of the corresponding electromotive force e_a over the product of the resistance r of the coil by the current $\frac{\mathrm{I}}{2}$; the current i in the galvanometer gives then

$$\epsilon_a = e_a - \frac{\mathrm{I}}{2} r = ig, \qquad \text{or} \qquad e_a = ig + \frac{\mathrm{I}}{2} r.$$

Starting from one of the brushes, we shall successively determine

$$e_\beta,\ e_{2\beta},\ e_{3\beta}, \ldots .\ e_{(2m-1)\beta},\ e_{2m\beta} = e_\pi,$$

and we shall get

$$\mathrm{E}_1 = \phi(\mathrm{I}) = e_\beta + e_{2\beta} + \ldots . + e_{(2m-1)\beta} + e_\pi.$$

In the present case, each of these values of e is the algebraical sum of the electromotive forces corresponding to the two flows of force of the field and of the ring.

4th. The electromotive force of a machine is sometimes estimated by the current I which it produces in a circuit of total resistance $\mathrm{R}+x$, by the help of the equation $\mathrm{E} = \mathrm{I}(\mathrm{R}+x)$, and from it is deduced the characteristic function

$$\phi(\mathrm{I}) = \frac{\mathrm{E}}{n} = \mathrm{I}\frac{\mathrm{R}+x}{n};$$

but this method of working is not quite accurate, for the resistance of the induced ring should be considered as containing a fictive term, almost proportional to the velocity of rotation, and this calculation cannot be neglected in calculating the electromotive force.

We may determine this fictive resistance y and the electromotive force E by two experiments, in which the inductors are excited by an external current I, by measuring the induced currents obtained for the same velocity with external resistances x and x'. The equations

$$E = i(R+x+y) = i'(R+x'+y),$$

give

$$y = \frac{i'x' - ix}{i - i'} - R, \qquad E = i\frac{i'}{i-i'}(x'-x).$$

1271. A knowledge of the function $\phi(I)$ enables us to determine all the properties of the machine.

In a series of researches on dynamos, Fröhlich has called this quantity the *active magnetism;* he found that if the angle of lead is unchanged, it may be represented by an expression of the form

$$\phi(I) = \frac{I}{A+BI},$$

in which A and B are constants to be determined experimentally.

Expressed geometrically by taking intensities for the abscissæ, this equation represents a hyperbola passing through the origin, and having a horizontal asymptote. It only holds really for mean strengths of the current, in regions in which the curvature of the characteristic is pretty large.

When the machine acts as an electromotor, and we add to the circuit an external resistance x, the current is

$$I = \frac{n\phi(I)}{R+x} = \frac{nI}{(A+BI)(R+x)},$$

or

$$A+BI = \frac{n}{R+x}, \qquad I = \frac{n}{B(R+x)} - \frac{A}{B}.$$

In order that the machine may prime itself, we must have

$$\frac{n}{A} > R+x.$$

The minimum velocity for an external resistance null, is then equal to AR.

The condition for the maximum useful work (1256) becomes, in the present case,

$$\frac{2IR}{n} = \frac{I(2A+BI)}{(A+BI)^2},$$

$$A+BI = \frac{n}{4R}\left(1+\sqrt{1+\frac{8AR}{n}}\right).$$

If we replace $A+BI$ by its value $\frac{n}{R+x}$, we have, in the case of the maximum useful value,

$$\frac{2R}{R+x} = \frac{1}{2}\left(1+\sqrt{1+\frac{8AR}{n}}\right).$$

As the second member is greater than unity, the external resistance should be less than that of the machine; but it is nearer to equality, the greater the velocity of the machine.

The relative efficiency for maximum work is

$$u_m = \frac{x}{R+x} = 1 - \frac{R}{R+x} = \frac{3}{4} - \frac{1}{4}\sqrt{1+\frac{8AR}{n}}.$$

If the magnetic field had its maximum value, $\phi(I)$ would be equal to $\frac{1}{B}$, and the current produced by the motion of the ring, in an equal resistance $R+x$, would be

$$I_1 = \frac{n}{B(R+x)}.$$

On the other hand, the current I_2 capable of giving the field an intensity one half of the maximum, is defined by the condition

$$\frac{I_2}{A+BI_2} = \frac{1}{2B}, \quad \text{or} \quad I_2 = \frac{A}{B};$$

from this follows the curious property

$$I = I_1 - I_2.$$

In other words, the real current I is equal to the excess of the current I_1 which is produced for the same velocity in a field of maximum intensity, over the current I_2, which would magnetise the inductors to half the maximum. The current I_1 only depends on the movable ring, while the current I_2 only depends on the inductors; the true current is thus expressed by two terms of very different characters.

1272. Resistances of the Inductor and of the Induced Body.—We may inquire what ratio should exist between the resistances of the inductor and induced body so as to obtain the greatest efficiency.* Suppose, first, that the two parts are not traversed by the same current, and that we are given the volumes occupied by the coiling of the wires.

For an annular coil, let

V be the total volume of the wire, including the insulator,
L the length of the wire,
S the section, comprising the insulating material,
$\frac{1}{m}$ the fraction of this section occupied by the conductor,
σ the specific resistance of the metal,
a the total resistance,
I the current.

If, in like manner, V′, L′, S′, m', σ', b and I′ are analogous quantities for the inductor, we have

$$V = SL, \qquad a = m\sigma\frac{L}{S} = m\sigma\frac{V}{S^2};$$

putting $K^2 = m\sigma V$, we may write

$$S^2 = \frac{K^2}{a}.$$

In like manner, putting $K'^2 = m\sigma' V'$, we shall have

$$S'^2 = \frac{K'^2}{b}.$$

* Sir W. Thomson. *Comptes rendus*, Vol. xciii., p. 474. 1881.

The field of the inducing current is proportional to I′L′, or to $V'\frac{I'}{S'}$; that is to say, to its mean density $\frac{I'}{S'}$, and the field produced is a function $\phi\left(\frac{I'}{S'}\right)$ of this density. As the electromotive force is proportional to the field, to the velocity, and to the length L or $\frac{V}{S}$ of the induced wire, we may write

$$E = C\frac{n}{S}\phi\left(\frac{I'}{S'}\right) = Cn\frac{\sqrt{a}}{K}\phi\left(\frac{I'\sqrt{b}}{K'}\right).$$

The quantities K and K′ are constants if the thickness of the insulator is proportional to the diameter of the wire; the factor C depends on the volumes V and V′ and on the quality of the iron, especially as regards the inductor.

The useful work is

$$U = EI - (aI^2 + bI'^2),$$

and the efficiency

$$u = 1 - \frac{aI^2 + bI'^2}{EI}.$$

When the inductor receives the whole of the current, we have

$$u = 1 - \frac{(a+b)I}{E},$$

and this efficiency is a maximum for the current, which makes the fraction

$$\frac{E}{(a+b)I} = \frac{Cn}{KI}\frac{\sqrt{a}}{a+b}\phi\left(\frac{I\sqrt{b}}{K'}\right)$$

a maximum.

If the electromotive force were simply proportional to the current of the inductors, it would be sufficient to make the ratio $\frac{\sqrt{ab}}{a+b} = \frac{1}{\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}}$ a maximum, which is the case when the two resistances a and b are equal to each other.

With Fröhlich's formula, we should put

$$\phi\left(\frac{\mathrm{I}\sqrt{b}}{\mathrm{K}'}\right)=\frac{\sqrt{b}}{\mathrm{K}'}\frac{\mathrm{I}}{\mathrm{A}+\frac{\mathrm{B}\sqrt{b}}{\mathrm{K}'}\mathrm{I}},$$

and we should have

$$\frac{\mathrm{E}}{(a+b)\mathrm{I}}=\frac{\mathrm{C}n}{\mathrm{K}}\frac{\sqrt{ab}}{a+b}\frac{1}{\mathrm{A}+\frac{\mathrm{B}\sqrt{b}}{\mathrm{K}'}\mathrm{I}}.$$

The maximum of the second member does not exactly correspond to the maximum of the factor $\frac{\sqrt{ab}}{a+b}$, in consequence of $\sqrt{b}$ occurring in the denominator of the following factor; we should thus diminish b, and therefore make $a>b$.

Suppose that the machine excited is shunt-wound; a part only I′ of the principal current is used for the inductors, and the other part $\mathrm{I}-\mathrm{I}'$ is taken at the terminals by a useful resistance x. We have then

$$\mathrm{I}'b=(\mathrm{I}-\mathrm{I}')x \quad , \quad \mathrm{I}'=\mathrm{I}\frac{x}{b+x},$$

$$\mathrm{E}=\mathrm{I}a+\mathrm{I}'b \quad = \quad \mathrm{I}\left(a+\frac{bx}{b+x}\right),$$

and the expression for the efficiency is

$$u=1-\frac{a+b\left(\frac{x}{b+x}\right)^2}{a+\frac{bx}{b+x}}=\frac{b^2}{(a+b)x+\frac{ab^2}{x}+b(2a+b)}.$$

This efficiency is a maximum, when the resistance x satisfies the condition

$$x^2=\frac{ab^2}{a+b}=\frac{ab}{\frac{a}{b}+1},$$

and it has the value

$$u_m = \frac{b^2}{b(2a+b) + 2b\sqrt{a(a+b)}} = \frac{1}{1 + 2\frac{a}{b} + 2\sqrt{\frac{a}{b}\left(1 + \frac{a}{b}\right)}}.$$

The efficiency is nearer unity the smaller is the ratio of the resistances a and b. Contrary to what takes place with the entire current, we should take $b>a$; if we assume that the ratio $\frac{a}{b}$ is extremely small, we have, as approximate value,

$$x = \sqrt{ab}, \qquad \text{and} \qquad u_m = \frac{1}{1 + 2\sqrt{\frac{a}{b}}}.$$

Given the value of u_m, we deduce from it, for the ratio of the resistances a and b,

$$\frac{a}{b} = \frac{1}{4}\left[\frac{1}{u_m} - 1\right]^2.$$

1273. Dimensions of Machines.—Let us consider two machines which are entirely similar, the number of windings being the same on the inductor and on the induced body, and the dimensions of the second being μ times that of the first.

Let us suppose that the machines are dynamos working with the entire current. A first practical condition of a mechanical kind, is that the absolute velocity of the ring at its circumference, whatever be the diameter shall not exceed a given value. As we endeavour to attain the greatest velocity in each case, the number n' of turns of the second machine will then be equal to $\frac{n}{\mu}$.

On the other hand, the magnetisation of the armatures is raised to a value near the maximum, or at any rate should be the same in each case. For the same current, the field of the wires in the second machine, other things being equal, is μ times feebler; the intensity I′ should then be equal to μI.

As the action of a magnet is proportional to its volume, and is inversely as the cube of the distance, the fields of two magnets which are similar, and of the same magnetisation, are equal to each other

at two homologous points. The field of the inductors is then the same in the two machines.

The surface traversed by the windings of the ring being μ^2 times greater, and the field being the same, the flow of force cut at each turn, or the characteristic function, becomes μ^2 times greater—that is to say, $\mu^2\phi(\mathrm{I})$. The electromotive force of the second machine

$$\mathrm{E}' = n'\mu^2\phi(\mathrm{I}) = \mu n\phi(\mathrm{I}) = \mu\mathrm{E}$$

is then μ times greater than that of the first.

Finally, the resistance of the inducing wire, being proportional to its length and inversely as its section, is inversely as μ; as the same ratio holds for the induced wire, the total resistance $a' + b'$ of the second machine is equal to $\frac{a+b}{\mu}$.

The energy lost by heating

$$(a' + b')\mathrm{I}'^2 = \mu(a + b)\mathrm{I}^2,$$

is then proportional to the ratio of the similitude. The heat lost by radiation is, for the same rise of temperature, proportional to its surface—that is, to μ^2. The heating will then be much less for the larger machine.

The ratio of the energies expended is then

$$\frac{\mathrm{W}'}{\mathrm{W}} = \frac{n'\mathrm{I}'\mu^2\phi(\mathrm{I})}{n\mathrm{I}\phi(\mathrm{I})} = \mu^2;$$

the useful work is

$$\mathrm{U}' = \mu^2\mathrm{W} - \mu(a + b)\mathrm{I}^2,$$

and the efficiency

$$u' = 1 - \frac{\mathrm{RI}}{\mu n\phi(\mathrm{I})}.$$

This efficiency increases with the dimensions. The fictive resistance which is equivalent to the effects of self-induction does not change, since the coefficient l for each ring is proportional to its dimensions—that is to say, to μ, and we have

$$mn'l' = mnl = a.$$

But as the resistance of the machine is diminished, the relative importance of the effects of self-induction increases with the dimensions.

The volume and the weight of a machine of given form are proportional to the cube of its dimensions. As the ratio $\frac{W'}{\mu^2}$ is nearly constant, it follows that the quotient of the energy expended by the volume—that is to say, the work for unit volume—is greater as the machine is smaller. Yet as the cost of the machine diminishes less rapidly than its total weight, there is no advantage in reducing the dimensions too much. Hence we should choose a form such that the price of the machine varies as the square of the dimensions for slight changes in the dimensions.

It follows from these remarks that if a particular machine gives excellent results, it does not follow that the same proportions should be maintained for another of larger dimensions. It is better to change the general form, and multiply the parts, by referring them to dimensions of the same order as those of the corresponding organs of the former machine.

If the inductors are excited by an extraneous current, the energy necessary for magnetisation is proportional to μ—that is, to the simple dimensions. The resistance of the coil becomes μ times less, and the electromotive force μ times as great. For the same external resistance, the ratio of the currents will be

$$\frac{I'}{I}=\frac{\mu E}{E}\frac{a+a_1+x}{\frac{a}{\mu}+a_1+x}=\frac{\mu^2(a+a_1+x)}{a+\mu(a_1+x)}.$$

This ratio is equal to μ if the resistance x is very great compared with that of the machine.

1274. Graphical Properties of the Characteristics.—Instead of calculating the effects of machines by the algebraic expression of $\phi(I)$, M. Marcel Deprez* determines them by graphical constructions on the *characteristic* curve—that is to say, on the experimental curve which represents this function. With Fröhlich's formula this curve is a hyperbola, but the results thus calculated do not completely agree with experiment.

When the inductors have no remanent magnetism, $\phi(I)$ is null for $I=0$, and tends towards a maximum value when I increases; the general form of the curve is then hyperbolical.

Let us assume that the resistance R of the machine is independent of the velocity—that is to say, that we neglect the fictive

* Marcel Deprez. *Comptes rendus*, Vol. xcii., p. 1152. 1881.

resistance which represents the effects of self-induction, and that the angle of lead of the brushes is unaltered; for a self-exciting machine with entire current, calling x the useful external resistance, we have the equation

$$I=\frac{n\phi(I)}{R+x}=\frac{nE_1}{R+x},$$

which may be written

$$\frac{E_1}{I}=\frac{\phi(I)}{I}=\frac{R+x}{n}.$$

As the characteristic is given by a curve E_1 as a function of I (Fig. 256), we take OA $=1$ and $AB=\frac{R+x}{n}$; that is to say, that we

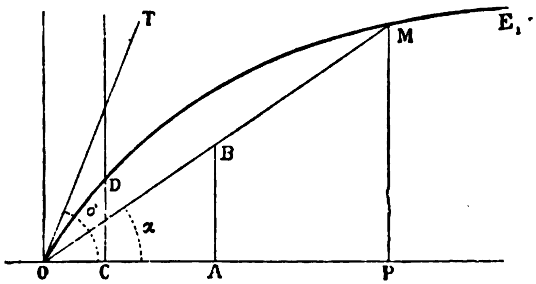


Fig. 256.

draw a straight line OB in a direction α defined numerically by the condition $\tan\alpha=\frac{R+x}{n}$. The point M at which this right line cuts the characteristic, gives the electromotive force of the current $E=n\times MP$, and the corresponding intensity OP of the current.

In order that the machine may produce a current in the given conditions, the right line OB must meet the curve—that is to say, the angle α must be less than the angle δ of the tangent OT at the origin.

If the inductors have an appreciable remanent magnetism, the curve E_1 no longer passes through the origin; we may consider this remanent magnetism as equivalent to that which an initial current i_0 would produce, and the characteristic is sensibly represented by

$\phi(I+i_0)$; it is the same curve (Fig. 257) the origin of which has been displaced by $OC=i_0$. We have again

$$\frac{E_1}{I}=\frac{\phi(I+i_0)}{I}=\frac{R+x}{n};$$

by making the same construction, starting from the point C, we

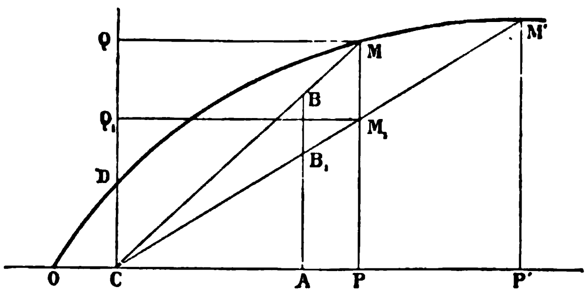


Fig. 257.

shall get the value MP of E_1, and the corresponding intensity CP. In this case the problem is always possible.

If the useful work is an electromotive force to be overcome e, the equation of the current

$$I=\frac{nE_1-e}{R}$$

gives then

$$I\frac{R}{n}=E_1-\frac{e}{n}.$$

Starting from a point C′, at a distance $CC'=\frac{e}{n}$ from the origin, we draw the right line C′B′ (Fig. 258) in a direction α, given by the condition $\tan\alpha=\frac{R}{n}$; we have

$$E_1-\frac{e}{n}=MP'.$$

$$E_1=MP \qquad \text{and} \qquad I=CP.$$

The problem is always possible when the point C′ is below the curve—that is to say, when we have $e < \phi(i_0)$. If this point is above the curve at C″, the right line of construction C″B″ may meet the curve at two points N and N′, the former of which represents an unstable equilibrium. Finally the problem is impossible if this right line does not meet the curve.

The same constructions give also a representation of the work.

The total energy expended in unit time $W = nE_1I$, is equal to n times the area of the rectangle PQ (Fig. 257).

The ratio of the useful to the useless work is $\frac{x}{R}$; by taking $AB_1 = \frac{R}{n}$, this ratio is equal to $\frac{BB_1}{B_1A}$ or $\frac{MM_1}{M_1P}$. The useful work

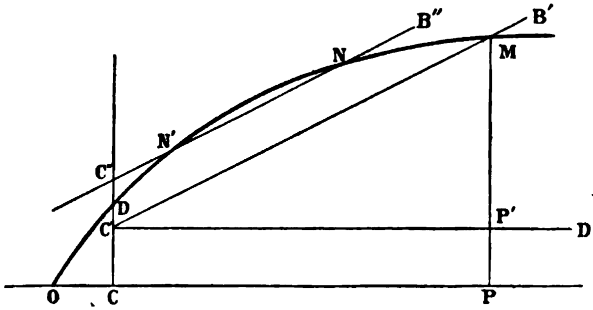


Fig. 258.

is then n times the area of the rectangle QM_1, and the value of the efficiency u is $\frac{MM_1}{MP}$.

If the right line CB_1 be prolonged until it meets the curve at M′, CP′ represents the current I_0 which we should have with the resistance R alone. To obtain the maximum useful work, the point M of the curve must be determined such that the surface of the rectangle QM_1 is a maximum.

1275. Analogous considerations enable us to represent the action of dynamos, when the inductors are excited in the shunt.

If I′ is the current of the inductors, $i = I - I'$ the external current, a the resistance of the ring, b that of the inductors, and putting

$$(1) \qquad y = (a+b)x + ab,$$

we have (1272)

(2) $$I = nE_1 \frac{b+x}{y},$$

(3) $$I' = I \frac{x}{b+x} = nE_1 \frac{x}{y},$$

(4) $$i = I' \frac{b}{x} = nE_1 \frac{b}{y}.$$

Drawing the right line L (Fig. 259), defined by equation (1), in which we take as co-ordinates the numerical values of x and of y, we

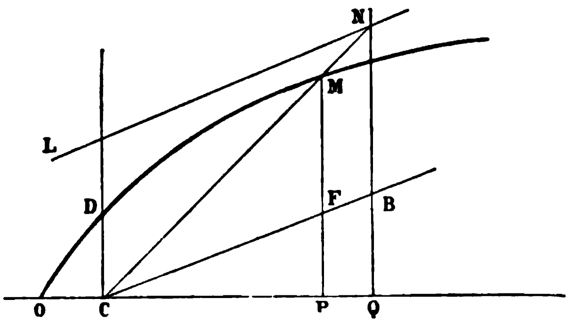


Fig. 259.

take $CQ = x$, draw the vertical QN, and join CN. The point M, at which this right line cuts the characteristic curve, gives the electromotive force E_1, and from equation (3) we have

$$I' = n \times CP.$$

The condition $ix = I'b$ (4) shows also that if we take $QB = b$, and join CB, the current i is represented by $n \times PF$. Finally, the total current is $I = n(CP + PF)$.

In order that the intensity i of the current shall be independent of the resistance x, the ratio $\frac{E_1}{y}$ must be constant—that is to say, the characteristic curve must be a right line parallel to L.

1276. TRANSFORMATION OF CHARACTERISTICS.—When the characteristic has been found for a machine, we may, by a simple geometrical construction, deduce from it the curve which would hold for the same machine, the wires of which had been changed, while retaining the same total volume and the same mode of winding.

In fact, for the inductor, the field of the current which traverses the wires is proportional to the number of turns, and to the intensity of the current. The field will then be the same, and therefore the magnetisation will not be modified, if we replace the number of turns N by N′, on condition that the new intensity I′ satisfies the equation

$$NI = N'I'.$$

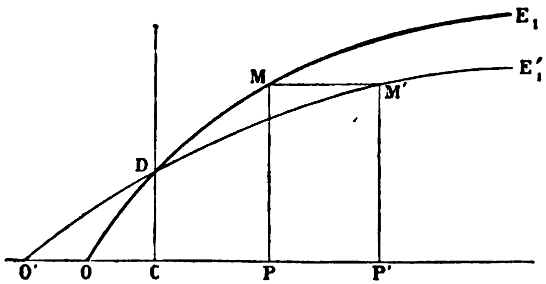


Fig. 260.

The original value of E_1 for the current I is then the same in the new machine for the current

$$I' = \frac{N}{N'} I,$$

and the characteristic curve E_1' of the new machine will be deduced from the characteristic curve E_1 relative to the former, if we take for the same ordinate MP (Fig. 260), abscissæ C′P′ and CP, which are in the ratio of N to N′.

In like manner, if the iron ring is not altered, and we replace the number of turns P by another number P′ of turns occupying the same volume, the corresponding characteristic functions are in the ratio of P to P′.

1277. DISTRIBUTION OF ENERGY.—It is often necessary to use fractions of the current of an electromotor—for instance, in lighting

by electricity or in transmitting force at a distance. This is the problem of the distribution of energy.

The simplest arrangement consists in putting all the apparatus in series in the circuit, by which any given fraction can be utilised in each. This has the twofold inconvenience that recourse must be had to very high electromotive forces, which may be dangerous, and give rise to important losses from defective insulation, and also that a break of the current in any one part stops the work in all the rest.

The apparatus should be so arranged that a break of the work in any one part does not hinder the general work, and as much as possible that a modification in the useful work does not alter the working. The problem is the same as that of the regulation of steam-engines; no exact solution is possible.

In incandescent lighting the lamps are usually arranged in parallel series with two conductors of large section in connection with the machine. Let A and B (Fig. 261) be the two poles of the machine; AA_1 and BB_1 the principal conductors; A_1B_1, A_2B_2

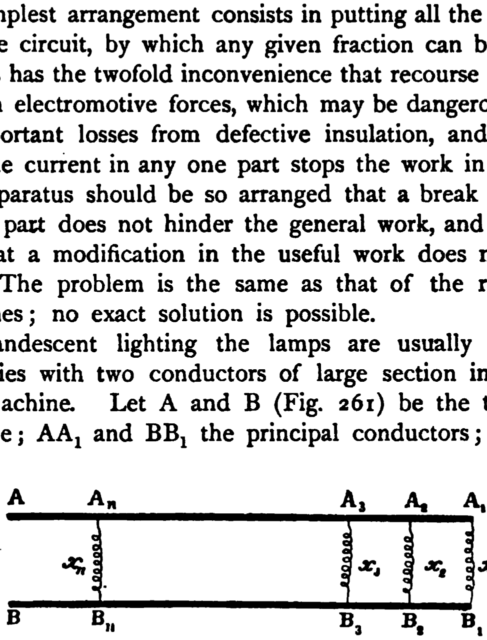


Fig. 261.

$\dots A_nB_n$ the points where the useful resistances x_1, $x_2, \dots x_3$ are connected; a_1, $a_2 \dots a_n$ the resistances of the principal conductors comprised between two successive contacts—that is to say, the resistances of the portions of the circuit $A_1A_2 + B_1B_2$, $A_2A_3 + B_2B_3, \dots$ $A_nA + B_nB$; y_1, $y_2, \dots y_n$, the total resistances between the corresponding points A_2B_2, A_3B_3, A_nB_n AB, not counting the branch directly between them. We have

$$y_1 = a_1 + x_1,$$

$$y_2 = a_2 + \frac{1}{\dfrac{1}{x_2} + \dfrac{1}{y_1}},$$

$$\vdots \quad \vdots$$

$$y_n = a_n + \frac{1}{\dfrac{1}{x_n} + \dfrac{1}{y_{n-1}}}.$$

If we replace the intermediate resistances y by their values, we get for the total resistance y_n, between the two poles A and B,

$$y_n = a_n + \cfrac{1}{\cfrac{1}{x_n} + \cfrac{1}{a_{n-1} + \cdots + \cfrac{1}{a_2 + \cfrac{1}{\cfrac{1}{x_2} + \cfrac{1}{a_1 + x_1}}}}}.$$

Finally, let $e_1, e_2, \dots e_n$ be the differences of potential at the corresponding points of contact; $i_1, i_2, \dots i_n$ the currents in the resistances $a_1, a_2 \dots a_n$. We have generally

$$e_p = i_{p-1} y_{p-1} = i_{p-1} a_{p-1} + e_{p-1},$$

or

$$e_{p-1} = e_p \left[1 - \frac{a_{p-1}}{y_{p-1}} \right].$$

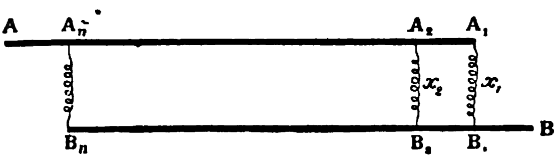


Fig. 262.

The disposable difference of potential in the successive parts decreases as we get away from the machine; the fall is only negligable provided the sum of the resistances is very small compared with that of any one of the parts ρ.

1278. A somewhat more satisfactory solution of the problem is obtained by connecting the poles of the machine with the opposite ends A and B (Fig. 262) of two principal conductors.

Let $a_1, a_2, \dots a_{n-1}$ be the resistances $A_1A_2, A_2A_3, \dots A_{n-1}A_n$; $b_1, b_2, \dots b_{n-1}$ the corresponding resistances of the conductors B; $\alpha_1, \alpha_2, \dots \alpha_{n-1}$ and $\beta_1, \beta_2, \dots \beta_{n-1}$ the respective currents; I the current. We have evidently

$$\alpha_{n-p} + \beta_{n-p} = \alpha_{n-p-1} + \beta_{n-p-1} = I,$$

for the sum of the currents which traverse any plane cut by the principal conductors is constant, and equal to I; we get from this

$$\Sigma\alpha + \Sigma\beta = (n-1)I.$$

Let $i_1, i_2, \ldots i_n$ be the currents in the resistances $x_1, x_2, \ldots x_n$. If all the resistances are respectively equal to each other, we have, from symmetry, at an equal distance from the ends,

$$i_{n-p} = i_{p+1},$$
$$a_{n-p} = \beta_{p+1},$$
$$\Sigma a = \Sigma \beta = \frac{n-1}{2} \mathrm{I} = \frac{n-1}{2} \Sigma i.$$

The shunt currents diminish from the ends of the principal conductors to the middle, for we have

$$aa_p + xi_p = xi_{p+1} + b\beta_p,$$
$$x(i_p - i_{p+1}) = b\beta_p - aa_p = (b+a)\beta_p - a\mathrm{I},$$
$$x(i_p - i_{p+1}) = 2a\left(\beta_p - \frac{\mathrm{I}}{2}\right).$$

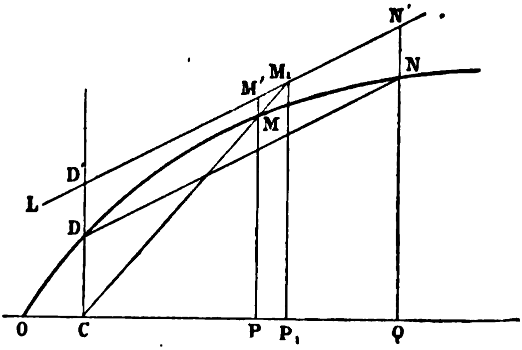


Fig. 263.

Now, the difference $\beta_p - \frac{\mathrm{I}}{2}$ is positive as long as $p > \frac{n}{2}$. In these conditions the different organs would receive in pairs the same energy. The benefit, however, of this arrangement is not great enough to compensate for its practical inconveniences.

In both cases the suppression of one of the shunted parts has the effect of increasing the general current. This inconvenience can be counteracted, either by a regulator of the current, which brings variable resistances into the circuit, or by the aid of an auxiliary current shunted in the circuit, and which keeps constant

the electromotive force of the machine by automatic means, by displacing the brushes, or acting on the supply of steam.

1279. The properties of characteristics have furnished M. Marcel Deprez* with two ingenious mechanical solutions for machines of constant velocity.

If all the parts utilised are arranged in series, and the inductors are shunted, the compensation may be very near when we utilise the characteristic curve in the part in which it is nearly parallel to the right line L

$$y=(a+b)x+ab.$$

Draw through the point D (Fig. 263) a right line DN parallel to the former. If we increase the resistance x—that is to say, the useful work starting from zero—the external current, which is expressed by $i=nb\,\frac{E_1}{y}$, varies as the ratio $\frac{E_1}{y}$.

The ratio is at first equal to $\frac{DC}{D'C}$ when the resistance is null; it then increases to a maximum $\frac{MP}{M'P}$ for the resistance $x=CP_1$; it then diminishes and acquires its original value N'Q for a resistance which will be given by the abscissa of the point where the right line CN cuts L.

As long as the resistance x is less than CP, the machine has the curious property that with a constant velocity the external work increases with the resistance, so that the introduction of a new apparatus into the circuit increases the work of each of the preceding ones. It will thus be understood that within certain limits it is possible to obtain a satisfactory regulation.

The second solution holds for the case in which the apparatus are shunted. Let:

e be the difference of potential of the poles A and B which is to be kept constant,
x the interposed resistance,
R the resistance of the machine,

we have

$$I=\frac{E-e}{R}.$$

* MARCEL DEPREZ. *Comptes rendus*, Vol. XCIII., pp. 892, 950. 1881.

If the electromotive force E were of the form $E = e + IR$, this equation will be satisfied for any given value of I, and the current

$$I = \frac{e}{x}$$

will be proportional to $\frac{1}{x}$, that is to say, to the number of parts interposed between the poles.

If the total electromotive force E be represented by a curve (Fig. 264), making the ordinary construction,

$$AB' = R,$$

$$B'B = x,$$

we have

$$E = MP, \qquad e = MM'.$$

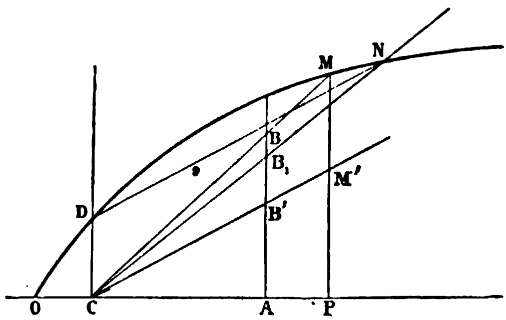


Fig. 264.

In order that the value MM′ shall be independent of x, the curve ODM must be a right line parallel to CB′—that is to say, represented by $e + IR$.

M. Marcel Deprez realises these conditions practically, by closing the inducing coil by two wires, one f traversed by a constant current i, the other F traversed by the entire current of the machine. If the magnetisation were proportional to the current, we should have

$$E = C(i + I) = C' + CI.$$

The value of the current i and the coiling of the wire are altered by trial so as to obtain an approximate solution. A curve is obtained parallel to the preceding, in which that portion DN is utilised which is virtually parallel to CB′.

For a very great resistance x—that is to say, with a single apparatus introduced—we have $e = DC$. As this resistance diminishes—that is to say, as the number of apparatus increases—the difference of potential at first increases, then diminishes, and resumes its original value for $x = B'B_1$. Starting from this point, the addition of fresh apparatus rapidly diminishes the work of each.

Machines which are thus provided with a double winding are often called *compound.*

1280. Machines with Variable Current.—The current produced by mechanical electromotors is of course alternating, and it may be used in this form.

Let:

L be the coefficient of self-induction of the circuit,
Q the flow of magnetic force which it traverses at the time t,
R its total resistance;

the induced current I (**518**) is defined by the equation

$$L\frac{dI}{dt} + RI + \frac{dQ}{dt} = 0. \tag{5}$$

It will, however, be remarked that if the induced coils contain soft iron, the coefficient L is no longer constant unless the magnetisation is proportional to the magnetising force. In the general case we must consider the flow of force LI as a function of the current which tends towards a maximum with the magnetisation, and replace the first term of equation (5) by

$$\frac{d(LI)}{dt} = \left(L + I\frac{dL}{dI}\right)\frac{dI}{dt}.$$

The following results can only then be applied with restriction to machines with electromagnets.

As the flow of force as well as its differential is periodical, the current presents the same characteristics. If this flow of force is sinusoidal, as for a frame which turned in a uniform field with a constant velocity ω (**535**), and if T is the duration of the period, we shall have

$$Q = Q_0 \cos 2\pi\frac{t}{T} = Q_0 \cos \omega t,$$

$$E = -\frac{dQ}{dt} = Q_0\omega \sin \omega t.$$

For a machine, the factor ω is the product of the angular velocity by half the number of reversals of the current at each turn.

The expression for the current, starting from the closing of the circuit, contains first an exponential; but, if the regime is permanent, the current itself is sinusoidal, and of the form

$$I = A \sin 2\pi\left(\frac{t}{T} - \phi\right) = A \sin \omega(t - \phi T),$$

with the conditions

$$A^2 = \frac{Q_0^2\omega^2}{R^2 + L^2\omega^2},$$

$$\tan 2\pi\phi = \frac{L\omega}{R}.$$

The coefficient A represents the mean intensity of the current, and $2\pi\phi$ the difference of phase, being the fraction of the period which elapses between the moment at which the electromotive force is null and that in which the current passes through zero.

The quantity $\sqrt{R^2 + L^2\omega^2}$ may be called the *apparent resistance;* the effect is as if the square of a fictive resistance $L\omega$ proportional to the velocity, was added to the square of the real resistance.

The mean square of the intensity

$$I''^2 = \frac{A^2}{2} = \frac{Q_0^2\omega^2}{2(R^2 + L^2\omega^2)}$$

is equal to the quotient of the mean square of the electromotive force by the square of the apparent resistance. This value does not increase indefinitely with the velocity, it tends towards a limit $\frac{Q_0^2}{2L^2}$ independent of the resistance.

The difference of phase increases with the velocity, and at the limit we have $2\pi\phi = \frac{\pi}{2}$, or $\phi = \frac{1}{4}$.

1281. The energy expended in unit time is

$$W = I''^2 R = R\frac{A^2}{2} = \frac{Q_0^2\omega^2}{2\left(R + \frac{L^2\omega^2}{R}\right)}.$$

For a given velocity this energy is a maximum when $R = L\omega$, which gives

$$W_m = \frac{Q_0^2}{4L}\omega,$$

$$I_m''^2 = \frac{Q_0^2}{4L^2},$$

$$\tan 2\pi\phi = 1, \qquad \text{or} \qquad \phi = \frac{1}{8}.$$

The resistance corresponding to the maximum energy and the value of this maximum are proportional to the velocity. The mean square of the intensity, and the retardation relative to the maximum, have constant values half those which they would have if the velocity were infinite.*

If the total resistance R is made up of two parts, the one a which comprises the machine and the connections, the other x corresponding to a useful work u, and having no self-induction, we have

$$U = I''^2 x = W\frac{x}{a+x},$$

$$U = \frac{Q_0^2\omega^2}{2}\frac{x}{(a+x)^2 + L^2\omega^2} = \frac{Q_0^2\omega^2}{2}\frac{1}{2a + x + \dfrac{a^2 + L^2\omega^2}{x}}.$$

It will be seen that at constant velocity the useful work is a maximum when the value of x is equal to the apparent resistance of the electromotor; the corresponding values of the useful work and of the efficiency are then

$$U_m = \frac{Q_0^2\omega^2}{4}\frac{1}{a + \sqrt{a^2 + L^2\omega^2}},$$

$$u_m = \frac{x}{a+x} = \frac{1}{1 + \dfrac{a}{\sqrt{a^2 + L^2\omega^2}}} > \frac{1}{2}.$$

The machines with alternating currents have the remarkable property that the relative efficiency for the maximum useful work

* JOUBERT. *Ann. de l'École Normale* [2], Vol. x., p. 131. 1881.

is greater than 50 per cent. This efficiency is nearer unity the greater the velocity and the greater the coefficient of self-induction.

1282. In order to rectify currents in the outer circuit a commutator is used, formed, for instance, of two half-rings B and B′, connected respectively with the two ends of the induced wire, and of two springs *b* and *b′* forming the termination of the external circuit. The springs should pass from one half to the other of the ring at the moment the current is null; with this object the diameter of the contacts *bb′* must be displaced in the direction of the motion through an angle $2\pi\phi$, starting from the position which the diameter of the section occupies at the moment the electromotive force is null.

If these conditions were exactly realised, the mean intensity I′ of the external current (535) would be

$$I'^2 = \frac{4A^2}{\pi} = \frac{4Q_0^2\omega^2}{\pi\left[(a+x)^2 + L^2\omega^2\right]}.$$

No sparks should be produced in the commutator; but they cannot be entirely avoided, for the current varies very rapidly in passing through zero, and at the moment of the inversion each of the springs is for an appreciable time in contact with the two half rings. The machine is then short circuited; there is accordingly a loss of energy by the heating of the circuit, and a production of sparks on breaking contact. The inconvenience of rectifying the current is due less to the loss of energy than to the rapid waste of the commutator.

1283. Let us suppose that the external circuit contains an electromotive force E′; the differential equation becomes

$$(6) \qquad L\frac{dI}{dt} + RI + \frac{dQ}{dt} = E'.$$

When this electromotive force is constant, like that of a battery, the intensity of the current for a permanent regime is expressed by

$$I = A \sin \omega(t - \phi T) + \frac{E'}{R}.$$

The induced current simply adds itself with its sign to the current

of the battery; the total current is always in the same direction, or alternately in opposite directions, according as E′ is greater or less than AR. In these conditions the machine would have no application; it can only produce a useful result if at each semi-period the direction of one of the electromotive forces is reversed.

This reversal is produced spontaneously when the external electromotive force is a polarization, like that of voltameter or an electric arc.

If the electromotive force is always less than its maximum value, the current is still sinusoidal, and the useful work is zero. When it is possible that the electromotive force of the machine may become greater than the maximum polarization, the current presents two very distinct characters. As long as the polarization is not overcome, if c is the capacity of the electrodes, the differential equation is first

$$(7) \qquad L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{I}{c}+\frac{d^2Q}{dt^2}=0.$$

We should then take equation (6) during the fraction of the period for which the maximum E′ of polarization is attained. The expression for the current is then very complicated, since the conditions are different for two portions of the same period, and the exponentials do not neutralize each other.

1284. ON TRANSFORMERS.—M. Jablochkoff* has had the idea of interposing in the circuit of an alternate-current machine a series of shunt condensers, so as to distribute the energy of the current on different parallel circuits, in the ratio of their resistance and the capacity of the corresponding condenser. He then applied the same arrangement to the secondary currents of an induction coil worked by an alternating primary current. If the same primary current supplies several coils arranged in series, the induced current of each takes its energy in an indirect form from the principal circuit; in this way a method of distributing energy is effected comparable with the use of shunt wires for continuous circuits. This is the principle of the method used for several years and known as *transformers.*

Let us suppose that a circuit containing a sinusoidal electromotive force $E_0 \sin \omega t$ acts on an adjacent closed circuit, assuming that there are no electromagnets, or at any rate that the magnetisation of iron is proportional to the magnetising force.

When the permanent regime is attained, the expressions for the primary current and the secondary one are of the form

$$I = A \sin 2\pi\left(\frac{t}{T} - \phi\right) = A \sin \omega(t - \phi T),$$

$$I' = A' \sin 2\pi\left(\frac{t}{T} - \phi'\right) = A' \sin \omega(t - \phi' T).$$

The constants of the circuits being defined as before (544), if we put

$$r = R + R' \frac{M^2\omega^2}{R'^2 + L'^2\omega^2},$$

$$l = L - L' \frac{M^2\omega^2}{R'^2 + L'^2\omega^2},$$

we have

$$A^2 = \frac{E_0^2}{r^2 + l^2\omega^2}, \qquad \tan 2\pi\phi = \frac{l\omega}{r},$$

$$A'^2 = A^2 \frac{M^2\omega^2}{R'^2 + L'^2\omega^2}, \qquad \tan 2\pi\phi' = -\frac{1}{\omega}\,\frac{1 - \dfrac{LL' - M^2}{RR'}\omega^2}{\dfrac{L}{R} + \dfrac{L'}{R'}}.$$

The energy expended over the whole of the circuits in unit time is

$$W = R\frac{A^2}{2} + R'\frac{A'^2}{2},$$

and the energy transformed

$$W' = R'\frac{A'^2}{2}.$$

The fraction of the total energy transformed, or the coefficient of transformation, is

$$w = \frac{W'}{W} = \frac{R'A'^2}{RA^2 + R'A'^2} = \frac{1}{1 + \dfrac{R}{R'}\,\dfrac{R'^2 + L'^2\omega^2}{M^2\omega^2}}.$$

This fraction increases with ω; the maximum, which corresponds to infinite velocity of the electromotor, is

$$w_m = \frac{1}{1 + \frac{R}{R'}\frac{L'^2}{M^2}}.$$

In the transformers of MM. Gaulard and Gibbs,* the two wires, primary and secondary, are wound together on a cylindrical core formed of a bundle of iron wires. The transformers of MM. Zipernowski, Deri and Blathy,† on the contrary, consist of a ring of iron wires on which the two circuits are coiled together as in the windings of a Gramme machine.

1285. The coefficient of self-induction of the primary circuit contains a term relative to the machine itself; we shall assume that it is very small compared with that which arises from the portion of the circuit comprised in the transformer. If the inducing and the induced circuit have the different numbers of turns N and N′, and the mean diameter is almost the same, we have sensibly

$$\frac{L}{N^2} = \frac{L'}{N'^2} = \frac{M}{NN'} \qquad \text{and} \qquad LL' - M^2 = 0;$$

and may then write

$$w_m = \frac{1}{1 + \frac{R}{R'}\frac{N'^2}{N^2}}.$$

Let us consider the resistance R′ of the secondary circuit as formed of two parts, the one b' comprising the transformer and the connections, and the other x which corresponds to the useful work U; we have

$$U = x\frac{A'^2}{2} = x\frac{A^2}{2}\frac{M^2\omega^2}{R'^2 + L'^2\omega^2}.$$

* L. GAULARD and J. D. GIBBS. *La Lumière Électrique*, Vol. VIII., p. 189. 1883.

† G. FERRARIS. *La Lumière Électrique*, Vol. XVII., p. 145. 1885.

If we assume that the duration of the period is very small, we have sensibly

$$l = L - \frac{M^2}{L'} = 0,$$

$$r = R + R'\frac{M^2}{L'^2} = R + R'\frac{N^2}{N'^2},$$

$$A = \frac{E_0}{R + R'\frac{N^2}{N'^2}}.$$

The efficiency is

$$u = \frac{U}{W} = \frac{xA'^2}{RA^2 + (b' + x)A'^2} = \frac{x}{b' + x + R\frac{A^2}{A'^2}}.$$

This efficiency increases still with ω, and its maximum value

$$u_m = \frac{x}{b' + x + R\frac{L'^2}{M^2}} = \frac{x}{b' + x + R\frac{N'^2}{N^2}}$$

gives for $N = N'$

$$u_m = \frac{x}{b' + x + R}.$$

In the case in which the two wires of the transformer are coiled in the same way, the efficiency is equal to the quotient of the useful resistance by the total resistance, as for permanent currents. The maximum useful work corresponds to the condition $x = b' + R$, and the efficiency is then 0·50.

It should be observed that the maximum efficiency u_m is the greater the smaller is the ratio $\frac{N'}{N}$. It is then advantageous that the secondary wire should have a smaller number of turns than the primary wire.

On the other hand, the primary circuit comprises two resistances, the one a in the electromotor, and the other b in the transformer; we may say that the efficiency for the transformer itself is expressed by

$$v = \frac{xA'^2}{(b' + x)A'^2 + bA^2}.$$

1286. If a series of similar apparatus are arranged on the same primary circuit, the problem will be determined, like the preceding, by the aid of a series of differential equations (**540**) relative to each of the transformers, and which should be satisfied at the same time:

$$\Sigma M\frac{dI'}{dt}+L\frac{dI}{dt}+RI-E=0,$$

$$M_1\frac{dI}{dt}+L_1'\frac{dI_1'}{dt}+R_1'I_1'=0,$$

$$M_2\frac{dI}{dt}+L_2'\frac{dI_2'}{dt}+R_2'I_2'=0, \text{ etc.}$$

Instead of treating the general case, we may suppose simply that all the transformers are identical, and that each of them has the same useless resistance.

The shunt-currents being equal, the result is the same as if all the induced circuits were joined in series, without the addition of fresh resistances. The expression for the efficiency is the same as above, provided we consider b' and x as representing, in the secondary wires, the total resistances of the transformers and of the useful work.

The effects are really more complicated, for we should consider the coefficients of induction as functions of the intensities.

1287. Induction Coils.—The induction coil is a true transformer, by which we endeavour to obtain, in the induced wire, considerable electromotive forces, capable of producing long sparks, of charging batteries, and reproducing the phenomena which are realised by means of electrostatic machines. The primary current is furnished by a constant electromotive force, and the induced currents by the opening or closing of the primary circuit.

These coils, as we know, are formed of a central cylinder of iron wire, on which is coiled a stout wire, through which the primary current passes, and then a secondary wire of far smaller diameter, in which are formed the induced currents, and the windings of which are carefully insulated, and terminate in two binding screws forming the poles.

The induced currents depend specially on the coefficient of mutual induction M. We have seen (**554**) that the best conditions are when the radius x of the core, the external radius y of the primary circuit, and the radius z of the secondary circuit are to

each other as the numbers 2, 3, and 4; the thicknesses $y-x$ and $z-y$ of the two coils are then equal, each being $\frac{z}{4}$.

Supposing that k is constant, the approximate value of M is

$$M=\left(\frac{19}{12}+4\pi k\right)\pi^2 n_1^2 n_2^2 l(z-y)(y-x)z^2.$$

Denoting, by N and N′, the numbers of turns $n_1^2 l(y-x)$ and $n_2^2 l(z-y)$ of the principal and of the secondary wire, we may write

$$M=\left(\frac{19}{12}+4\pi k\right)\pi^2 NN'\frac{z^2}{l}.$$

The coefficient of self-induction of the secondary coil is

$$L'=\left(\frac{19}{12}+4\pi k\right)\pi^2 N'^2\frac{z^2}{l},$$

and that of the primary wire

$$L=\left(\frac{19}{12}+4\pi k\right)n^2 N^2\frac{z^2}{l}.$$

If the secondary current is closed, the discharge induced by making or breaking the primary current I will be equal to $\frac{MI}{R'}$, and independent of the duration of the variable state; but, when the secondary circuit is broken, we must diminish as much as possible the duration of the variable state, so that the discharge may pass as a spark across the break between the poles.

It will be seen that the result will be better on breaking than on making the principal current. Nevertheless, as the coefficient L is very large, the extra current on breaking produces a powerful spark, the effect of which is to increase the time of variation of the inducing current, and therefore to diminish the induced electromotive force.

1288. One of the first improvements, due to Foucault,* consists in replacing the metal contact-breakers by a platinum point, which

* FOUCAULT. *Comptes rendus*, Vol. XLII., p. 215. 1856.

dips in a layer of mercury covered with water or alcohol. The spark is shorter, for the alcohol rapidly condenses the metallic vapours which facilitate the passage of the extra current.

A more important improvement was effected by M. Fizeau.* The ends of the principal circuit, on either side the point where the break is made, communicate separately with the armatures of a condenser. The difference of potential of the two points is thus reduced, for the electricity due to the extra current rushes into the condenser. This condenser then discharges itself when the contact is made, so that, by increasing its capacity, we always diminish the spark on breaking; but we finish by increasing those on making. Experiment shows what capacity gives the best results.

Finally, the windings of the secondary coil are usually arranged in successive layers, which occupy the entire length coil, and which are separated from each other by insulating layers. For coils of great dimensions, this arrangement has the disadvantage that the adjacent windings of two consecutive layers are separated by a great length of induced wire; there is thus a great difference of potential between them, which can perforate the insulating layer—or, at any rate, produce losses of electricity. To Poggendorff† is due the idea of *partitioning* the coils—that is to say, arranging the secondary wire in a series of successive compartments separated from each other by insulating plates perpendicular to the axis of the coil. The potential of the secondary wire thus increases from one end to the other of the coil, without producing too great a difference between two adjacent layers of the same compartment.

When the ends of a secondary wire are connected with the armatures of a condenser, the maximum difference of potential diminishes, for the capacity of the induced wire increases; but this connection is equivalent to the introduction of a conductor; the discharge is accordingly increased. If the poles are separated by so small a distance that the spark can pass across it, the quantity of electricity corresponding to the discharge increases then with the capacity of the condenser. As the striking distance is, in each case, raised to a maximum, the corresponding energy also increases as the distance diminishes.

If the striking distance is not to be too much diminished by the introduction of a condenser, the capacity of this condenser must be so small that it can bear a great difference of potential. In that case

* FIZEAU. *Comptes rendus*, Vol. XXXVI., p. 418. 1853.

† POGGENDORFF. *Pogg. Ann.*, Vol. XCIV., p. 289. 1850.

it is advantageous, as Cazin* has done, to use batteries or Leyden jars, arranged in cascade (95).

1289. It is somewhat difficult to take account of all the effects of an induction coil, since the spark, on breaking, introduces into the principal circuit a rapidly varying resistance, which plays the most important part in the character of the induced discharge. We may form an idea of the play of the apparatus by supposing that the two circuits are closed, and that the coil is used as distributor, with a sinusoidal electromotive force of very short period in the primary circuit. We have then sensibly $A'L' = AM$.

The maximum electromotive force on the induced wire is

$$A'R' = AR\frac{M}{L'}\cdot\frac{R'}{R}.$$

If U and U' are the volumes of the wires, y and y' their diameters, if the thickness of the insulating layer is proportional to the diameter of the wire in the two coils, the ratio of the resistances R' and R is (726)

$$\frac{R'}{R} = \frac{U'}{U}\left(\frac{y}{y'}\right)^4.$$

But we have (1287)

$$\frac{U'}{U} = \left(\frac{4}{3}\right)^2, \qquad N'y'^2 = Ny^2;$$

consequently

$$\frac{R'}{R} = \left(\frac{4}{3}\right)^2\frac{N'^2}{N^2}.$$

Lastly, the ratio of the coefficients M and L' is sensibly equal to the ratio of N to N', which gives

$$\frac{A'R'}{AR} = \left(\frac{4}{3}\right)^2\frac{N'}{N}.$$

The maximum electromotive force developed on the induced wire is then proportional to the number of turns.

* CAZIN. *Comptes rendus*, Vol. XLVI., p. 307. 1863.

The ratio of the energy disengaged in the secondary wire to that which is expended in the principal wire is

$$\frac{A'^2R'}{A^2R}=\frac{A'^2R'^2}{A^2R^2}\frac{R}{R'}=\left(\frac{4}{3}\right)^4\frac{N'^2}{N^2}\left(\frac{3}{4}\right)^2\frac{N^2}{N'^2}=\left(\frac{4}{3}\right)^2=\frac{16}{9},$$

and the efficiency

$$w=\frac{16}{25}=0{\cdot}64.$$

Nevertheless, these conditions assume that the magnetisation of iron is proportional to the magnetising force—a hypothesis which is, undoubtedly, far from the truth.

CHAPTER II.

NUMERICAL CONSTANTS.

1290. Chemical Action.—Experiment shows that a coulomb or an ampère per second reduces 1·1173 mgr. of silver and decomposes 0·09316 mgr. of water (**918**); it follows that for the

ACTION OF AN AMPÈRE.

		During a Minute.	During an Hour.
		Mgr.	Gr.
Silver reduced	Ag = 107·93	67·04	4·022
Copper ,, ...	Cu = 31·98	19·74	1·184
Water decomposed	HO = 9·0	5·59	0·3354
Hydrogen.	H = 1·0	0·6211	0·03726
		C.C.	C.C.
Volume of Hydrogen at 76 cm. and 0°.	—	6·933	416·0
,, Detonating Gas	—	10·40	624·0

1291. Resistances.—All the following resistances are arranged in reference to the legal ohm. If we assume that the resistance of 10^9 C.G.S. units were represented by a column of mercury of 106·25 cm. instead of 106 cm., we should multiply the conductivities by the ratio $\frac{106{\cdot}25}{106} = 1{\cdot}0024$ and divide the resistances by the same number.

METALS AND ALLOYS.

At the Temperature of 0°.

Nature of Conductors.	Value in C.G.S. Units.		Resistance in Ohms.	
	Specific Resistance.	Specific Conductivity.	1 Metre Weighing 1 Gramme.	100 Metres 1 Mm. in Diameter.
Silver annealed . . .	$1{\cdot}492.10^3$	$67{\cdot}03.10^{-6}$	0·1517	1·899
,, hard drawn . .	1·620	61·73	0·1650	2·062
Copper annealed . . .	1·584	63·13	0·1415	2·017
,, hard drawn . .	1·621	61·69	0·1443	2·063
Gold annealed . . .	2·041	49·00	0·4007	2·598
,, hard drawn . . .	2·077	48·14	0·4076	2·644

Nature of Conductors.	Value in C.G.S. Units.		Resistance in Ohms.	
	Specific Resistance.	Specific Conductivity.	1 Metre Weighing 1 Gramme.	100 Metres 1 Mm. in Diameter.
Aluminum annealed . .	2·889	34·61	0·0743	3·678
Zinc compressed . . .	5·580	17·92	0·3995	7·105
Platinum annealed . .	8·981	11·14	1·925	11·435
Iron ,, . .	9·636	10·38	0·7518	12·27
Nickel ,, . .	12·356	8·093	1·052	15·73
Tin ,, . .	13·103	7·632	0·9564	16·68
Lead compressed . . .	19·465	5·137	2·217	24·78
Antimony ,, . . .	35·21	2·84	2·370	44·83
Bismuth ,, . . .	130·10	0·769	12·80	165·60
Mercury liquid . . .	94·34	1·06	12·826	120·11
Alloy 2Pt + 1Ag . . .	24·187	4·135	2·907	30·79
,, 2Au + 1Ag . . .	10·776	9·280	1·638	13·72
,, 9Pt + 1Ir . . .	21·633	4·627	4·651	27·54
German Silver . . .	20·76	4·817	1·817	26·43

The numbers of the first column, omitting the factor 10^8, represent the specific resistances estimated in microhms. The same numbers represent in ohms the resistance of a wire 100 metres in length, having a section of a square millimetre.

Influence of Temperature.

The change of resistance and of alloys with the temperature may in general (998) be expressed by the formula

$$R = R_0(1 + \alpha t + \beta t^2).$$

For mercury in glass, up to 100°, the values of the coefficients α and β are

α	β
0·000,864,9	0·000,001,12 *
0·000,857,7	0·000,000,897 †

For other conductors the results vary with different specimens, arising no doubt from traces of impurities or changes of physical

* Mascart, De Nerville, and Benoît, *Ann. de Chim. et de Phys.* [6], Vol. VI., p. 77. 1884.

† R. Lenz and N. Restzoff. *Études Électrométrologiques*, Vol. II. 1884.

condition, and we can at present give anything scarcely but the mean values.

Mean Coefficient between 0° and 20°.

Silver	$0{\cdot}377 . 10^{-2}$ to $0{\cdot}405 . 10^{-2}$*
Copper	0·388
Gold	0·365
Aluminum	0·390†
Platinum	0·247†
Iron	0·453†
Tin	0·365
Lead	0·387
Antimony	0·389
Bismuth	0·354
Mercury	0·0887‡
Alloy 2Pt + 1Ag	0·022 to 0·0311 §
„ 2Au + 1Ag	0·065
„ 9Pt + 1Ir	0·133*
German Silver	0·028 to 0·044 §

The numbers without any reference are deduced from the experiments of Matthiesen.

Low Temperatures. ||

Silver	$0{\cdot}305 . 10^{-2}$	between	+ 30°	and	− 120°
Aluminum	0·388	„	28	„	− 90
Magnesium	0·390	„	0	„	− 88
Tin	0·424	„	0	„	− 85
Copper	0·418	„	0	„	− 58
„	0·425	„	− 69	„	− 123
Iron	0·490	„	9	„	− 92
Platinum	0·342	„	0	„	− 95
Solid Mercury	0·407	„	40	„	− 92

For very high temperatures, and as far as 1000°, Sir W. Siemens finds that the resistance, as a function of the absolute temperature T

* CH. RIVIÈRE. 1884.

† BENOÎT. *Comptes rendus*, Vol. LXXVI., p. 345. 1873.

‡ Lord RAYLEIGH takes 0·0861 and M. LENZ 0·0879.

§ MASCART, DE NERVILLE, and BENOÎT, *loc. cit.*

|| CAILLETET and BOUTY. *Comptes rendus*, Vol. C., p. 1188. 1885.

(998), is given by the following expressions, in which R_0 represents the resistance at the temperature of 0° centigrade,

Platinum	$R=R_0[0{\cdot}16315\sqrt{T}+0{\cdot}008968T-1]$,
Copper	$R=R_0[0{\cdot}11682\sqrt{T}+0{\cdot}013821T-1]$,
Iron	$R=R_0[0{\cdot}05852\sqrt{T}+0{\cdot}003076T-1]$.

LIQUIDS.

The conductivity of a solution increases at first with its strength, and passes generally through a maximum, especially in the case of very soluble bodies. The conductivity, moreover, increases with the temperature.

In tables for liquids, the specific resistances are estimated in ohms.

By the method indicated (990), M. Paalzow has obtained the following numbers for different solutions:—

Aqueous Solutions.

Substance Dissolved.	Composition.	Temperature.	Specific Resistance.
Sulphuric Acid . . .	SO^3HO	15°	9·146
	$SO^3HO + 14HO$	19	1·336
	$SO^3HO + 13HO$	22	1·256
	$SO^3HO + 499HO$	22	17·431
Sulphate of Zinc . . .	$ZnOSO^3 + 23HO$	23	18·31
	$ZnOSO^3 + 24HO$	23	18·02
	$ZnOSO^3 + 105HO$	23	33·04
,, Copper . .	$CuOSO^3 + 45HO$	22	19·10
	$CuOSO^3 + 105HO$	12	31·42
,, Magnesia .	$MgOSO^3 + 34HO$	22	18·44
	$MgOSO^3 + 107HO$	22	30·06
Hydrochloric Acid . .	$HCl + 7{\cdot}5HO$	23	1·285
	$HCl + 250HO$	23	8·177

By means of sinusoidal currents (992) MM. Kohlrausch and Nippoldt* have investigated a great number of solutions, and have expressed their results in reference to percentages of solution.

* KOHLRAUSCH. *Leitfaden der Praktischen Physik*, p. 298.

Specific Resistances of Aqueous Solutions.

Weight of Body in 100 Parts.	Common Salt.	Sal Ammoniac.	Sulphate of Soda.	Sulphate of Zinc.	Potass.
5	14·97	10·82	24·82	52·40	5·86
10	8·34	5·683	14·74	31·44	3·20
15	6·16	3·898	11·37	24·19	2·37
20	5·15	2·995	—	21·94	2·02
25	4·72	2·501	—	21·43	1·87
30	—	—	—	23·01	1·86
35	—	—	—	28·58	1·98
	Sulphuric Acid.	Nitric Acid.	Hydrochloric Acid.	Nitrate of Silver.	Iodide of Potassium.
5	4·84	3·92	2·56	39·31	29·49
10	2·58	2·19	1·59	21·39	14·75
15	1·86	1·65	1·35	14·74	9·62
20	1·55	1·42	1·32	11·64	6·92
25	1·41	1·31	1·39	9·53	5·39
30	1·38	1·29	1·52	8·13	4·38
35	1·39	1·31	1·71	7·20	3·67
40	1·49	1·38	1·95	6·46	2·25
50	1·87	1·59	—	5·45	2·57
60	2·70	1·96	—	4·81	2·27
70	4·67	2·55	—	—	—
80	9·15	3·78	—	—	—

Maximum Conductivity.

Bodies Dissolved.	Weight in Hundredths.	Density.	Specific Resistance.
Nitric Acid	20·7	1·185	1·287
Hydrochloric Acid	18·3	1·1092	1·315
Sulphuric Acid	30·4	1·224	1·364
Potass	28·0	1·274	1·850
Sulphate of Zinc	23·5	1·286	21·35

Specific Resistances of Solutions as a Function of their Density.

Sulphuric Acid at 22° (Kohlrausch and Nippoldt).

Density of the Solution.	Proportion of Acid.	Specific Resistance.	Relative Increase of Conductivity for 1°.
0·9985	0·0	70·41	$0{\cdot}47 \,.\, 10^{-2}$
1·0000	0·2	41·05	0·47
1·0504	8·3	3·252	0·653
1·0989	14·2	1·787	0·646
1·1431	20·2	1·414	0·799
1·2045	28·0	1·239	1·317
1·2631	35·2	1·239	1·259
1·3163	41·5	1·347	1·410
1·3547	46·0	1·487	1·674
1·3994	50·4	1·672	1·582
1·4482	55·2	1·962	1·417
1·5026	60·3	2·412	1·794

Sulphate of Copper at 10° (Ewing and MacGregor).*

Density.	Specific Resistance.	Density.	Specific Resistance.
1·0167	164·4	1·1386	35·0
1·0216	134·8	1·1432	34·1
1·0318	98·7	1·1679	31·7
1·0622	59·0	1·1829	30·6
1·0858	47·3	1·2051	29·3
1·1174	38·1	(saturated).	

Sulphate of Zinc at 10° (Ewing and MacGregor).

Density.	Specific Resistance.	Density.	Specific Resistance.
1·0140	182·9	1·2709	28·5
1·0187	140·5	1·2891	28·3
1·0278	111·1	1·2895	28·5
1·0540	63·8	1·2987	28·7
1·0760	50·8	1·3288	29·2
1·1019	42·1	1·3530	31·0
1·1582	33·7	1·4053	32·1
1·1845	32·1	1·4174	33·4
1·2186	30·3	1·4220	33·7
1·2562	29·2	(saturated).	

Nitric Acid (density = 1·36).

Temperature.	Specific Resistance.	Temperature.	Specific Resistance.
2°	1·74	16°	1·39
4	1·83	20	1·30
8	1·65	24	1·22
12	1·50	28	1·28

Solution of Chloride of Potassium.

M. Bouty† has determined, with particular care, the resistance of solutions of chloride of potassium, which might be used as standards in comparative experiments. Putting $R_0 = R(1 + \alpha t)$, we have, between 0° and 30°:

Number of Equivalents per litre. 1 eq. = 74·59.	Specific Resistance at 0°. R_0.	Coefficient of Variation. α.	Number of Equivalents per litre. 1 eq. = 74·59.	Specific Resistance at 0°. R_0.	Coefficient of Variation. α.
3	5·172	0·0230	0·2	72·23	0·0326
2	7·785	0·0259	0·1	141·0	0·0327
1	15·415	0·0291	0·01	1325	0·0333
0·5	30·49	0·0302	0·001	12697	0·0333

* These values are considerably higher than those of M. Kohlrausch.

† BOUTY. *Comptes rendus*, Vol. CII., p. 1097. 1886.

Water.

The properties of water are greatly influenced by traces of foreign substances which cannot be detected by analysis. The specific resistance varies, according to different observers, from 0·3 megohm (Foussereau) to 7 megohms (Kohlrausch). Ice has at least 100,000 times the resistance of liquid water at the same temperature.

DIELECTRICS.

The specific resistance in ohms may be expressed as a function of the temperature by the formula:*

$$\text{Log. } R = A - Bt + Ct^2.$$

	Density.	A.	B.	C.	Extreme Temperature.
Bohemian Glass . .	2·431	13·783	0·0495	$7·11.10^{-6}$	$-15°$ to $50°$
Ordinary ,, . .	2·539	15·005	0·0527	0·37	-17 ,, 60
Crystal	2·933	19·224	0·0880	28·07	45 ,, 105
Porcelain	2·933	17·734	0·0520	7·21	50 ,, 210
Phosphorus, solid		11·2103	0·01475	$-22·55$	10 ,, 42
,, liquid		6·5035	0·00523	$-4·34$	25 ,, 100
Ice		9·6006	0·08797	$-127·2$	-1 ,, -17

At the temperature zero:

	Specific Resistance.		
Bohemian Glass	$6·07.10^{18}$	or 60·7	millions of megohms.
Ordinary ,, ...	$1·012.10^{15}$	,, 1012	,, ,,
Flint ,, ...	$1·675.10^{19}$	,, 1675.10^4	,, ,,
Porcelain	$5·421.10^{17}$	,, 5421.10^2	,, ,,

After several Minutes of Electrification.†

	Specific Resistance.	Temperature.
Mica	$0·084.10^9$ megohms	20°
Gutta-percha	0·45 ,,	24
Shellac	9·0 ,,	28
Ebonite	28·0 ,,	46
Paraffine	34·0 ,,	46

1292. ELECTROMOTIVE FORCES.—The volt, taken as unit, is the electromotive force capable of maintaining a current of an ampère in a legal ohm.

* FOUSSEREAU. *Ann. de Chim. et de Phys.* [6], Vol. v., p. 317. 1885.
† AYRTON and PERRY. *Proceedings of the Roy. Soc.*, March 21st, 1878.

ELECTROMOTIVE FORCES OF CONTACT.

If 100 represents the difference of potential of the contact of zinc and copper, the following are the values for the electromotive force of contact of zinc with different metals; from them can be deduced the electromotive force of contact of the metals with each other:

Contact of Metals.

	Volta.	Kohlrausch.	Hankel.	Ayrton and Perry.
Zinc | Platinum	—	123	123	131
,, Carbon	—	—	122	146
,, Palladium	—	—	115	—
,, Gold	—	115	110	—
,, Silver	109	109	118	—
,, Copper	100	100	100	—
,, Iron	82	75	84	80
,, Mercury	—	—	81	—
,, Bismuth	—	—	72	—
,, Antimony	—	—	69	—
,, Tin	55	—	23	37
,, Lead	45	—	44	28
,, Cadmium	—	—	24	—
,, Aluminum	—	—	−25	—

According to Kohlrausch, the contact zinc-copper is about half a Daniell, or 0·5 volt. M. Pellat* has found, on the contrary, 0·80 volt by direct electrostatic measures. It appears, on the other hand, from an experiment of Sir W. Thomson,† that the difference of potential of zinc and copper is not modified when the direct contact is replaced by a drop of water; the contact of zinc and copper would then be sufficient to give the total electromotive force of a Volta's element, which appears to be about 0·85 volt, in the first few minutes in which the fresh metals are in contact with water. Professors Ayrton and Perry finally estimate the electromotive force of the zinc-copper contact at 0·75 volts.

The results obtained by various experimenters, for the contact of metals with liquids, or of liquids with each other, do not agree closely enough to enable us to state them in a simple manner.

* PELLAT. *Journal de Phys.*, Vol. IX., p. 123. 1880.

† JENKIN. *Electricity and Magn.*, p. 46. 1873.

*Electromotive Forces of Contact, in volts at 25°, deduced from the Peltier Effect.**

Copper | Antimony†	-0·0151
,, ,, commercial	-0·0056
,, Iron	-0·0029
,, Cadmium	-0·00053
,, Zinc	-0·00045
,, Argentan	+0·00287
,, Bismuth, pure	+0·0222
,, ,, ‡	+0·0300
Iron | Zinc, at 13·8°	+0·0025§
Copper | Sulphate of Copper, at 12°	+0·212‖
Zinc | Sulphate of Zinc, at 12°	+0·241

These numbers are very different from those which have been obtained by electrostatic measurements.

ELECTROMOTIVE FORCES OF THE PRINCIPAL LIQUID ELEMENTS.

Standard Elements at 15°.¶

Sir W. Thomson.	Zinc Saturated solution of Sulphate of Zinc Semi-saturated solution of Sulphate of Copper Copper	1·074
Latimer Clark**.	Zinc Sulphate of Zinc Sulphate of Mercury in paste Mercury	1·435

Ordinary Couples.

Volta††	Zinc Ordinary Water Copper	0·98

* LE ROUX. *Ann. de Chim. et de Phys.* [4], Vol. x., p. 248. 1867.

† Alloy containing 1 eq. of antimony, 1 eq. of cadmium, and ⅛ of the total weight of cadmium.

‡ 10 bismuth + 1 antimony.

§ BELLATI. *Atti del R. I. V.* [5], Vol. v. 1879.

‖ BOUTY. *Journal de Physique,* Vol. IX., p. 229. 1880.

¶ Lord RAYLEIGH. *Phil. Trans., 1884,* pt. ii., p. 459.

** The electromotive force diminishes as the temperature increases. The variation for 1° is 0·00082 according to Lord Rayleigh, and 0·000781 according to M. Pellat.

†† The electromotive force is at first feeble, and attains the constant value 0·98 after some time. The same result is at once arrived at by adding to the liquid a little sulphate of zinc.

Leclanché	Amalgamated Zinc Solution of Sal-Ammoniac Binoxide of Manganese and Carbon	1·46
Poggendorff	Amalgamated Zinc Solution of Chromate of Potass Carbon	1·08
Warren de la Rue	Zinc Solution of Sal-Ammoniac Chloride of Silver and Silver	1·02
Daniell	Amalgamated Zinc 1 Sulphuric Acid + 4 Water Saturated solution of Sulphate of Copper Copper	1·07
Daniell	Amalgamated Zinc 1 Sulphuric Acid + 12 Water Saturated solution of Sulphate of Copper	0·97
J. Regnault	Amalgamated Zinc 1 Sulphuric Acid + 12 Water 1 Sulphate of Cadmium Cadmium	0·34
Marié-Davy	Amalgamated Zinc 1 Sulphuric Acid + 12 Water Paste of Sulphate of Mercury Carbon	1·51
Bunsen	Amalgamated Zinc 1 Sulphuric Acid + 12 Water Fuming Nitric Acid Carbon	1·94
Bunsen	Amalgamated Zinc 1 Sulphuric Acid + 12 Water Nitric Acid ($d. = 1·38$) Carbon	1·87
Grove	Amalgamated Zinc 1 Sulphuric Acid + 4 Water Fuming Nitric Acid Platinum	1·95
Poggendorff	Amalgamated Zinc 12 Bichromate of Potass + 25 Sulphuric Acid + 100 Water Carbon	2·01

1293. Thermoelectric Couples.—The thermoelectric power of two metals is a linear function of the temperature (**280**). Since,

according to M. La Roux (284 and 1034), the specific heat of electricity of lead is null, then, referring all metals to lead, we may write

$$\phi(t) = \frac{dE}{dt} = A + Bt = k(t_n - t).$$

The coefficient $k = -B$, on Professor Tait's hypothesis (284), is the constant ratio of the specific heat of electricity to the absolute temperature; the constant $t_n = -\frac{A}{B}$ represents the neutral point of the couple formed by the metal compared with lead.

For two different metals M and M′ the thermolectric power is

$$\frac{dE}{dt} = A + Bt - (A' + B't);$$

the electromotive force E_1^2, between two temperatures t_1 and t_2, the mean of which is $\theta = \frac{t_1 + t_2}{2}$, is expressed by

$$E_1^2 = (B' - B)(t_2 - t_1)\left[\frac{A - A'}{B' - B} - \theta\right].$$

This electromotive force is equal to the product of the difference of temperatures $t_2 - t_1$ of the two functions by the difference of the thermoelectric powers of the metals in respect of the mean temperature θ; we see thus that the neutral point of the couples is

$$t_n = -\frac{A - A'}{B - B'}.$$

It is not surprising that the numbers found by different experimenters are not in close agreement, for traces of foreign substances and the least change of physical state are sufficient to modify the thermoelectric power. It is difficult, for instance, to obtain bismuth-copper couples, in which the bismuth bar has been prepared by fusion, which are identical to within $\frac{1}{10}$th; the addition of a tenth of antimony to the bismuth almost doubles the electromotive force of the couple.

The following table has been calculated from Tait's experiments,* assuming that the electromotive force of the Grove's element, which was taken as standard, is 1·95 volts.

THERMOELECTRIC POWER IN MICROVOLTS.

Substance.	Referred to Lead. A+Bt. A	B	At 20°.	At 50°.	Referred to Copper at 50°.
Cadmium	-2·63	-4·24.10^{-2}	-3·48	-4·75	-2·94
Zinc	-2·32	-2·38	-2·79	-3·51	-1·70
Silver	-2·12	-1·47	-2·41	-2·86	-1·05
Gold	-2·80	-1·01	-3·00	-3·30	-1·49
Copper	-1·34	-0·94	-1·52	-1·81	—
Alloy (85Pt+15Ir)	-7·90	-0·62	-8·03	-8·21	-6·40
,, (95Pt+5Ir)	-6·15	-0·55	-6·26	-6·42	-4·61
Tin	+0·43	-0·55	+0·32	+0·16	+1·97
Aluminum	+0·76	-0·39	+0·68	+0·56	+2·37
Platinum hardened	-2·57	+0·74	-2·42	-2·20	-0·39
Magnesium	-2·22	+0·94	-2·03	-1·75	+0·06
Platinum malleable	+0·60	+1·09	+8·82	+1·15	+2·96
Alloy (90Pt+10Ir)	-5·90	+1·33	-5·63	-5·23	-3·42
Steel	-11·27	+3·25	-10·62	-9·65	-7·84
Palladium	+6·18	+3·55	+6·90	+7·96	+9·77
Iron	-17·15	+4·82	-16·20	-14·74	-12·93
Argentan	+11·94	+5·06	+12·95	+14·47	+16·28
Nickel (-18° to 175°)	+21·80	+5·06	+22·80	+24·33	+26·14
,, (250° to 300°)	+83·57	-23·84	+78·80	+71·65	+73·46
,, above 340°	+3·04	+5·06	+8·00	+5·57	+7·38

Fleeming Jenkin† deduced from Matthiesen's experiments the thermoelectric powers of different bodies in respect of lead at the temperature of 20°:

Bismuth, Pressed commercial Wire	+97	Pressed Antimony Wire	-2·8
,, Pure pressed Wire	+89	Silver, Pure hard	-3·0
,, Crystal axial	+65	Zinc ,, pressed	-3·7
,, ,, equatorial	+45	Copper, Galvanoplastic	-3·8
Cobalt	+22	Antimony, Commercial pressed Wire	-6·0
Argentan	+11·75	Arsenic	-13·56
Mercury	+0·413	Iron, Pianoforte Wire	-17·5
Lead	0·0	Antimony, Axial	-22·6
Tin	-0·1	,, Equatorial	-26·4
Copper, Commercial	-0·1	Red Phosphorus	-29·9
Platinum	-0·9	Tellurium	-502·0
Gold	-1·2	Selenium	-807·0

* TAIT. *Phil. Trans. Roy. Soc. Edin.*, *1873*, Vol. XXVII., p. 125.

† FLEEMING JENKIN. *Electricity and Magnetism*, p. 176. 1873.

In conclusion, the measurements of M. Ed. Becquerel* give for the couples formed by various metals or alloys, taking 1·07 volt for the sulphate of copper couple to which they are referred:

Couple formed with Copper and one of the following Metals.	Electromotive Force† between 0° and 100°.		Thermoelectric Power at 50° in Microvolts.	
	In Thousandths of a Daniell.	In Thousandths of a Volt.	Referred to Copper.	Referred to Lead.
Tellurium	+39·95	+42·74	−427·4	−429·3
Fused Sulphuret of Copper	+32·76	+35·05	−350·5	−352·4
Antimony and Cadmium, equal equivalents	+18·13	+19·40	−194·0	−195·0
Antimony and Zinc, in equal equivalents	+9·02	+9·65	−96·5	−98·4
Ordinary Antimony	+1·41	+1·51	−15·1	−17·0
Commercial Iron	+0·95	+1·02	−10·2	−12·1
,, ,, another spcmn.	+0·674	+0·72	−7·2	−9·1
Fused Cadmium	+0·033	+0·035	−0·35	−2·45
Silver Wire	+0·026	+0·028	−0·28	−2·18
Ordinary fused Zinc	−0·018	−0·019	+0·19	−1·71
,, ,, another spcmn.	−0·037	−0·039	+0·39	−1·51
Platinum Wire	−0·090	−0·096	+0.96	−0·94
,, ,, another specimen	−0·378	−0·404	+4·04	+2·14
Gas Graphite	−0·142	−0·152	+1·52	−0·38
Commercial Tin	−0·147	−0·157	+1·57	−0·33
,, Lead	−0·187	−0·19	+1·9	—
Mercury	−0·483	−0·52	+5·2	+3·3
Palladium Wire	−0·82	−0·88	+8·8	+6·9
Argentan ,,	−1·26	−1·35	+13·6	+12·7
Nickel ,,	−1·63	−1·74	+17·4	+15·5
Commercial Bismuth	−3·91	−4·18	+41·8	+39·9
10 Bismuth + 1 Antimony	−6·20	−6·63	+66·3	−64·4

The bismuth-copper couple has often been taken as a standard, its course being pretty regular between the temperatures of 0° and 100°, but the results obtained for this couple are very discordant.

According to M. Regnault,* a volt is equal to 3 sulphate of zinc and cadmium couples, and each of these again to 55 bismuth-copper couples, between the temperatures of 0° and 100°; so that a volt would be equal to about 165 bismuth-copper couples. The measurements of M. Ed. Becquerel would give 239 couples, and those of Fleeming Jenkin a number varying between 100 and 210, according to the state of the bismuth.

* Ed. Becquerel. *Ann. de Chim. et de Phys.* [4], Vol. VIII., p. 389. 1866.

† The current goes from the metal to the copper by the hot junction, or conversely, according as the electromotive force is positive or negative.

‡ J. Regnault. *Ann. de Chem. et de Phys.* [3], Vol. LXIV., p. 453. 1854.

1294. Specific Inductive Powers.—The duration of the charge has so great an influence that it is difficult to obtain with exactitude the value of the specific inductive power, especially for solids or liquids. In most cases the following numbers can only be considered as approximate values.

SOLIDS AND LIQUIDS.

Flint Glass, density 2·87	6·57*
,, ,, ,, 3·2	6·85
,, ,, ,, 3·66	7·4
,, ,, ,, 4·65	10·1
Ordinary Glass	5·83 to 6·34†
Flint Glass, extra dense	3·164‡
,, ,, dense	3·050
,, ,, light	3·013
Crown Glass	3·108
Paraffine	1·85 to 2·47§
Ebonite	2·21 to 3·15‖
Sulphur	2·579 to 3·21¶
,, for the three axes	4·770; 3·970; 3·811**
Shellac	3·15
Turpentine	2·15 to 2·22††
Benzole	2·199
Petroleum	2·039 to 2·071

Gases.‡‡

Air	1·000
Vacuum	0·9985
Carbonic acid	1·0008
Hydrogen	0.9998
Coal gas	1·0004
Sulphurous acid	1·0007

1295. Magnetic Constants.—The magnetic magnitudes are measured in C.G.S. units, which lead to much simpler numbers than the practical units.

* Hopkinson. *Phil. Trans.*, *1877*, p. 23.

† Wüllner. *Wied. Ann.*, Vol. I., p. 247. 1877.—Schiller. *Pogg. Ann.*, Vol. CLII., p. 535. 1874.

‡ Gordon. *Proceedings of the Roy. Soc.*, *1878*.

§ Gibson and Barclay. *Phil. Trans.*, *1871*, p. 173.—Hopkinson. Wüllner, etc., *loc. cit.*

‖ Wüllner, Schiller, Gordon, *loc. cit.*—Boltzmann. *Carl's Repertorium*, Vol. X., p. 92.

¶ Wüllner, Boltzmann, Gordon, *loc. cit.*

** Boltzmann. *Wiener Berichte*, Vol. LXX., p. 342. 1879.

†† Silow. *Pogg. Ann.*, Vols. CLVI., CLVIII.

‡‡ Ayrton and Perry, Gordon. *Treatise on Electricity*, Vol. I., p. 134.

Elements of Terrestrial Magnetism.

At Paris, January 1st, 1886.

Declination	16° 3·5′
Inclination	65° 15·7′
Horizontal component	0·1943
Vertical ,,	0·4217
Total force	0·4644

Maximum Intensity of Magnetisation.

Iron	1400
Soft Steel	780
Cobalt	800
Nickel	494

COEFFICIENTS OF MAGNETISATION.

Diamagnetic Bodies.

Values calculated, taking $14·6.10^{-6}$ for bismuth.

Glass	$0·135.10^{-6}$ *	
Wax		$0·38.10^{-6}$ †
Zinc	0·56	0·17
Ether	0·56	
Absolute Alcohol	0·58	0·53
Camphor	0·61	
Linseed Oil	0·63	
Olive ,,	0·64	
Pitch	0·65	
Nitric Acid	0·65	
Water	0·72	0·67
Solution of Ammonia	0·73	
Bisulphide of Carbon	0·74	0·89
Sulphuric Acid	0·77	
Sulphur	0·87	0·76
Chloride of Arsenic	0·91	
Fused Borate of Lead	1·02	
Lead		1·03
Phosphorus	1·24	1.10
Selenium	1·25	1·11
Copper	1·27	
Silver	1·74	
Gold	2·60	

Solutions. ‡

	Density.	Coefficients.
Water	1	$-0·67.10^{-6}$
Chloride of Sodium	1·2080	$-0·75$
Chloride of Magnesium	1·3197	$-0·81$

* FARADAY. *Experimental Researches*, Vol. III., p. 497.

† E. BECQUEREL. *Ann. de Chim. et de Phys.* [3], Vol. XXVIII., p. 283. 1850.

‡ E. BECQUEREL. *Ann. de Chim. et de Phys.* [3], Vol. XLIV., p. 209. 1855.

	Density.	Coefficients.
Sulphate of Copper	1·1265	$+0·54.10^{-6}$
,, Nickel	1·0827	+1·46
,, Iron	1·1923	+14·17
,,	1·1728	+12·09
Protochloride of Iron	1·0695	+6·17
	1·2767	+24·21
	1·4334	+44·16
Sulphate of Sesquioxide of Iron	1·1587	+9·24

Gases.

Values in respect of water, the gases being taken at the pressure 76 cm.

	Faraday.†	E. Becquerel.
Oxygen	+0·1295	+0·1823
Air	+0·0253	+0·0383
Nitrogen	+0·0022	0
Vacuum	0	
Carbonic Acid	0	−0·0051
Hydrogen	−0·0007	
Ammonia	−0·0037	−0·002
Cyanogen	−0·0067	
Nitrous Oxide		−0·018
Nitric Oxide		+0·0498
Ethylene	+0·0045	−0·0082
Chlorine		−0·0046
Sulphurous Acid		−0·0005

Specific Magnetism (for Equal Weights).

From Plücker.‡

Iron	100000
Magnetic Oxide of Iron	40227
Sesquioxide of Iron	286
Hematite	134
Specular Iron Ore	533
Hydrate of Sesquioxide of Iron	156
Sulphate ,, ,,	111
Sulphate of Protoxide of Iron	78
Nitrate of Iron in concentrated solution	34
Perchloride of Iron in concentrated solution	98
Persulphate ,, ,, ,,	58
Protochloride ,, ,, ,,	84
Protosulphate ,, ,, ,,	126
Oxide of Nickel	35
Hydrate of Oxide of Nickel	106
Hydrated Oxide of Manganese	70
Protosesquioxide of Manganese	167

† FARADAY. *Experimental Researches*, Vol. III., p. 497. 1853.

‡ PLÜCKER. *Pogg. Ann.*, Vol. LXXIV., p. 321. 1888. For want of terms of comparison, these numbers cannot be reduced to absolute measurements.

1296. MAGNETIC ROTATORY POWERS.—The most accurate experiments on the determination in absolute value of what is known as *Verdet's constant*—that is to say, the rotation of the plane of polarization between two points whose difference of magnetic potential is equal to the C.G.S. unit—were made by bisulphide of carbon.

For temperature zero, and the ray D, the values given previously (**916**), according to Gordon, H. Becquerel, and Lord Rayleigh, vary from 0·0426′ to 0·0463′. The most recent experiments of M. Kœpsel* and of M. H. Becquerel† have given respectively 0·04297′ and 0·04341′.

Bisulphide of Carbon.

Variation with temperature, according to M. Bichat.‡

$$R = R_0[1 - 0{\cdot}00104t - 0{\cdot}000014t^2].$$

Rotation for the various rays of the spectrum at 25°.§

C.	D.	E.	F.	G.
0·0319′	0·0415′	0·0637′	0·0667′	0·0920′

Liquids.||

(Ray D. Temperature, 15°.)

Sulphuric Acid, monohydrated	0·0104′
Methylic Alcohol	106
Propylic ,,	117
Nitric Acid, ordinary	123
Butylic Alcohol	124
Distilled Water	130
Amylic Alcohol	131
Chloroform	160
Chloride of Carbon	170
,, Silicon	187
Hydrochloric Acid, pure	206
Xylene	221
Toluene	242
Benzine	268

* KŒPSEL. *Wied. Ann.*, Vol. XXVI., p. 456. 1885. Observation made at 18°, and reduced by Bichat's formula.

† H. BECQUEREL. *Ann. de Chim. et de Phys.* [6], Vol. VI., p. 245. 1885. The observation was made near zero.

‡ BICHAT. *Journal de Physique*, Vol. VIII., p. 204. 1879.

§ VERDET. *Ann. de Chim. et de Phys.* [3], Vol. LII., p. 129. 1857.

|| H. BECQUEREL. *Ann. de Chim. et de Phys.* [5], Vol. XII., p. 34. 1877.

Protochloride of Phosphorus	275
Bichloride of Carbon	321
,, Sulphur SCl.	393
Protochloride of Sulphur S^2Cl.	415
Bisulphate of Carbon	422
Chloride of Arsenic	422
Bichloride of Zinc	437
Melted Sulphur (114°)	803
,, Phosphorus (33°)	1316
Bichloride of Titanium	151
Liquid Sulphurous Acid	153*

Solids.†

(Ray D. Temperature, 15°.)

Crown Glass	0·0203′
Flint ,,	325
,, ,,	416
,, ,, dense	574
,, ,, ,,	647
Sylvine (KCl)	283
Rock Salt (NaCl)	355
Blende	2234
Fluor Spar (1)	87
,, (2)	98
Spinelle (coloured by chrome)	209
Diamond	127

For the ray B, the rotation of selenium is 10·96 times, and the rotation of sub-oxide of copper (*zigueline*) 14 times that of bisulphide of carbon.

Gases.

(Ray D, under the atmospheric pressure, and at the ordinary temperature.)

Oxygen	$6{\cdot}28'.10^{-6}$‡
Atmospheric Air	6·83‡
Nitrogen	6·92‡
Carbonic Acid	13·00‡
Nitrous Oxide	16·90‡
Sulphurous Acid	31·39‡
Ethylene	34·48‡
Sulphurous Acid (at 20°, pressure 2460 mm.)	38·40§
Bisulphide of Carbon (at 70°, pressure 740 mm.)	23·49§

* BICHAT. *Journal de Physique*, Vol. IX., p. 275. 1880.

† H. BECQUEREL. *Ann. de Chim. et de Phys.* [5], Vol. XII., p. 39. 1877.

‡ H. BECQUEREL. *Journal de Physique*, Vol. IX., p. 270. 1880.

§ BICHAT. *Journal de Physique*, Vol. IX., p. 275. 1880.

SOLUTIONS.

NATURE OF THE SOLUTION.	Density.	Weight of Anhydrous Salt contained in unit weight of the solution.	Weight of Anhydrous Salt contained in unit volume of the solution.	Total Rotatory Magnetic Power.	Molecular Rotatory Magnetic Power of the Salt dissolved.	
				′	′	
Protochloride of Tin	1·3280	0·302	0·401	+0·0266	+0·0362	VERDET, *Ann.*
,,	1·1637	0·170	0·198	+0·0198	+0·0365	*de Chim. et de*
,,	1·1112	0·120	0·133	+0·0175	+0·0352	*Phys.* [3], Vol.
Chloride of Zinc	1·2851	0·266	0·342	+0·0196	+0·0214	LII., p. 129.
,,	1·1595	0·150	0·174	+0·0161	+0·0207	,,
Sal Ammoniac	1·0718	0·247	0·265	+0·0178	+0·0279	,,
,,	1·0493	0·129	0·135	+0·0154	+0·0260	,,
Chloride of Sodium	1·2052	—	0·316	+0·0180	+0·0206	H. BECQUEREL,
,,	1·1058	—	0·158	+0·0155	+0·0195	*Ann. de Chim.*
,,	1·0546	—	0·079	+0·0144	+0·0224	*et de Phys.* [5],
Chloride of Potassium	1·6000	—	0·264	+0·0163	+0·0177	Vol. XII., p. 48.
Iodide of Potassium*	1·6743	—	0·964	+0·0338	+0·0254	,,
,,	1·3398	—	0·482	+0·0237	+0·0258	,,
,,	1·1705	—	0·241	+0·0182	+0·0256	,,
,,	1·0871	—	0·120	+0·0158	+0·0270	,,
Bichloride of Copper	1·5158	—	0·6408	+0·0221	+0·0168	,,
,,	1·2789	—	0·3204	+0·0186	+0·0191	,,
,,	1·1330	—	0·1602	+0·0156	+0·0188	,,
Protochloride of Antimony dis. in Hydrochloric Acid	2·4755	—	2·1611	+0·0603	+0·0250	,,
,,	1·8573	—	1·0805	+0·0449	+0·0270	,,
,,	1·5195	—	0·5402	+0·0347	+0·0273	,,
,,	1·3420	—	0·2701	+0·0277	+0·0212	,,
Chloride of Bismuth dis. in Hydrochloric Acid...	2·0822	—	1·3204	+0·0509	+0·0296	,,
,,	1·6550	—	0·6602	+0·0406	+0·0359	,,
,,	1·4156	—	0·3301	+0·0305	+0·0350	,,
Protochlor. of Iron in Water	1·4331	—	0·5283	+0·0025	−0·0174	,,
,,	1·2141	—	0·2461	+0·0099	−0·0091	,,
,,	1·1093	—	0·1320	+0·0118	−0·0068	,,
,,	1·0548	—	0·0660	+0·0124	−0·0067	,,
Perchloride of Iron in Water	1·6933	—	1·0247	−0·2026	−0·2063	,,
,,	1·5315	—	0·7657	−0·1140	−0·1618	,,
,,	1·3230	—	0·4410	−0·0348	−0·1047	,,
,,	1·1681	—	0·2205	−0·0015	−0·0627	,,
,,	1·0864	—	0·1102	+0·0081	−0·0418	,,
,,	1·0445	—	0·0551	+0·0113	−0·0282	,,
,,	1·0232	—	0·0275	+0·0122	−0·0280	,,
Chromate of Potass	1·3598	0·319	0·504	+0·0098	−0·0026	VERDET, *loc. cit.*
Bichromate of Potass	1·0786	0·101	0·109	+0·0126	−0·0095	,,
Chromic Acid	1·3535	0·341	0·470	+0·0040	−0·0157	,,
Nitrate of Uranium	2·0267	—	1·2727	+0·0053	−0·0035	H. BECQUEREL,
,,	1·7640	—	0·9245	+0·0078	−0·0034	*loc. cit.*
,,	1·3865	—	0·4622	+0·0105	−0·0033	,,
,,	1·1963	—	0·2311	+0·0117	−0·0033	,,
Chloride of Nickel	1·4685	—	0·5311	+0·0270	+0·0279	,,
,,	1·2432	—	0·2651	+0·0196	+0·0260	,,
,,	1·1233	—	0·1327	+0·0162	+0·0252	,,
,,	1·069	—	0·0663	+0·0146	+0·0252	,,

* According to MM. CORNU and POTIER, *Comptes rendus*, Vol. CII., p. 387, 1886, the saturated solution of iodide of mercury and iodide of potassium has thrice the rotatory power of bisulphide of carbon.

Magnetic Bodies with Negative Rotatory Powers (VERDET).

Salts of Protoxide of Iron.
,, Sesquioxide of Iron.
,, ,, Manganese.
Chromic Acid.
Bichromate of Potass.
Bichloride of Titanium.
Salts of Cerium.

Magnetic Bodies of Positive Rotatory Power (id.)

Salts of Nickel.
,, Cobalt.
,, Protoxide of Manganese.
Ferricyanide of Potassium.

Diamagnetic Substances of Positive Rotatory Power (id.)

Almost all diamagnetic bodies.
Molybdate of Soda.
,, Ammonia.
Salts of Aluminum.

Diamagnetic Bodies of Negative Rotatory Power (id.)

Chlorate of Potass.
Bichloride of Titanium.
Nitrate of Uranium.
Salts of Magnesium.

THE END.

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